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ON THE CRITERION OF THE WELL-POSEDNESS OF THE MODIFIED INITIAL PROBLEM FOR SINGULAR LINEAR GENERALIZED ORDINARY DIFFERENTIAL SYSTEMS

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Abstract. Effective necessary and sufficient conditions are established for the well-possedness of the initial problem with weight for linear systems of generalized ordinary differential equations with singularities.

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1. Statement of the problem, basic notation and formulations of the basic results

Let $[a, b] \subset \mathbb{R}$ be a finite and closed interval non-degenerate in the point.

Consider the modified initial problem for linear system of generalized ordinary differential equations with singularities

$$dx = dA(t) \cdot x + df(t) \quad \text{for} \quad t \in [a, b[\tag{1.1}$$

$$\lim_{t \to b-} (\Phi^{-1}(t)\, x(t)) = 0, \tag{1.2}$$

where $A = (a_{ik})_{i,k=1}^{n}$ is an $n \times n$-matrix valued function and $f = (f_k)_{k=1}^{n}$ is an n-vector valued function, both of them have a locally bounded variation on $[a, b[$; $\Phi = \operatorname{diag}(\varphi_1, \dots, \varphi_n)$ is a diagonal $n \times n$-matrix valued function is defined on $[a, b[$ and having an inverse $\Phi^{-1}(t)$ for each $t \in [a, b[$.

Along with system (1.1) consider the perturbed singular systems

$$dx = dA_m(t) \cdot x + df_m(t) \quad \text{for} \quad t \in [a, b[\tag{1.3}$$

$(m = 1, 2, \dots)$ under the conditions (1.2), where A_m is an $n \times n$-matrix valued function and f_m is an n-vector valued function, both of them have a locally bounded variation on $[a, b[$.

We are interested to established the necessary and sufficient conditions whether the unique solvability of problem (1.1), (1.2) guarantees the unique solvability of problem (1.3), (1.2) and nearness of its solution in the definite sense if matrix-functions A_m and A and vector-functions f_m and f are nearly among themselves.

We assume

$$A(a) = A_m(a) = O_{n\times n} \ \text{ and } \ f(a) = f_m(a) = 0_n$$

$(m = 1, 2, \dots)$ without loss of generality.

The same and related problems for ordinary differential systems with singularities

$$\frac{dx}{dt} = P(t)\,x + q(t) \quad \text{for} \quad t \in [a,b], \tag{1.4}$$

where $P \in L_{loc}([a,b[,\mathbb{R}^{n\times n})$, $q \in L_{loc}([a,b[,\mathbb{R}^{n})$, have been investigated in [11], [14] (see also references therein).

The singularity of system (1.4) is considered in the sense that the matrix P and vector q functions, in general, are not integrable at the point b. In general, the solution of problem (1.4), (1.2) is not continuous at the point b and, therefore, it can not be a solution in the classical sense. But its restriction on every interval from $[a,b[$ is solution of system (1.4). In connection with this we remind the following example from [14].

Let $\alpha > 0$ and $\varepsilon \in]0,\alpha[$. Then $x(t) = |t|^{\varepsilon-\alpha}\operatorname{sgn} t$ is the unique solution of the problem

$$\frac{dx}{dt} = -\frac{\alpha\, x}{t} + \varepsilon |t|^{\varepsilon-1-\alpha}, \ \ \lim_{t\to 0}(t^{\alpha}x(t)) = 0.$$

The function x is not solution of the equation on the set $I = \mathbb{R}$, however x is a solution to the above equation only on $\mathbb{R}\setminus\{0\}$.

The singularity of system (1.1) consists in the fact that both A and f need not have bounded variations on any interval containing the point b.

The solvability question of generalized differential problem (1.1), (1.2) has been investigated in [9]. The well-posedness of problem (1.1), (1.2) with singularity has been considered in [10]. To our knowledge, the necessary and sufficient conditions for well-posedness of problem (1.1), (1.2) with singularity has not been investigated up to now.

Some singular boundary problems for generalized differential system (1.1) are investigated in [3] – [5].

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see [1]–[7], [12], [13], [15], [16] and references therein).

In the paper, we give necessary and sufficient conditions for the so called strongly Φ-well-posedness of problem (1.1), (1.2).

Throughout the paper we use the following notation and definitions.

$\mathbb{R} =]-\infty,+\infty[$.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X = (x_{ik})_{i,k=1}^{n,m}$ with the norm $\|X\| = \max\limits_{k=1,\dots,m}\sum\limits_{i=1}^{n}|x_{ik}|$.

If $X = (x_{ik})_{i,k=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{ik}|)_{i,k=1}^{n,m}$,

$$[X]_- = \frac{1}{2}(|X| - X), \ \ [X]_+ = \frac{1}{2}(|X| + X).$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all column n-vectors $x = (x_i)_{i=1}^{n}$.

$O_{n\times m}$ (or O) is the zero $n\times m$-matrix, 0_n (or 0) is the zero n-vector.

I_n is identity $n\times n$-matrix.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X.

The inequalities between the matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

If $X : \mathbb{R} \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\overset{b}{\underset{a}{\mathrm{V}}}(X)$ is the sum of total variations on $[a,b]$ of its components x_{ik} $(i = 1,\dots,n;\ k = 1,\dots,m)$; $\overset{b-}{\underset{a}{\mathrm{V}}}(X) = \lim\limits_{t\to b-} \overset{t}{\underset{a}{\mathrm{V}}}(X)$; if $a > b$, then we assume $\overset{b}{\underset{a}{\mathrm{V}}}(X) = -\overset{a}{\underset{b}{\mathrm{V}}}(X)$;

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$ at the point t $(X(a-) = X(a), X(b+) = X(b))$; $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$;

$\mathrm{BV}([c,d],\mathbb{R}^{n\times m})$ is the set of all bounded variation matrix-functions $X : [c,d] \to \mathbb{R}^{n\times m}$ (i.e., such that $\overset{d}{\underset{c}{\mathrm{V}}}(X) < \infty$).

Let $I = [a,b[$.

$\mathrm{BV}_{loc}(I;D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all $X : I \to D$ for which the restriction on $[a,c]$ belong to $\mathrm{BV}([a,c];D)$ for every $c \in I$;

$L([a,c];\mathbb{R}^{n\times m})$ is the set of all integrable matrix-functions on $[a,c]$.

$L_{loc}(I;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : I \to D$ for which the restriction on $[a,c]$ belong to $L([a,c];\mathbb{R}^{n\times m})$ for every $a, c \in I$.

If $X = (x_{ik})_{i,k=1}^{n,m} \in \mathrm{BV}_{loc}([a,b[;\mathbb{R}^{n\times m})$, then $V(X)(t) = (v(x_{ik})(t))_{i,k=1}^{n,m}$, where $v(x_{ik})(t) \equiv \overset{t}{\underset{a}{\mathrm{V}}}(x_{ik})$.

$$[X(t)]_-^v \equiv \frac{1}{2}(V(X)(t) - X(t)),\quad [X(t)]_+^v \equiv \frac{1}{2}(V(X)(t) + X(t)).$$

$s_1, s_2, s_c : BV_{loc}([a,b[;\mathbb{R}) \to BV_{loc}([a,b[;\mathbb{R})$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0,\quad s_c(x)(a) = x(a);$$
$$s_1(x)(t) = s_1(x)(a) + \sum_{a<\tau\le t} d_1x(\tau),\quad s_2(x)(t) = s_2(x)(a) + \sum_{a\le\tau<t} d_2x(\tau)$$
$$s_c(x)(t) = s_c(x)(a) + x(t) - x(a) - \sum_{j=1}^{2} s_j(x)(t) \ \text{ for } \ a < t < b.$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function and $x : [a,b] \to \mathbb{R}$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,ds_c(g)(\tau) + \sum_{s<\tau\le t} x(\tau)d_1g(\tau)$$
$$+ \sum_{s\le\tau<t} x(\tau)d_2g(\tau) \ \text{ for } \ s < t;\ s,t \in [a,b],$$

where $\int_{]s,t[} x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure corresponding to the function $s_c(g)$. So $\int_s^t x(\tau)\,dg(\tau)$ is the Kurzweil integral ([15], [16]).

If $a = b$, then $\int_a^b x(t)\,dg(t) = 0$; if $a > b$, then $\int_a^b x(t)\,dg(t) = -\int_b^a x(t)\,dg(t)$.

Moreover, we put

$$\int_s^{t-} x(\tau)\,dg(\tau) = \lim_{\delta\to 0+} \int_s^{t-\delta} x(\tau)\,dg(\tau).$$

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_a^t x(\tau)\,dg(\tau) \equiv \int_a^t x(\tau)\,dg_1(\tau) - \int_a^t x(\tau)\,dg_2(\tau).$$

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to \mathbb{R}^{l\times n}$ is a bounded variation matrix-function and $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to \mathbb{R}^{n\times m}$, then

$$\int_a^t dG(\tau)\cdot X(\tau) \equiv \bigg(\sum_{k=1}^{n}\int_a^t x_{kj}(\tau)dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m},$$

$$S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \ \ (j=1,2), \ \ S_c(G)(t) \equiv \big(s_c(g_{ik})(t)\big)_{i,k=1}^{l,n}.$$

Somewhere we use the following designation $\int_a^{\cdot} dG(s)\cdot X(s)$ for the integral $\int_a^t dG(s)\cdot X(s)$ as the vector-function to variable t.

We introduce the operators $\mathcal{A}(X,Y)$, $\mathcal{B}(X,Y)$ and $\mathcal{I}(X,Y)$ in the following way:

a) if $X \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $\det(I_n + (-1)^j d_j X(t)) \neq 0$ for $t \in I$ $(j=1,2)$, and $Y \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m})$, then

$$\mathcal{A}(X,Y)(a) = O_{n\times m},$$
$$\mathcal{A}(X,Y)(t) \equiv Y(t) - Y(a) + \sum_{a<\tau\le t} d_1X(\tau)\cdot(I_n - d_1X(\tau))^{-1}\,d_1Y(\tau)$$
$$- \sum_{a\le\tau<t} d_2X(\tau)\cdot(I_n + d_2X(\tau))^{-1}\,d_2Y(\tau);$$

b) if $X \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ and $Y : I \to \mathbb{R}^{n\times m}$, then

$$\mathcal{B}(X,Y)(a) = O_{n\times m},$$
$$\mathcal{B}(X,Y)(t) \equiv X(t)Y(t) - X(a)Y(a) - \int_a^t dX(\tau)\cdot Y(\tau);$$

c) if $X \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$, $\det(X(t)) \neq 0$, and $Y : I \to \mathbb{R}^{n\times n}$, then

$$\mathcal{I}(X,Y)(a) = O_{n\times m},$$
$$\mathcal{I}(X,Y)(t) \equiv \int_a^t d\big(X(\tau) + \mathcal{B}(X,Y)(\tau)\big)\cdot X^{-1}(\tau).$$

The operators $\mathcal{B}(X,Y)$ and $\mathcal{I}(X,Y)$ have the following properties (see, Lemma 1.2.1 from [6]):

$$\mathcal{B}(X,\mathcal{B}(Y,Z))(t) \equiv \mathcal{B}(XY,Z)(t), \tag{1.5}$$
$$\mathcal{B}\Big(X, \int_a^{\cdot} dY(s)\cdot Z(s)\Big) \equiv \int_a^t \mathcal{B}(X,Y)(s)\cdot Z(s), \tag{1.6}$$
$$\mathcal{I}(X,\mathcal{I}(Y,Z))(t) \equiv \mathcal{I}(XY,Z)(t). \tag{1.7}$$

In addition, let $\mathcal{V}_j(\Phi, A_*, \cdot) : \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times l}) \to \mathbb{R}$ $(j=1,2)$ be operators defined, respectively, by

$$\mathcal{V}_1(\Phi, A_*, F)(t,\tau) = \int_t^\tau \Phi^{-1}(s)\,d\,\mathrm{V}(\mathcal{A}(A_*,F))(s)\cdot\Phi(s) \ \text{ and}$$
$$\mathcal{V}_2(\Phi, A_*, F)(t,\tau) = \int_t^\tau \Phi^{-1}(s)\,d\,\mathrm{V}(\mathcal{A}(A_*,A_*))(s)\cdot|F(s)| \ \text{ for } \ a\le t<\tau<b.$$

Here the use will be made of the following formulas from [16]:

$$\int_a^b f(t)\,dg(t) = \int_a^b f(t)\,dg(t-) + f(b)d_1g(b) = \int_a^b f(t)\,dg(t+) + f(a)d_2g(a), \tag{1.8}$$

$$\int_a^b f(t)\,d\Big(\int_a^t h(s)dg(s)\Big) = \int_a^b f(t)h(t)dg(t) \ \text{(substitution formula)};$$

$$\int_a^b f(t)ds_j(g)(t) = \sum_{a\le t\le b} f(t)d_jg(t) \ \ (j=1,2), \tag{1.9}$$

$$\int_a^b f(t)dg(t) + \int_a^b f(t)dg(t) = f(b)g(b) - f(a)g(a) + \sum_{a<t\le b} d_1f(t)\cdot d_1g(t)$$
$$- \sum_{a\le t<b} d_2f(t)\cdot d_2g(t) \ \text{(integration-by-parts formula)}. \tag{1.10}$$

where f, g and $h \in \mathrm{BV}([a,b],\mathbb{R})$. Further, we use these formulas without special indication.

A vector-function $x : I \to \mathbb{R}^n$ is said to be a solution of system (1.1) if $x \in \mathrm{BV}_{loc}(I,\mathbb{R}^n)$ and

$$x(t) = x(a) + \int_a^t dA(\tau)\cdot x(\tau) + f(t) - f(a) \ \text{ for } \ t \in I.$$

We assume that

$$\det(I_n + (-1)^j d_jA(t)) \neq 0 \quad \text{for} \quad t \in I \quad (j=1,2).$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems (see, [15], [16] and the references therein), i.e., for the case when $A \in \mathrm{BV}([a,c];\mathbb{R}^{n\times n})$ and $f \in \mathrm{BV}([a,c];\mathbb{R}^n)$ for every $c \in I$.

Let a matrix-function $A_* = (a_{*ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ be such that

$$\det(I_n + (-1)^j d_jA_*(t)) \neq 0 \ \text{ for } \ t \in I \ \ (j=1,2). \tag{1.11}$$

Then a matrix-function $C_* : I \times I \to \mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the homogeneous system

$$dx = dA_*(t)\cdot x, \tag{1.12}$$

if, for each interval $J \subset I$ and $\tau \in J$, the restriction of the matrix-function $C_*(.,\tau) : I \to \mathbb{R}^{n\times n}$ on J is the fundamental matrix of system (1.12), satisfying the condition

$$C_*(\tau,\tau) = I_n.$$

Therefore, C_* is the Cauchy matrix of system (1.12) if and only if the restriction of C_* on $J \times J$ is the Cauchy matrix of the system in the regular case. Let $X_*(\tau) \equiv C_*(.,\tau)$.

Definition 1.1. Problem (1.1), (1.2) is said to be weakly Φ-well-posed with respect to the matrix-function A_* if it has the unique solution x_0 and for every sequences of matrix-and vector-functions A_m and f_m $(m = 1,2,\dots)$ such that

$$\det(I_n + (-1)^j d_jA_m(t)) \neq 0 \ \ \text{for} \ \ t \in I \ \ (j=1,2), \tag{1.13}$$

for each sufficiently large m, and conditions

$$\lim_{m\to+\infty} \|\mathcal{V}_1(\Phi, A_*, A_m - A)(t, b-)\| = 0, \tag{1.14}$$

$$\lim_{m\to+\infty} \|\mathcal{V}_2(\Phi, A_*, f_m - f)(t, b-)\| = 0 \tag{1.15}$$

and

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)(f_m(t) - f(t)) - \Phi^{-1}(b-)(f_m(b-) - f(b-))\| = 0 \tag{1.16}$$

hold uniformly on I, problem (1.3),(1.2) has the unique solution x_m for each sufficiently large m and the condition

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)\,(x_m(t) - x_0(t))\| = 0 \tag{1.17}$$

holds uniformly on I.

Definition 1.2. Problem (1.1), (1.2) is said to be strongly Φ-well-posed with respect to the matrix-function A_* if it has the unique solution x_0 and for every sequences of matrix- and vector-functions A_m and f_m ($m = 1, 2, \dots$) such that condition (1.13) holds for every sufficiently large m and the conditions (1.15) and

$$\lim_{m\to+\infty} \|\mathcal{V}_1(\Phi, A_*, f_m - f)(t, b-)\| = 0$$

hold uniformly on I, problem (1.3),(1.2) has the unique solution x_m for each sufficiently large m and condition (1.17) holds uniformly on I.

Remark 1.1. By Lemma 2.3 (see, below) if problem (1.1), (1.2) is strongly well-posed, then it is weakly well-posed, as well, because

$$\begin{aligned}\|\mathcal{V}_1(\Phi, A_*, f_m - f)(t, \tau)\| &\le \|\Phi^{-1}(t)(f_m(t) - f(t)) - \Phi^{-1}(\tau)(f_m(\tau) - f(\tau))\| \\ &\quad + \|\mathcal{V}_2(\Phi, A_*, f_m - f)(t, \tau)\| \ \text{ for } \ a \le t < \tau < b.\end{aligned}$$

Definition 1.3. We say that the sequence (A_m, f_m) ($m = 1, 2, \dots$) belongs to the set $\mathcal{S}_{A_*}(A, f; \Phi, b)$, i.e.,

$$\big((A_m, f_m)\big)_{m=1}^{+\infty} \in \mathcal{S}_{A_*}(A, f; \Phi), \tag{1.18}$$

if problem (1.3),(1.2) has the unique solution x_m for each sufficiently large m and condition (1.17) holds uniformly on I.

Let $I(\delta) = [b - \delta, b[$ for every $\delta > 0$.

Theorem 1.1. *Let there exist a matrix-function $A_* \in \mathrm{BV}_{loc}([a, b[, \mathbb{R}^{n\times n})$ and constant matrices B_0 and B from $\mathbb{R}_+^{n\times n}$ such that conditions* (1.11) *and*

$$r(B) < 1 \tag{1.19}$$

hold, and the estimates

$$|C_*(t, \tau)| \le \Phi(t)\, B_0\, \Phi^{-1}(\tau) \quad \textit{for} \ \ b - \delta \le t \le \tau < b \tag{1.20}$$

and

$$\left| \int_t^{b-} |C_*(t, s)| d\,\mathrm{V}(\mathcal{A}(A_*, A - A_*))(s) \cdot \Phi(s) \right| \le H(t)\, B \ \ \textit{for} \ \ t \in I(\delta) \tag{1.21}$$

fulfilled for some $\delta > 0$*, where* C_* *is the Cauchy matrix of system (1.12). Let, moreover,*

$$\lim_{t\to b-}\left\|\int_t^{b-}\Phi^{-1}(t)\,C_*(t,\tau)d\mathcal{A}(A_*,f)(\tau)\right\| = 0. \tag{1.22}$$

Then problem (1.1), (1.2) is weakly Φ*-well-posed with respect to the matrix-function* A_**.*

Theorem 1.2. *Let there exist a constant matrix* $B = (b_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$ *such that conditions (1.19) and*

$$[(-1)^j d_j a_{ii}(t)]_+ > -1 \ \text{ for } \ t \in I \ \ (j = 1, 2;\ i = 1,\dots,n) \tag{1.23}$$

hold, and the estimates

$$c_i(t,\tau) \le b_0\frac{h_i(t)}{h_i(\tau)} \quad \text{for} \quad b-\delta \le t \le \tau < b \ \ (i = 1,\dots,n); \tag{1.24}$$

$$\left|\int_t^{b-} c_i(t,\tau)h_i(\tau)d[a_{ii}(\tau)]_-^v\right| \le b_{ii}\,h_i(t) \ \text{ for } \ t\in I(\delta) \ \ (i=1,\dots,n) \tag{1.25}$$

and

$$\left|\int_t^{b-} c_i(t,\tau)h_k(\tau)d\,\mathrm{V}(\mathcal{A}(a_{*ii},a_{ik}))(\tau)\right| \le b_{ik}\,h_i(t) \quad \text{for} \ \ t\ \in I(\delta) \ \ (i\ne k;\ i,k=1,\dots,n)$$

are fulfilled for some $b_0 > 0$ *and* $\delta > 0$*. Let, moreover,*

$$\lim_{t\to b-}\int_t^{b-}\frac{c_i(t,\tau)}{h_i(t)}d\,\mathrm{V}(\mathcal{A}(a_{*ii},f_i))(\tau) = 0 \ \ (i = 1,\dots,n), \tag{1.26}$$

where $a_{*ii}(t) \equiv [a_{ii}(t)]_+^v$ $(i = 1,\dots,n)$*, and* c_i *is the Cauchy function of the equation*

$$dx = x\,da_{*ii}(t).$$

Then problem (1.1), (1.2) is weakly Φ*-well-posed with respect to the matrix-function* $A_*(t) \equiv \mathrm{diag}(a_{*11}(t),\dots,a_{*nn}(t))$*.*

Remark 1.2. The Cauchy functions $c_i(t,\tau)$, $c_i(t,t) = 1$ $(i = 1,\dots,n)$, mentioned in the theorem, have the form (see, [12])

$$c_i(t,\tau) = \begin{cases} \exp(s_c(a_{0ii})(t) - s_c(a_{0ii})(\tau)) \displaystyle\prod_{\tau<s\le t}(1-d_1a_{0ii}(s))^{-1}\times \\ \qquad \displaystyle\prod_{\tau\le s<t}(1+d_2a_{0ii}(s)) \ \text{ for } \ t>\tau, \\ \exp(s_c(a_{0ii}(t) - s_c(a_{0ii}(\tau))) \displaystyle\prod_{t<s\le\tau}(1-d_1a_{0ii}(s))\times \\ \qquad \displaystyle\prod_{t\le s<\tau}(1+d_2a_{0ii}(s))^{-1} \ \text{ for } \ t<\tau. \end{cases}$$

for $t,\tau \in I$.

Theorem 1.3. *Let conditions of Theorem (1.1) be fulfilled and let there exist a sequence of the non-degenerated matrix-functions* $H_m \in \mathrm{BV}_{loc}([a,b[;\mathbb{R}^{n\times n})$ $(m = 1,2,\dots)$ *such that*

$$\lim_{m\to+\infty}\|\Phi^{-1}(t)H_m^{-1}(t)\Phi(t) - I_n\| = 0, \tag{1.27}$$

$$\lim_{m\to+\infty}\|\mathcal{V}_1(\Phi,A_*,A_m^* - A)(t,b-)\| = 0, \tag{1.28}$$

$$\lim_{m\to+\infty}\|\mathcal{V}_2(\Phi,A_*,f_m^*-f)(t,b-)\|=0 \tag{1.29}$$

and

$$\lim_{m\to+\infty}\|\Phi^{-1}(t)(f_m^*(t)-f(t))-\Phi^{-1}(b-)(f_m^*(b-)-f(b-))\|=0 \tag{1.30}$$

hold uniformly on I, where $A_m^*(t)\equiv\mathcal{I}(H_m,A_m)(t)$ and $f_m^*(t)\equiv\mathcal{B}(H_m,f_m)(t)$. Then the inclusion

$$\big((A_m^*,f_m^*)\big)_{m=1}^{+\infty}\in\mathcal{S}_{A_*}(A,f;\Phi) \tag{1.31}$$

holds.

Theorem 1.3 has the following form if we assume that $H_m(t)\equiv I_n$ $(m=1,2,\dots)$ therein.

Corollary 1.1. Let conditions of Theorem 1.1 be fulfilled and conditions (1.14)–(1.16) hold uniformly on I. Then inclusion (1.18) holds.

Theorem 1.4. *Let conditions of Theorem 1.1 be fulfilled and let, moreover,*

$$\|B_0\|\,\|(I_n-B)^{-1}\|<1 \tag{1.32}$$

and

$$\limsup_{t\to b-}\bigg\|\Phi^{-1}(t)\int_t^{b-}dV(A)(s)\cdot\Phi(s)\bigg\|<+\infty. \tag{1.33}$$

Then inclusion (1.18) *holds if and only if there exists the sequence of matrix functions $H_m\in$ $\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ such that*

$$\limsup_{t\to b-}\bigg\|\int_t^{b-}\Phi^{-1}(s)\,d\,\mathrm{V}(\mathcal{A}(A_*,A_*))(s)\cdot\Phi(s)\bigg\|<+\infty \ \ \text{for } a\le t<\tau<b,$$

$$\limsup_{t\to b-}\Bigg(\|\Phi^{-1}(t)(f_m^*(t)-f(t))\| +\bigg\|\Phi^{-1}(t)\int_t^{b-}d\,\mathrm{V}(A)(s)\cdot|f_m^*(s)-f(s)|\bigg\|\Bigg)=0 \tag{1.34}$$

and the conditions (1.27)–(1.30) *hold uniformly on I, where the matrix-and vector functions A_m^* and f_m^* $(m=1,2,\dots)$ are defined as in Theorem 1.3.*

Theorem 1.4′. *Let conditions of Theorem 1.4 be fulfilled. Then inclusion* (1.18) *holds if and only if the conditions* (1.29), (1.30) *and*

$$\lim_{m\to+\infty}\|\Phi^{-1}(t)(X_m(t)-X_0(t))\|=0$$

hold uniformly on I, where X_0 and X_m are the fundamental matrices of systems (1.1) *and* (1.3), *respectively, and $f_m^*(t)\equiv\mathcal{B}(X_0X_m^{-1},f_m)(t)$ $(m=1,2,\dots)$.*

Remark 1.3. As to the forms of the fundamental matrixes X_0 and X_m $(m=1,2,\dots)$, one can find, for example, in [6], [7], [13].

Remark 1.4. In Theorem 1.4, condition (1.32) is essential and it cannot be neglected, i.e., if the condition is violated, then the conclusion of the the theorem is not true, in general. Below we present an example.

Let $I = [0, 1]$, $n = 1$, $b = 1$, $B = 0$, $B_0 = 1$, $\Phi(t) \equiv 1 - t$;

$$A(t) = A_m(t) = A_*(t) \equiv \ln(1-t) \ \ (m = 1, 2, \dots);$$

$$f(t) \equiv 0, \ \ f_m(t) \equiv -\frac{1}{m}\int_0^t \cos\frac{\ln(1-t)}{m} \ \ (m = 1, 2, \dots).$$

Then

$$C_*(t,\tau) \equiv \frac{1-t}{1-\tau}, \ \ x_0(t) \equiv 0, \ \ x_m(t) \equiv (1-t)\sin\frac{\ln(1-t)}{m} \ \ (m = 1, 2, \dots).$$

So, all conditions of Theorem 1.4 are fulfilled, except of (1.32), but condition (1.17) is not fulfilled uniformly on I.

Remark 1.5. The results analogous to Theorems 1.1-1.4$'$, and Corollary 1.1 are proved in [10] for the strongly well-posed case, as well. But, in the circumscribed paper, the necessary and sufficient conditions for well-posed in the strongly case are not considered.

Remark 1.6. Some corollaries with effective conditions of solvability of problem (1.1), (1.2) one can find in [9]. Moreover, in some additional conditions the solution of the problem, where $\Phi(t) \equiv \operatorname{diag}((b-t)^{\mu_1}, \dots, (b-t)^{\mu_n})$, belongs to $\mathrm{BV}([a,b],\mathbb{R}^n)$ (see, for example [9], as well).

2. Auxiliary propositions

We use new type of the Cauchy formula, differing from earlier one [16], for the representation of the solutions of the generalized systems and the lemma on the a priori estimate of the solutions of system (1.1) (Lemmas 2.1, 2.2). These propositions are proved in [9].

Lemma 2.1 *Let $A_* \in \mathrm{BV}_{loc}(I,\mathbb{R}^{n\times n})$ be such that*

$$\det(I_n + (-1)^j d_j A_*(t)) \neq 0 \quad \textit{for} \quad t \in I \quad (j = 1, 2),$$

and $f_* \in \mathrm{BV}_{loc}(I,\mathbb{R}^n)$. Then every solution $x \in \mathrm{BV}_{loc}(I,\mathbb{R}^n)$ of the system

$$dx = dA_*(t)\cdot x + df_*(t) \quad \textit{for} \quad t \in I \tag{2.1}$$

admits the representation

$$x(t) = C_*(t,s)x(s) + \int_s^t C_*(t,\tau)d\mathcal{A}(A_*,f_*)(\tau) \quad \textit{for} \quad s,t \in I, \tag{2.2}$$

where C_* is the Cauchy matrix of system (2.1).

Lemma 2.2 *Let the matrix-function $A_* \in \mathrm{BV}_{loc}(I,\mathbb{R}^{n\times n})$ and constant matrices B_0 and B from $\mathbb{R}_+^{n\times n}$ be such that conditions (1.11), (1.19), (1.20) and (1.21) hold for some $\delta > 0$, where C_* is the Cauchy matrix of system (1.12). Let, moreover,*

$$\gamma(t) = \sup\left\{\left\|\int_s^{b-} \Phi^{-1}(s)C_*(s,\tau)d\mathcal{A}(A_*,f)(\tau)\right\| : t \le s < b\right\} < +\infty \quad \textit{for} \quad t \in I(\delta).$$

Then each solution $x \in \mathrm{BV}_{loc}(J,\mathbb{R}^n)$ of system (1.1) admits the estimate

$$\|\Phi^{-1}(t)x(t)\| \le \rho\left(\|B_0\|\cdot\|\Phi^{-1}(s_0)x(s_0)\| + \gamma(t)\right) \ \textit{for} \ t \in J, \ \ t \le s < b,$$

where $\rho = \|(I_n - B)^{-1}\|$, and $J \subset I(\delta)$ and $s_0 \in J$ are an arbitrary interval and point.

Lemma 2.3 *Let a matrix-function $B \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ be such that*

$$\det(I_n + (-1)^j d_j B(t)) \neq 0 \ \ for \ \ t \in [a,b] \ \ (j=1,2)$$

and let X $(X(a)=I_n)$ be the fundamental matrix of the system

$$dX = dB(t)\cdot X \ \ for \ \ t\in[a,b].$$

Then

$$\int_a^b X^{-1}(s)d\mathcal{A}(B,F) = X^{-1}(s)\,F(s)\Big|_a^b + \int_a^b X^{-1}(s)d\mathcal{A}(B,B)(s)\cdot F(s) \tag{2.3}$$

for every $F\in \mathrm{BV}([a,b],\mathbb{R}^{n\times m})$.

Proof. Let

$$D_1(t) \equiv d_1B(t)\,(I_n - d_1B(t))^{-1}, \ \ D_2(t)\equiv d_2B(t)\,(I_n + d_2B(t))^{-1}$$

Due to definition of the operator $\mathcal{A}$ and integration-by-parts formula (1.10) we conclude

$$G \equiv \int_a^b X^{-1}(s)d\mathcal{A}(B,F)(s) = \int_a^b X^{-1}(s)dF(s)$$
$$+\int_a^b X^{-1}(s)d\Big(\sum_{a<\tau\le s} D_1(\tau)\,d_1F(\tau) - \sum_{a\le\tau<s} D_2(\tau)\,d_2F(\tau)\Big) = X^{-1}(s)\,F(s)\Big|_a^b$$
$$+\int_a^b dX^{-1}(s)\cdot F(s) + \sum_{a<s\le b} d_1X^{-1}(s)\cdot d_1F(s) - \sum_{a\le s<b} d_2X^{-1}(s)\cdot d_2F(s)$$
$$+\int_a^b X^{-1}(s)d\Big(\sum_{a<\tau\le s} D_1(\tau)\,d_1F(\tau) - \sum_{a\le\tau<s} D_2(\tau)\,d_2F(\tau)\Big).$$

Using now the equalities

$$dX^{-1}(t) \equiv -X^{-1}(t)d\mathcal{A}(B,B)(t),$$
$$d_jX^{-1}(t) \equiv -X^{-1}(t)d_j\mathcal{A}(B,B)(t) \ \ (j=1,2)$$

and

$$d_j\mathcal{A}(B,B)(t) \equiv d_jB(t)\left(I_n + (-1)^j d_jB(t)\right)^{-1} \ \ (j=1,2) \tag{2.4}$$

([6], see Proposition 1.1.4) we find

$$G \equiv \int_a^b X^{-1}(s)d\mathcal{A}(B,F)(s) = \int_a^b X^{-1}(s)dF(s)$$
$$+\int_a^b X^{-1}(s)d\Big(\sum_{a<\tau\le s} D_1(\tau)\,d_1F(\tau) - \sum_{a\le\tau<s} D_2(\tau)\,d_2F(\tau)\Big)$$
$$= X^{-1}(s)\,F(s)\Big|_a^b + \int_a^b X^{-1}(s)d\mathcal{A}(B,B)(s)\cdot F(s)$$
$$-\sum_{a<s\le b} X^{-1}(s)\,D_1(s)\,d_1F(s) + \sum_{a\le s<b} X^{-1}(s)\,D_2(s)\,d_2F(s)$$

$$+\int_a^b X^{-1}(s)d\Big(\sum_{a<\tau\le s} D_1(\tau)\,d_1F(\tau)-\sum_{a\le\tau<s} D_2(\tau)\,d_2F(\tau)\Big).$$

So that, thanks to (1.9) equality (2.3) is valid.

3. Proofs of results

Proof of Theorem 1.1. By conditions (1.19) – (1.22) problem (1.1), (1.2) has the unique solution x ([9], see Theorem 1.1). On the other hand, because I is the finite interval there exists $\overline{\rho}\in\mathbb{R}_+^n$ such that

$$|x(t)|\le\Phi(t)\overline{\rho}\quad\text{for}\quad t\in I. \tag{3.1}$$

It is clear that

$$\rho_1=\sup\{\rho_1(\delta):\delta\in]0,b-a]\}<+\infty, \tag{3.2}$$

where

$$\rho_1(\delta)=\|\mathcal{V}_1(\Phi,A_*,A-A_*)(a,b-\delta)\|.$$

Let B_1 be the $n\times n$-matrix whose every element equals to 1 and let $\widetilde{B}=B+\eta_0B_0\,B_1$. Then due to (1.19) there exists $\eta_0\in]0,1[$ such that

$$r(\widetilde{B})<1. \tag{3.3}$$

Let $\varepsilon>0$ be an arbitrary fixed number. Then, taking into account (3.2), we get that there there exists $\eta\in]0,\eta_0[$ such that

$$\rho_0\big[1+(1-\eta)^{-1}\|B_0\|\,\exp\big((1-\eta)^{-1}(\eta+\rho_1)\|B_0\|\big)\big]<\varepsilon, \tag{3.4}$$

where

$$\rho_0=\eta\,(1+\|\overline{\rho}\|)(1+\|(I_n-\widetilde{B})^{-1}\|\,\|B_0\|).$$

Let $A_m\in\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ and $f_m\in\mathrm{BV}_{loc}(I;\mathbb{R}^n)$ $(m=1,2,\dots)$ be an arbitrary matrix- and vector-functions satisfying conditions (1.13), (1.14) and (1.15). In first, we have to show that the matrix-and vector-functions A_m and f_m $(m=1,2,\dots)$ satisfy conditions (1.21) and (1.22).

By (1.20) and (1.21) we find that, without loss of generality, for every natural m

$$\begin{aligned}
&\left|\int_t^\tau |C_*(t,s)|d\,\mathrm{V}(\mathcal{A}(A_*,A_m-A_*))(s)\cdot\Phi(s)\right|\\
&\le\left|\int_t^\tau |C_*(t,s)|d\,\mathrm{V}(\mathcal{A}(A_*,A-A_*))(s)\cdot\Phi(s)\right|\\
&+\left|\int_t^\tau |C_*(t,s)|d\,\mathrm{V}(\mathcal{A}(A_*,A_m-A))(s)\cdot\Phi(s)\right|\\
&\le\Phi(t)B+\Phi(t)B_0|\mathcal{V}_1(\Phi,A_*,A_m-A)(t,\tau)|\\
&\le\Phi(t)\widetilde{B}\ \text{ for }\ a\le t<\tau<b.
\end{aligned}$$

Therefore, the matrix-function $\widetilde{B}$ satisfies condition (1.21).

In addition, due to (1.14) – (1.16) we can assume without loss of generality that, for every $m > m_0$,

$$\|\mathcal{V}_1(\Phi, A_*, A_m - A)(t, b-)\| < \eta \ \text{ for } \ t \in I, \tag{3.5}$$

and

$$\begin{aligned} &\|\Phi^{-1}(t)(f_m(t) - f(t)) - \Phi^{-1}(b-)(f_m(b-) - f(b-))\| \\ &\quad + \|\mathcal{V}_2(\Phi, A_*, f_m - f)(t, b-)\| < \eta \ \text{ for } \ t \in I. \end{aligned} \tag{3.6}$$

Below, we assume that $m > m_0$ is an arbitrary fixed natural number.

Now, using (1.20) and (2.3) we show that

$$\begin{aligned} &\left|\int_t^\tau \Phi^{-1}(t)\, C_*(t,s) d\mathcal{A}(A_*, f_m)(s)\right| \le \left|\int_t^\tau \Phi^{-1}(t)\, C_*(t,s) d\mathcal{A}(A_*, f)(s)\right| \\ &+ \Phi^{-1}(t)\left|\int_t^\tau C_*(t,s) d\mathcal{A}(A_*, f_m - f)(s)\right| \\ &= \left|\int_\tau^t \Phi^{-1}(t) C_*(t,s) d\mathcal{A}(A_*, f)(s)\right| \\ &+ \Phi^{-1}(t)\left| X_*(t) \int_\tau^t X_*^{-1}(s) d\mathcal{A}(A_*, f_m - f)(s)\right| \\ &\le \left|\int_t^\tau \Phi^{-1}(t) C_*(t,s) d\mathcal{A}(A_*, f)(s)\right| + B_0 \left|\big(\Phi^{-1}(s)(f_m(s) - f(s))\big)\Big|_t^\tau\right| \\ &+ B_0 |\mathcal{V}_2(\Phi, A_*, f_m - f)(t, \tau)| \ \text{ for } \ a \le t < \tau < b. \end{aligned}$$

From this, in view of the conditions (1.15), (1.16) and (1.22) it follows that the vector-function f_m satisfies condition (1.22), as well.

Hence, according to Theorem 1.1 from [9], the last two conditions together with condition (1.13) guarantee the unique solvability of problem (1.3), (1.2).

Let x_m be the solution of problem (1.3), (1.2) and let

$$z(t) \equiv x(t) - x_m(t) \ \text{ and } \ u(t) \equiv \|\Phi^{-1}(t) z(t)\|.$$

Then z will be a solution of the system

$$dz = dA_m(t) \cdot z + d\varphi_m(t)$$

under the condition

$$\lim_{s_0 \to b-} (\Phi^{-1}(s_0)\, z(s_0)) = 0,$$

where

$$\varphi_m(t) = g_m(t) + f(t) - f_m(t), \ \ g_m(t) = \int_a^t d(A(\tau) - A_m(\tau)) \cdot x(\tau).$$

In view of Lemma 2.2, conditions (3.3) and (3.12) guarantee the estimate

$$u(t) \le \|(I_n - \widetilde{B})^{-1}\|\,\gamma(t) \quad \text{for} \quad t \in I(\delta), \tag{3.7}$$

where

$$\gamma(t) = \sup\left\{\left\|\int_s^{b-} \Phi^{-1}(s)C_*(s,\tau)d\mathcal{A}(A_*, f_*)(\tau)\right\| : t \le s < b\right\} \ \text{for} \ t \in I(\delta).$$

It is not difficult to verify that

$$\int_s^{s_0} \Phi^{-1}(s)C_*(s,\tau)d\mathcal{A}(A_*, g_m)(\tau)$$
$$= \int_s^{s_0} \Phi^{-1}(s)C_*(s,\tau)d\mathcal{A}(A_*, A - A_m)(\tau)\cdot x(\tau) \ \text{for} \ b-\delta \le s < s_0 < b.$$

From this we conclude that

$$\left|\int_s^{s_0} \Phi^{-1}(s)C_*(s,\tau)d\mathcal{A}(A_*, f)(\tau)\right| = \left|\int_s^{s_0} \Phi^{-1}(s)C_*(s,\tau)d\mathcal{A}(A_*, g_m)(\tau)\right.$$
$$\left. + \int_s^{s_0} \Phi^{-1}(s)C_*(s,\tau)d\mathcal{A}(A_*, f - g_m)(\tau)\right|$$
$$\le \left|\int_s^{s_0} \Phi^{-1}(s)|C_*(s,\tau)|d\,\mathrm{V}(\mathcal{A}(A_*, A - A_m))(\tau)\cdot|f(\tau) - g_m(\tau)|\right|$$
$$+ B_0\left|\left(\Phi^{-1}(s)(f(s) - g_m(s))\right)\Big|_t^\tau\right| + B_0|\mathcal{V}_2(\Phi, A_*, f_m - f)(t,\tau)|$$
$$\text{for} \ b-\delta \le s < s_0 < b$$

and, therefore, due to (1.20), (3.5), (3.6) and (3.7) we find

$$\gamma(t) \le \eta\,(1 + \|\overline{\rho}\|)\|B_0\|$$

and

$$u(t) \le \eta\,(1 + \|\overline{\rho}\|)\,\|(I_n - \widetilde{B})^{-1}\|\,\|B_0\| < \rho_0 \ \text{for} \ t \in I(\delta). \tag{3.8}$$

Let $b - \delta > a$. Consider the case where $t \in [a, b-\delta]$.

Due to (3.12) the vector-function $z(t)$ satisfies the system

$$dz = dA_*(t)\cdot z + d(A_m(t) - A_*(t))\cdot z + d\varphi_m(t).$$

Therefore, according to Lemma 2.1 (see, (2.2)) we find

$$\Phi^{-1}(t)z(t) = \Phi^{-1}(t)C_*(t, b-\delta)z(b-\delta)$$
$$+ \int_t^{b-\delta} \Phi^{-1}(t)C_*(t,\tau)d\mathcal{A}(A_*, A_m - A_*)(\tau)\cdot z(\tau)$$
$$+ \int_t^{b-\delta} \Phi^{-1}(t)C_*(t,\tau)d\mathcal{A}(A_*, A - A_m)(\tau)\cdot x(\tau)$$

$$+\int_t^{b-\delta}\Phi^{-1}(t)C_*(t,\tau)d\mathcal{A}(A_*,f-f_m)(\tau)\ \text{ for }\ t\in[a,b-\delta].$$

Further, using (2.3) we have

$$\begin{aligned}
&\left\|\int_t^{b-\delta}\Phi^{-1}(t)C_*(t,\tau)d\mathcal{A}(A_*,f-f_m)(\tau)\right\|\\
&=\left\|\Phi^{-1}(t)X_*(t)\int_t^{b-\delta}X_*^{-1}(\tau)d\mathcal{A}(A_*,f-f_m)(\tau)\right\|\\
&=\left\|\Phi^{-1}(t)X_*(t)\bigg\{\big(X_*^{-1}(\tau)\,(f(\tau)-f_m(\tau))\big)\Big|_t^{b-\delta}\right.\\
&\left.+\int_t^{b-\delta}X_*^{-1}(\tau)d\mathcal{A}(A_*,A_*)\cdot(f(\tau)-f_m(\tau))\bigg\}\right\|\\
&\leq\left\|\Phi^{-1}(t)C_*(t,\tau)\,(f(\tau)-f_m(\tau))\Big|_t^{b-\delta}\right\|\\
&+\left\|\int_t^{b-\delta}\Phi^{-1}(t)C_*(t,\tau)d\mathcal{A}(A_*,A_*)\cdot(f(\tau)-f_m(\tau))\right\|\ \text{ for }\ t\in[a,b-\delta].
\end{aligned}$$

From this due to (1.20), if we take consideration that by (1.20) and (3.1)

$$\begin{aligned}
&\|\Phi^{-1}(t)z(t)\|\leq\|B_0\|\|\Phi^{-1}(t)*b-\delta\|\\
&\quad+\|B_0\|\left\|\int_t^{b-\delta}\Phi^{-1}(\tau)\,d\,\mathrm{V}(\mathcal{A}(A_*,A_m-A_*))(\tau)\cdot\Phi(\tau)\,|\Phi^{-1}(\tau)z(\tau)|\right\|\\
&\quad+\|B_0\|\left\|\int_t^{b-\delta}\Phi^{-1}(\tau)d\,\mathrm{V}(\mathcal{A}(A_*,A-A_m))(\tau)\cdot|x(\tau)|\right\|\\
&\quad+\left\|\int_t^{b-\delta}\Phi^{-1}(t)C_*(t,\tau)d\mathcal{A}(A_*,f-f_m)(\tau)\right\|\ \text{ for }\ t\in[a,b-\delta],
\end{aligned}$$

we conclude

$$\begin{aligned}
&\|\Phi^{-1}(t)z(t)\|\leq\|B_0\|\|\Phi^{-1}(b-\delta)z(b-\delta)\|\\
&+\|B_0\|\left\|\int_t^{b-\delta}dW_m(\tau)\cdot|\Phi^{-1}(\tau)z(\tau)|\right\|\\
&+\|B_0\|\left\|\int_t^{b-\delta}\Phi^{-1}(\tau)d\,\mathrm{V}(\mathcal{A}(A_*,A-A_m))(\tau)\cdot|x(\tau)|\right\|\\
&+\|B_0\|\left\|\Phi^{-1}(\tau)\,(f(\tau)-f_m(\tau))\Big|_t^{b-\delta}\right\|\\
&+\|B_0\|\,\|\mathcal{V}_2(\Phi,A_*,f-f_m)(b-\delta,t)\|\ \text{ for }\ t\in[a,b-\delta],
\end{aligned}$$

where

$$W_m(t)\equiv\int_a^t\Phi^{-1}(\tau)\,d\,\mathrm{V}(\mathcal{A}(A_*,A_m-A_*))(\tau)\cdot\Phi(\tau).$$

Let $w_m(t) \equiv \|W_m(t)\|$. Due to $(3.1), (3.5), (3.6)$ and (3.8), we find

$$u(t) \le \rho_0 \|B_0\| + \|B_0\| \int_t^{b-\delta} u(\tau) dw_m(\tau) \quad \text{for} \quad t \in [a, b-\delta]. \tag{3.9}$$

Moreover, by (3.5) we have

$$|d_j w_m(t)| \le \eta < 1 \quad \text{for} \quad t \in I \;\; (j = 1, 2). \tag{3.10}$$

Let now

$$w_m^+(t) = w_m(t+) \;\text{ for }\; t \in [a, b-\delta[\;\text{ and }\; w_m^+(b-\delta) = w_m(b-\delta).$$

Then due to (3.10) and the equality

$$\int_t^{b-\delta} u(\tau) dw_m(\tau) = \int_t^{b-\delta} u(\tau) dw_m^+(\tau) + u(t) d_2 w_m(t) \;\text{ for }\; t \in [a, b-\delta]$$

(see, (1.8)), from (3.9) it follows

$$u(t) \le (1-\eta)^{-1} \|B_0\| \left(\rho_0 + \int_t^{b-\delta} u(\tau) dw_m^+(\tau) \right) \;\text{ for }\; t \in [a, b-\delta].$$

Therefore, according to Gronwalls inequality (see, [16], Theorem I.4.30) the estimate holds

$$u(t) \le \rho_0 (1-\eta)^{-1} \|B_0\| \, \exp\big((1-\eta)^{-1} \|B_0\| (w_m^+(b-\delta) - w_m^+(t))\big) \\ \text{for }\; t \in [a, b-\delta]. \tag{3.11}$$

It is evident that the function w is nondecreasing. Using these and (3.5) we get

$$w_m^+(b-\delta) - w_m^+(t) \le \left\| \int_a^{b-\delta} \Phi^{-1}(\tau) \, d\,\mathrm{V}(\mathcal{A}(A_*, A_m - A))(\tau) \cdot \Phi(\tau) \right\| \\ + \left\| \int_a^{b-\delta} \Phi^{-1}(\tau) \, d\,\mathrm{V}(\mathcal{A}(A_*, A - A_*))(\tau) \cdot \Phi(\tau) \right\| < \eta + \rho_1 \;\text{ for }\; t \in [a, b-\delta].$$

So, thanks to (3.11)

$$u(t) \le \rho_0 (1-\eta)^{-1} \|B_0\| \, \exp\big((\eta + \rho_1)(1-\eta)^{-1} \|B_0\|\big) \;\text{ for }\; t \in [a, b-\delta].$$

By this and (3.8), thanks to (3.4), we have

$$\|\Phi^{-1}(t) z(t)\| < \varepsilon \;\text{ for }\; t \in [a, b].$$

Therefore, estimate (1.17) holds uniformly on I.

Proof of Theorem 1.2. Let us assume

$$\mathcal{A}(A_*, A - A_*)(t) \equiv (\widetilde{a}_{ik}(t))_{i,k=1}^n.$$

By the definition of the operator $\mathcal{A}$ we find

$$\widetilde{a}_{ik}(t) = \mathcal{A}(a_{*ik}, a_{ik} - a_{*ik})(t) = a_{ik}(t) - a_{*ik}(t)$$

$$+\sum_{0<\tau\le t} d_1 a_{*ii}(\tau)\cdot(1-d_1a_{*ii}(\tau))^{-1}d_1(a_{ii}(\tau)-a_{*ik}(\tau))$$
$$-\sum_{0\le\tau<t} d_2 a_{*ii}(\tau)\cdot(1+d_2a_{*ii}(\tau))^{-1}d_2(a_{ii}(\tau)-a_{*ik}(\tau))$$
$$\text{for } t\in I \ \ (i,k=1,\dots,n), \tag{3.12}$$

where $a_{*ik}(t)=0$ if $i\neq k$ $(i,k=1,\dots,n)$.

Consider the case where $i=k$ $(i=1,\dots,n)$. It is evident that

$$a_{ii}(t)-a_{*ii}(t)=-[a_{ii}(t)]_-^v \ \text{ for } \ t\in I \ \ (i=1,\dots,n).$$

Therefore,

$$d_ja_{0ii}(t)=[d_ja_{ii}(t)]_+ \ \text{ and } \ d_j(a_{ii}(t)-a_{0ii}(t))=-[d_ja_{ii}(t)]_-$$
$$\text{for } \ t\in I \ \ (j=1,2;\ i=1,\dots,n). \tag{3.13}$$

So that, due to (3.13)

$$d_ja_{0ii}(t)\cdot d_j(a_{ii}(t)-a_{0ii}(t))=0 \ \text{ for } \ t\in I_{t_0}(\delta) \ \ (j=1,2;\ i=1,\dots,n).$$

Thus from (3.12) we have

$$\widetilde{a}_{ii}(t)=-[a_{ii}(t)]_-^v \ \text{ for } \ t\in I \ \ (i=1,\dots,n).$$

On the other hand, it is evident that

$$\widetilde{a}_{ik}(t)\equiv\mathcal{A}(a_{0ii},a_{ik})(t) \quad \text{for } \ i\neq k \ \ (i,k=1,\dots,n).$$

The Cauchy matrix of system (1.12) has the form

$$C(t,\tau)\equiv\operatorname{diag}(c_1(t,\tau),\dots,c_n(t,\tau)).$$

In addition, due to $(1.23), (3.13)$ and (3.7), conditions (1.11) and

$$c_i(t,\tau)>0 \quad \text{for} \quad (t-t_0)(\tau-t_0)>0 \ \ (i=1,\dots,n)$$

hold. By this results, $(1.24), (1.25)$ and (1.26) we conclude that conditions $(1.20), (1.21)$ and (1.22) of Theorem 1.1 are valid. Hence the theorem immediately follows from Theorem 1.1.

Proof of Theorem 1.3. For each natural m, consider the system

$$dy=dA_m^*(t)\cdot y+f_m^*(t) \ \text{ for } \ t\in[a,b[. \tag{3.14}$$

Due to (1.19) there exists $\eta_0\in]0,1[$ such that $r(\widetilde{B})<1$, where $\widetilde{B}=B+\eta_0B_0\,\mathfrak{I}_{\mathfrak{n}\times\mathfrak{n}}$.

Let us show that, for each sufficiently large m, the matrix-function A_m^* and the vector-function f_m^* satisfy, respectively, conditions (1.14) and (1.15) for constant matrix $\widetilde{B}$, where C_* is the Cauchy matrix of system (1.12).

Indeed, due to (1.14) we have

$$\bigg\|\int_t^{b-}\Phi^{-1}(s)\,d\operatorname{V}(\mathcal{A}(A_*,A_m^*-A))(s)\cdot\Phi(s)\bigg\|<\eta_0 \ \text{ for } \ t\in[b-\delta,b[$$

for each sufficiently large m.

On the other hand, in view of (1.20) and (1.21) we have

$$\begin{aligned}
&\left\| \int_t^{b-} |C_*(t,s)|\Phi^{-1}(s)\, d\,\mathrm{V}(\mathcal{A}(A_*, A_m^* - A_*))(s)\cdot\Phi(s)\right\| \\
&\quad \le \left\| \int_t^{b-} |C_*(t,s)|\Phi^{-1}(s)\, d\,\mathrm{V}(\mathcal{A}(A_*, A_m^* - A))(s)\cdot\Phi(s)\right\| \\
&\quad + \left\| \int_t^{b-} |C_*(t,s)|\Phi^{-1}(s)\, d\,\mathrm{V}(\mathcal{A}(A_*, A - A_*))(s)\cdot\Phi(s)\right\| \\
&\quad \le \Phi(t)\, B \left\| \int_t^{b-} \Phi^{-1}(s)\, d\,\mathrm{V}(\mathcal{A}(A_*, A_m^* - A))(s)\cdot\Phi(s)\right\| + \Phi(t)B
\end{aligned}$$

for each sufficiently large m and, therefore, we conclude that, without loss of generality, for every natural m,

$$\left\| \int_t^{b-} |C_*(t,s)|\Phi^{-1}(s)\, d\,\mathrm{V}(\mathcal{A}(A_*, A_m^* - A_*))(s)\cdot\Phi(s)\right\| \le \Phi(t)\widetilde{B} \quad \text{for } t \in [b-\delta, b[.$$

Similarly, we show that

$$\lim_{t\to b-} \left\| \int_t^{b-} H^{-1}(t)\, C_*(t,\tau) d\mathcal{A}(A_*, f_m^*)(\tau)\right\| = 0.$$

for each natural m.

In addition, by (1.13) and the equality

$$I_n + (-1)^j d_j A_m^*(t) \equiv \big(\Phi(t) + (-1)^j d_j\Phi(t)\big)(I_n + (-1)^j d_j A_m^*(t))\Phi(t) \quad (j=1,2;\ m=1,2,\dots),$$

we conclude that matrix-functions A_m^* $(m = 1,2,\dots)$ satisfy condition (1.13), as well. So, according to Theorem 1.1, system (3.14), under condition

$$\lim_{t\to b-} (\Phi^{-1}(t)\, y(t)) = 0,$$

has the unique solution y_m for every m and

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)\,(y_m(t) - x_0(t))\| = 0 \tag{3.15}$$

uniformly on I (here the value of the left hand equals 0 at the point b).

On the other hand, it is not difficult to verify that x_m is a solution of system (1.3) if and only if the vector-function $y_m(t) = H_m(t)x_m(t)$ is a solution of system (3.14) for each natural m. In addition, by (1.27) and the equality

$$\Phi^{-1}(t)x_m(t) = (\Phi^{-1}(t)H_m^{-1}(t)\Phi(t))\,\Phi^{-1}(t)y_m(t)$$

$(m = 1,2,\dots)$, the vector-function x_m satisfy condition (1.2) if and only if the vector-function y_m satisfy the same condition.

So that, the vector-functions $x_m(t) = H_m^{-1}(t)\, y_m(t)$ $(m = 1, 2, \dots)$ will be solutions of problems (1.1), (1.2), respectively.

Let us show that that condition (1.17) holds uniformly on I.

We have

$$\begin{aligned}\Phi^{-1}(t)\,(x_m(t) - x_0(t)) &= \Phi^{-1}(t)\,\big(H_m^{-1}(t)y_m(t) - x_0(t)\big)\\ &= \Phi^{-1}(t)\,\big(H_m^{-1}(t)\Phi(t)\,\Phi^{-1}(t)y_m(t) - \Phi(t)\,\Phi^{-1}(t)x_0(t)\big)\\ &+ \Phi^{-1}(t)\,\big(H_m^{-1}(t)\Phi(t)\,\Phi^{-1}(t)x_0(t) - \Phi(t)\,\Phi^{-1}(t)x_0(t)\big)\\ &= \Phi^{-1}(t)H_m^{-1}(t)\Phi(t)\,\big(\Phi^{-1}(t)y_m(t) - \Phi^{-1}(t)x_0(t)\big)\\ &+ \big(\Phi^{-1}(t)(H_m^{-1}(t) - I_n)\Phi(t)\big)\Phi^{-1}(t)x_0(t) \ \text{ for } \ t \in [a, b[\end{aligned}$$

and, therefore,

$$\begin{aligned}\|\Phi^{-1}(t)\,(x_m(t) - x_0(t))\| &\le \|\Phi^{-1}(t)H_m^{-1}(t)\Phi(t)\|\,\|\Phi^{-1}(t)(y_m(t) - x_0(t))\|\\ &+ \|\Phi^{-1}(t)(H_m^{-1}(t) - I_n)\Phi(t)\|\,\|\Phi^{-1}(t)x_0(t)\| \ \text{ for } \ t \in I,\end{aligned}$$

because the left side of the inequality equals to 0 for $t = b$ (by definition).

From the estimate, due to (1.27) and (3.15), we conclude that (1.17) holds uniformly on I. Hence inclusion (1.31) holds.

Proof of Theorem 1.4. The sufficiency follows from Theorem 1.3.

Let us show the necessity.

Let $\delta > 0$ be such that conditions of Lemma 2.2 are fulfilled.

For each $m \in \{0, 1, \dots\}$, let X_m $(X_m(a) = I_n)$ with columns x_{mj} $(j = 1, \dots, n)$ be a fundamental matrix of system (1.3) (if $m = 0$, then under the system we understand system (1.1) on the interval $[a, b[$).

Due to Lemma 2.2 we have the estimates

$$\begin{aligned}\|\Phi^{-1}(t)x_{mj}(t)\| \le \rho\|B_0\|\,\|\Phi^{-1}(s_0)x_{mj}(s_0)\| \ \text{ for } \ b - \delta \le t < s_0 < b\\ (j = 1, \dots, n;\ m = 0, 1, \dots),\end{aligned} \tag{3.16}$$

where $\rho = \|(I_n - B)^{-1}\|$.

Passing to the limit when $s_0 \to b-$ in the right hand of (3.16), we obtain

$$\begin{gathered}\|\Phi^{-1}(t)x_{mj}(t)\| \le \rho\|B_0\| \limsup_{s_0 \to b-} \|\Phi^{-1}(s_0)x_{mj}(s_0)\|\\ \text{for } \ b - \delta \le t < b \ \ (j = 1, \dots, n;\ m = 0, 1, \dots).\end{gathered}$$

Therefore,

$$\begin{gathered}\limsup_{t \to b-} \|\Phi^{-1}(t)x_{mj}(t)\| \le \rho\|B_0\| \limsup_{s_0 \to b-} \|\Phi^{-1}(s_0)x_{mj}(s_0)\|\\ (j = 1, \dots, n;\ m = 0, 1, \dots).\end{gathered}$$

From this, in view of (1.32), we have

$$\limsup_{t \to b-} \|\Phi^{-1}(t)x_{mj}(t)\| = 0 \ \ (j = 1, \dots, n;\ m = 0, 1, \dots).$$

Hence

$$\lim_{t \to b-} \|\Phi^{-1}(t)x_{mj}(t)\| = 0 \ \ (j = 1, \dots, n;\ m = 0, 1, \dots). \tag{3.17}$$

Let $H_m(t) \equiv X_0(t)\,X_m^{-1}(t)$ $(m = 0,1,\dots)$. It is evident that $H_m \in \mathrm{BV}_{loc}([a,b[;\mathbb{R}^{n\times n})$ $(m = 0,1,\dots)$.

Let us verify conditions (1.28) and (1.29).

In view of equality (1.5) and the equalities

$$X_m^{-1}(t) \equiv I_n - \mathcal{B}(X_m^{-1}, A_m)(t) \ \ (m = 0,1,\dots)$$

(see, Proposition 1.1.4 from [6]), we have

$$\begin{aligned}
&H_m(t) + \mathcal{B}(H_m, A_m)(t) = X_0(t)X_m^{-1}(t) + \mathcal{B}(X_0, \mathcal{B}(X_m^{-1}, A_m))(t)\\
&= X_0(t)X_m^{-1}(t) + \mathcal{B}(X_0, I_n - X_m^{-1})(t) = I_n + \mathcal{B}(X_0, I_n)(t) + \int_a^t dX_0(s)\cdot X_m^{-1}(s)\\
&= I_n + \int_a^t dA(s)\cdot X_0(s)\cdot X_m^{-1}(s) = I_n + \int_a^t dA(s)\cdot H_m(s)\\
&\qquad\qquad\qquad \text{for } t \in [a,b[\ \ (m = 1,2,\dots).
\end{aligned} \tag{3.18}$$

Consequently, due to (1.7), we find

$$A_m^*(t) = \mathcal{I}(H_m, A_m)(t) = \int_a^t dA(s)\cdot H_m(s)H_m^{-1}(s) \equiv A(t) \ \ (m = 0,1,\dots).$$

So that, condition (1.28) is valid uniformly on I.

It is clearly, conditions of Corollary 1.1 are fulfilled for the homogeneous systems corresponding to systems (1.1) and (1.3) $(m = 1,2,\dots)$, i.e., when $f(t) \equiv 0_n$ and $f_m(t) \equiv 0_n$ $(m = 1,2,\dots)$.

Now, if we take account (3.17), thanks to Corollary 1.1 we get

$$\lim_{m\to+\infty} \big(\Phi^{-1}(t)X_m(t) - \Phi^{-1}(t)X_0(t)\big) = O_{n\times n} \tag{3.19}$$

uniformly on I. So condition (1.27) holds.

Moreover, due (3.19) we have

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)H_m^{-1}(t)\Phi(t) - I_n\| = \lim_{m\to+\infty} \|\Phi^{-1}(t)X_m(t)X_0^{-1}(t)\Phi(t) - I_n\| = 0 \tag{3.20}$$

uniformly on I.

Consider now condition (1.29).

Let x_m $(m = 0,1,\dots)$ be the unique solution of problem (1.3), (1.2). Let $y_m(t) \equiv H_m(t)x_m(t)$ $(m = 0,1,\dots)$, as in the proof of Theorem 1.3, be the solution of system (3.14).

Due to (1.17) we have

$$\lim_{m\to+\infty} \big(\Phi^{-1}(t)x_m(t) - \Phi^{-1}(t)x_0(t)\big) = 0_n \tag{3.21}$$

uniformly on I.

Besides, due to (3.20), we have

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)H_m(t)\Phi(t) - I_n\| = 0$$

uniformly on I. From this and (3.21), by equalities

$$y_m(t) \equiv \Phi(t)\big(\Phi^{-1}(t)H_m(t)\Phi(t)\big)\,\big(\Phi^{-1}(t)x_m(t)\big) \ \ (m = 1,2,\dots),$$

we conclude that the function y_m satisfies condition (1.2) if and only if the function x_m satisfies the same one and, moreover,

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)y_m(t) - \Phi^{-1}(t)x_0(t)\| = 0$$

and

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)z_m(t)\| = 0 \tag{3.22}$$

uniformly on I, where $z_m(t) \equiv y_m(t) - x_0(t)$.

Further, using (1.6), we conclude

$$\begin{aligned} f_m^*(t) =&\mathcal{B}(H_m, f_m)(t) = \mathcal{B}\Big(H_m, x_m - \int_a^{\cdot} dA_m(s)\cdot x_m(s)\Big)(t)\\ &= \mathcal{B}(H_m, x_m)(t) - \mathcal{B}\Big(H_m, \int_a^{\cdot} dA_m(s)\cdot x_m(s)\Big)(t)\\ &= \mathcal{B}(H_m, x_m)(t) - \mathcal{B}(H_m, x_m)(a) - \int_a^t d\mathcal{B}(H_m, A_m)(s)\cdot x_m(s)\\ &\qquad\qquad \text{for } \ t\in [a,b[\ \ (m=1,2,\dots). \end{aligned}$$

Hence, due to (3.18),

$$f_m^*(t) \equiv H_m(t)x_m(t) - H_m(a)x_m(a) - \int_a^t dA(s)\cdot H_m(s)\,x_m(s).$$

So that,

$$f_m^*(t) - f(t) \equiv z_m(t) - \int_a^t dA(s)\cdot z_m(s)) \ \ (m=1,2,\dots)$$

and

$$\Phi^{-1}(t)(f_m^*(t) - f(t)) \equiv \Phi^{-1}(t)z_m(t) - \Phi^{-1}(t)\int_a^t dA(s)\cdot\Phi(s)(\Phi^{-1}(s)z_m(s)) \quad (m=1,2,\dots).$$

By this and (1.33), there exists a positive r_0 such that

$$\|\Phi^{-1}(t)(f_m^*(t) - f(t))\| \le \|\Phi^{-1}(t)z_m(t)\| + r_0\|\Phi^{-1}\, z_m\|_\infty \ \ (m=1,2,\dots).$$

Therefore, in view of (3.22), we conclude that

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)(f_m^*(t) - f(t))\| = 0 \tag{3.23}$$

uniformly on I. So, we get that condition (1.29) holds uniformly on I.

Moreover, by (1.34), there exists $r_1 > 0$ such that

$$\|\mathcal{V}_2(\Phi, A_*, f_m^* - f)(t, b-)\| \le r_1\|\Phi^{-1}(t)(f_m^*(t) - f(t))\| \quad \text{for } \ t\in [a,t[\ \ (m=1,2,\dots).$$

Consequently, due (3.23), cobdition (1.30) holds uniformly on I, as well.

The theorem is proved. □

The Theorem 1.4′ immediately follows from the proof of the necessary of Theorem 1.4, because we can choose $H_m(t) \equiv X_0(t)X_m^{-1}(t)$ $(m = 1, 2, \dots)$.

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