Seminar of I. Vekua Institute of Applied Mathematics REPORTS, Vol. 49, 2023

ON SALEM TEST FOR GENERAL DIRICHLET INTEGRALS

Akhobadze T., Ivanadze K.

Abstract. The Salem test on uniform convergence of trigonometric Fourier series is well-known. In this paper the analog of the Salem test and its corollaries for generalized Dirichlet integrals are proved. In particular, Dini, Dini-Lipschits and Kita tests for generalized Dirichlet integrals are obtained.

Keywords and phrases: General Dirichlet integral, Fourier series, generalized bounded variation, uniformly continuous functions.

AMS subject classification (2010): 26A45, 42A20, 42A38.

1. Introduction

It is well known ([1] (vol I, Ch.8), [2] (Ch.IV, section 4) or [3] (Ch.I, section 2)) that the Fourier series of a continuous function not only need not converge uniformly but it can even diverge. Therefore, the additional conditions are necessary under which the series does converge uniformly. The condition imposed probably on the function under which its Fourier series is uniformly convergent, for example, is Salem test [4] ([5], [6]).

Let f be a locally integrable function and let l be a positive number. Taberski [7] considered the sums

$$S_n^l(x,f)=\frac{a_0}{2}+\sum_{k=1}^{n}\left(a_k^l\cos\frac{k\pi t}{l}+b_k^l\sin\frac{k\pi t}{l}\right),$$

where

$$a_n^l=\frac{1}{l}\int_{-l}^{l}f(t)\cos\frac{n\pi t}{l}dt,\quad b_n^l=\frac{1}{l}\int_{-l}^{l}f(t)\sin\frac{n\pi t}{l}dt, \tag{1.1}$$

$x\in(-\infty;\infty)$ and $n\in\{1,2,3,\dots\}$. The last sums can be represented by Dirichlet integrals as follows

$$S_n^l(x,f)=\frac{1}{l}\int_{-l}^{l}f(u)D_n^l(u-x)du,$$

where

$$D_n^l(t)=\frac{1}{2}+\sum_{k=1}^{n}\cos\frac{k\pi t}{l}=\frac{\sin(2n+1)\frac{\pi t}{2l}}{2\sin\frac{\pi t}{2l}}.$$

The main purpose of this paper is to prove the analog of Salem [4] test and its corrolaries for the generalized Dirichlet integrals.

2. Auxiliary lemmas

Lemma 2.1. *Let f be a locally integrable function that is uniformly continuous over the intervals $(-\infty,-H]$, $[H,\infty)$ $(H\geq 0)$ and $c\geq 0$. Then*

$$\frac{1}{l}\int_{\theta_l}^{\theta_l+\frac{l}{n}}|f(t)|dt\to 0\quad\text{as}\quad\frac{n}{l}\to\infty\ (l\geq 1),$$

where θ_l is an arbitrary number chosen from $[-l-c; l+c]$.

If $\delta > 0$ is a fixed number then there exists a number $M > 0$ such that

$$\frac{1}{T}\int_T^{T+c} |f(t)|dt \le M; \quad \frac{1}{T}\int_{-T-c}^{-T} |f(t)|dt \le M \tag{2.1}$$

for every $T \ge \delta$.

Proof. Since f is a locally integrable function and $l \ge 1$ implies that for $\theta_l \in [-H-1; H]$

$$\frac{1}{l}\int_{\theta_l}^{\theta_l+\frac{l}{n}} |f(t)|dt \le \int_{\theta_l}^{\theta_l+\frac{l}{n}} |f(t)|dt \to 0 \quad \text{as} \quad \frac{n}{l} \to \infty.$$

If $\theta_l \in [H; \infty)$, then

$$\frac{1}{l}\int_{\theta_l}^{\theta_l+\frac{l}{n}} |f(t)|dt \le \frac{1}{l}\int_{\theta_l}^{\theta_l+\frac{l}{n}} |f(t)-f(H)|dt + \frac{1}{l}\int_{\theta_l}^{\theta_l+\frac{l}{n}} |f(H)|dt$$

$$\le \frac{1}{l}\omega\left(f; \theta_l + \frac{l}{n} - H\right)\frac{l}{n} + \frac{|f(H)|}{l}\cdot\frac{l}{n} \le \omega\left(f; \frac{\theta_l}{n} + \frac{l}{n^2}\right) + \frac{|f(H)|}{n}$$

$$\le \omega\left(f; \frac{l+c}{n} + \frac{l}{n^2}\right) + \frac{|f(H)|}{n} \to 0 \quad \text{if} \quad \frac{n}{l} \to \infty.$$

Now we prove the validity of the first inequality of (2.1), the second one can be proved analogously. We consider two cases: $\delta \le T \le H+1$ and $T > H+1$.

If $\delta \le T \le H+1$ then

$$\frac{1}{T}\int_T^{T+c} |f(t)| \le \frac{1}{\delta}\int_0^{H+c+1} |f(t)|dt =: M_1. \tag{2.2}$$

If $T > H+1$ then

$$\frac{1}{T}\int_T^{T+c} |f(t)|dt \le \frac{1}{T}\int_T^{T+c} |f(t)-f(H)|dt + \frac{1}{T}\int_T^{T+c} |f(H)|dt$$

$$\le \frac{c}{T}\omega(f; T+c-H) + \frac{cf(H)}{T} \le 2c\omega\left(f; 1 + \frac{c}{T} - \frac{H}{T}\right) + cf(H)$$

$$\le 2c\omega(f; 1 + |c-H|) + cf(H) =: M_2. \tag{2.3}$$

Estimations (2.2) and (2.3) imply (2.1) where $M = \max\{M_1, M_2\}$.

Lemma 2.2. *Let f be a locally integrable function that is uniformly continuous over the intervals $(-\infty, -H]$, $[H, \infty)$ $(H \ge 0)$ and let f be bounded in $[a,b]$ $(-\infty < a \le b < \infty)$. Then*

$$\frac{1}{l}\int_{\theta_l}^{\theta_l+\frac{l}{n}} |\phi_x(t)|dt \to 0 \quad \text{as} \quad \frac{n}{l} \to \infty \ (l \ge 1)$$

uniformly in $x \in [a,b]$, where θ_l is an arbitrary chosen from $[-l; l]$.

If $c \ge 0$ is a fixed number then there exists a number $M > 0$ such that

$$\frac{1}{T}\int_T^{T+c} |\phi_x(t)|dt \le M$$

for every $T \ge \delta$ and $x \in [a,b]$, where δ is a fixed positive number.

Proof. Since f is bounded on $[a,b]$, the validity of Lemma 2.2 can be checked easily similarly to Lemma 2.1.

Remark 2.3. It can be easily proved that lemma 2.2 is also true in the case if we consider any one of the functions $f_x^+(t)=f(x+t)-f(x)$ or $f_x^-(t)=f(x-t)-f(x)$ in place of $\phi_x(t)$.

Lemma 2.4. *Let f be a locally integrable function that is uniformly continuous over the intervals $(-\infty,-H]$, $[H,\infty)$ $(H\geq 0)$, let f be bounded in $[a,b]$ $(-\infty<a\leq b<\infty)$. Let $s(u)$ be $f(u)$ or $\phi_x(u)=f(x+u)+f(x-u)-2f(x)$ and let $r(u)$ be $\sin u$ or $\cos u$. Then*

$$\frac{1}{l}\int_{\theta_l}^{l} s(t)r\left(\frac{n\pi t}{l}\right)dt\to 0,\ \frac{1}{l}\int_{-l}^{-\theta_l} s(t)r\left(\frac{n\pi t}{l}\right)dt\to 0 \tag{2.4}$$

as $n,\frac{n}{l}\to\infty$ $(l\geq 1)$, where θ_l is arbitrarily chosen from $[0;l]$. In the case, when $s(u)=\phi_x(u)$ we have uniform convergence in $x\in[a,b]$.

Proof. We prove only the first part of (2.4), the second one will be proved by the similar method.

1^0 Suppese firstly that f is uniformly continuous in $(-\infty,\infty)$. Let p be an odd integer such that $p=\min\{i: i\in N, (i-\frac{1}{2})\frac{l}{n}\geq\theta_l\}$ and let q be an odd integer such that $q=\max\{i: i\in N, i-\frac{1}{2}\leq n\}$. The sign $\sum'$ signifies a sum where index k runs through odd values only. It is easy see that

$$p\frac{l}{n}-\theta_l\leq\frac{5l}{2n}<1,\quad l-q\frac{l}{n}<\frac{2l}{n}<1,\quad q-p<n+\frac{1}{2}. \tag{2.5}$$

Then

$$\frac{1}{l}\int_{\theta_l}^{l} s(t)r\left(\frac{n\pi t}{l}\right)dt=\frac{1}{l}\int_{\theta_l}^{p\frac{l}{n}} s(t)r\left(\frac{n\pi t}{l}\right)dt+\frac{1}{l}\int_{p\frac{l}{n}}^{q\frac{l}{n}} s(t)r\left(\frac{n\pi t}{l}\right)dt+$$
$$\frac{1}{l}\int_{q\frac{l}{n}}^{l} s(t)r\left(\frac{n\pi t}{l}\right)dt=:I_1+I_2+I_3. \tag{2.6}$$

By the first and the second inequalities of (2.5), by lemmas 2.1 and 2.2 we get

$$|I_1|\leq\frac{2}{2l}\int_{\theta_l}^{\theta_l+\frac{5l}{2n}}|s(t)|\to 0,\quad |I_3|\leq\frac{2}{2l}\int_{l-\frac{2l}{n}}^{l}|s(t)|dt\to 0 \tag{2.7}$$

when $\frac{n}{l}\to\infty$.

In the case $s(t)=\phi_x(t)$ we have uniform convergence in $x\in[a,b]$.

It is clear that

$$I_2=\sum_{k=p}^{q-1}\frac{1}{l}\int_{k\frac{l}{n}}^{(k+1)\frac{l}{n}} s(t)r\left(\frac{n\pi t}{l}\right)dt.$$

In both cases $r(u)=\sin u$ or $r(u)=\cos u$ we have $r(u+(k-1)\pi)=(-1)^{k-1}r(u)$ and $\omega(s;\delta)\leq\max\{\omega(\phi_x;\delta),\omega(f;\delta)\}\leq 2\omega(f;\delta)$. Using simple transformations and (2.5), we get

$$I_2=\sum_{k=p}^{q-1}\frac{1}{l}\int_{\frac{l}{n}}^{2\frac{l}{n}} s\left(u+(k-1)\frac{l}{n}\right)r\left(\frac{n\pi u}{l}+(k-1)\pi\right)du$$

$$=\sum_{k=p}^{q-2}{}'\frac{1}{l}\int_{\frac{l}{n}}^{2\frac{l}{n}}\left[s\left(u+(k-1)\frac{l}{n}\right)-s\left(u+k\frac{l}{n}\right)\right]r\left(\frac{n\pi u}{l}\right)du$$

$$\leq \frac{q-p}{n} 2\omega\left(f;\frac{l}{n}\right) \leq \frac{2n+1}{n}\omega\left(f;\frac{l}{n}\right) \to 0 \quad \text{as} \quad n, \frac{n}{l} \to \infty$$

(when $s(t)=\phi_x(t)$ we have uniform convergence in $x \in [a,b]$).

Hence, by (2.6) and (2.7) imply (2.4).

2^0 In the general case, there exists an uniformly continuous function g over $(-\infty,\infty)$ such that $g(t)=f(t)$, when $|t| \geq H$ and $\int_{-H}^{H}|f(t)-g(t)|dt < \frac{\varepsilon}{3}$. Let $s^*(t)=g(t)$ and $s^*(t)=g(x+t)+g(x-t)-2g(x)$ respectively, when $s(t)=f(t)$ or $s(t)=f(x+t)+f(x-t)-2f(x)$. It is clear that

$$\int_{-\infty}^{\infty}|s(t)-s^*(t)|dt \leq 3\int_{-H}^{H}|f(t)-g(t)|dt < \varepsilon,$$

$$\left|\frac{1}{l}\int_{\theta_l}^{l} s(t) r\left(\frac{n\pi t}{l}\right)dt\right| \leq \left|\frac{1}{l}\int_{\theta_l}^{l} s^*(t) r\left(\frac{n\pi t}{l}\right)dt\right| + \varepsilon.$$

Corollary 2.5. Let f be a locally integrable function that is uniformly continuous over the intervals $(-\infty,-H]$, $[H,\infty)$ $(H \geq 0)$. Then

$$a_n^l \to 0, \quad b_n^l \to 0 \quad \text{as} \quad \frac{n}{l} \to \infty \ (l \geq 1),$$

where a_n^l and b_n^l are Fourier coefficients, defined in (1.1).

Lemma 2.6. *Let f be a locally integrable function that is uniformly continuous over the intervals $(-\infty,-H]$, $[H,\infty)$ $(H \geq 0)$ and let f be bounded in $[a,b]$ $(-\infty < a \leq b < \infty)$. Then*

$$I := \frac{1}{l}\int_0^l \phi_x(t)\sin\frac{n\pi t}{l}\left[\frac{1}{2\tan\frac{\pi t}{2l}} - \frac{l}{\pi t}\right]dt \to 0 \quad \text{as} \quad \frac{n}{l} \to \infty \ (l \geq 1)$$

uniformly in $x \in [a,b]$.

Proof. We set $\psi(t)=(2\tan\frac{t}{2})^{-1}-t^{-1}$, then

$$I = \frac{1}{l}\int_0^l \phi_x(t)\sin\frac{n\pi t}{l}\psi\left(\frac{\pi t}{l}\right)dt.$$

Since $\psi\left(\frac{\pi t}{l}\right)$ is nondecreasing in $[0,l]$, $\lim\limits_{t\to 0}\psi\left(\frac{\pi t}{l}\right)=0$ and $\lim\limits_{t\to l}\psi\left(\frac{\pi t}{l}\right)=-\frac{1}{\pi}$, by second mean value theorem we get $I=-\frac{1}{\pi l}\int_{\theta_l}^{l}\phi_x(t)\sin\frac{n\pi t}{l}dt$, where $\theta_l \in [0;l]$. By lemma 2.4 $I \to 0$ as $\frac{n}{l} \to \infty$ uniformly in $x \in [a,b]$.

Lemma 2.7. *If $f(x)$ is a locally integrable function, bounded in finite $[a,b]$ then*

$$\int_{\delta}^{\Delta}\phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt \to 0, \quad \text{as} \quad \frac{n}{l} \to \infty$$

uniformly in $x \in [a,b]$, where δ and Δ are fixed positive numbers.

Proof. The proof is similar to the proof of lemma 6.4 (see [1], Ch. II).

3. The main results

Let's introduce a notation $\phi_x(t)=f(x+t)+f(x-t)-2f(x)$.

Theorem 3.1. *Let f be a locally integrable function that is uniformly continuous over the intervals $(-\infty,-H]$, $[H,\infty)$ $(H \geq 0)$ and let f be bounded in $[a,b]$ $(-\infty < a \leq b < \infty)$. Then*

$$S_n^l(x,f)-f(x) = \frac{1}{\pi}\int_0^l \phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt + o(1), \quad l, \frac{n}{l} \to \infty$$

uniformly in $x \in [a,b]$

Proof. We can easily get by corollary 2.5

$$S_n^l(x,f) = \frac{1}{l}\int_{-l}^{l} f(u) D_n^{*l}(u-x)du + o(1),$$

where

$$D_n^{*l}(t) := \frac{D_{n-1}^l(t) + D_n^l(t)}{2} = \frac{\sin\frac{n\pi t}{l}}{2\tan\frac{\pi t}{2l}},$$

Now, using Taberski ([7], p.500) method we can get

$$S_n^l(x,f) - f(x) = \frac{1}{l}\int_0^l \phi_x(t)\frac{\sin\frac{n\pi t}{l}}{2\tan\frac{\pi t}{2l}}dt + o(1) = \frac{1}{\pi}\int_0^l \phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt$$

$$+\frac{1}{l}\int_0^l \phi_x(t)\sin\frac{n\pi t}{l}\left[\frac{1}{2\tan\frac{\pi t}{2l}} - \frac{1}{\frac{\pi t}{l}}\right]dt + o(1) =: A_1 + A_2 + o(1).$$

By lemma 2.6 $A_2 \to 0$ as $l, \frac{n}{l} \to \infty$ uniformly in $x \in [a,b]$. Thus Theorem 3.1 is proved.

Let f be a locally integrable function that is uniformly continuous over the intervals $(-\infty,-H]$, $[H,\infty)$ $(H \geq 0)$. We define the modulus of continuity of f:

$$\omega(f;\delta) = \omega^H(f;\delta) := \max\{a;b\}$$

where

$$a = \sup_{x,y\in(-\infty;-H]}\{f(x) - f(y) : |x-y| \leq \delta\};$$

and

$$b = \sup_{x,y\in[H;\infty)}\{f(x) - f(y) : |x-y| \leq \delta\}.$$

Let for a nonnegative number Δ

$$T_{n,l,\Delta}^{\pm}(u) := \sum_{k=m}^{q}{}' \frac{f\left(u \pm (k-1)\frac{l}{n}\right) - f\left(u \pm k\frac{l}{n}\right)}{k}, \tag{3.1}$$

where m is an odd integer such that $m = \min\{i : i \in N, (i-\frac{1}{2})\frac{l}{n} \geq \Delta\}$ and q is an odd integer such that $q = \max\{i : i \in N, i - \frac{1}{2} \leq n\}$. The sign $\sum'$ signifies that k runs through odd values only. It is clear

$$T_{n,l,0}^{\pm}(u) := \sum_{k=1}^{q}{}' \frac{f\left(u \pm (k-1)\frac{l}{n}\right) - f\left(u \pm k\frac{l}{n}\right)}{k}. \tag{3.2}$$

Theorem 3.2. *Let f be a locally integrable function, bounded in $[a,b]$ $(-\infty < a \leq b < \infty)$, uniformly continuous over the intervals $(-\infty,H]$, $[H,\infty)$ $(H \geq 0)$.*

If $T_{n,l,\Delta}^{\pm}(x)$ uniformly converges to 0 in some open interval containing $[a,b]$ when $n, \frac{n}{l} \to \infty$, then

$$\int_\delta^l \phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt \to 0 \quad \text{as} \quad \frac{n}{l} \to \infty \ (l \geq 1)$$

uniformly in $x \in [a,b]$ $(-\infty < a \leq b < \infty)$, where $\Delta = \delta = 0$ when $H = 0$, and $\delta > 0$ is any fixed number for some fixed $\Delta \geq H + \max\{|a|,|b|\}$ when $H > 0$.

Proof. It is clear by lemma 2.7

$$\int_\delta^l \phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt = \int_\Delta^l \phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt + o(1) =: I + o(1),$$

when $\frac{n}{l}\to\infty$ uniformly in $x\in[a,b]$.

For $t\geq\Delta\geq 0$

$$x+t\in[H,\infty);\quad x-t\in(-\infty,-H] \tag{3.3}$$

for every $x\in[a,b]$. This means that $f_x^{\pm} := f(x\pm t)$ are uniformly continuous over $[\Delta;\infty)$ and

$$\omega(f_x^{\pm})\leq\omega(f). \tag{3.4}$$

Let m be an odd integer such that $m=\min\{i : i\in N, (i-\frac{1}{2})\frac{l}{n}\geq\Delta\}$ and let q be an odd integer such that $q=\max\{i : i\in N, i-\frac{1}{2}\leq n\}$. Then

$$I = \int_{(m-\frac{1}{2})\frac{l}{n}}^{(q-\frac{1}{2})\frac{l}{n}} \phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt + \int_\Delta^{(m-\frac{1}{2})\frac{l}{n}} \phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt$$

$$+\int_{(q-\frac{1}{2})\frac{l}{n}}^{l} \phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt =: I_1+I_2+I_3.$$

If $H>0$ then Δ is a fixed positive number and since $(m-\frac{1}{2})\frac{l}{n}-\Delta\leq\frac{2l}{n}$, f is integrable on $[-(\Delta+c),\Delta+c]$ and f is bounded in $[a,b]$ $(-\infty<a\leq b<\infty)$, we get

$$|I_2|\leq\frac{1}{\Delta}\int_\Delta^{(m-\frac{1}{2})\frac{l}{n}}|\phi_x(t)|dt\leq\frac{1}{\Delta}\int_{x+\Delta}^{x+(m-\frac{1}{2})\frac{l}{n}}|f(u)|du$$

$$+\frac{1}{\Delta}\int_{x-(m-\frac{1}{2})\frac{l}{n}}^{x-\Delta}|f(u)|du+\frac{4|f(x)|l}{n\Delta}\to 0\quad\text{as}\quad\frac{n}{l}\to\infty$$

uniformly in $x\in[a,b]$.

If $H=0$ then

$$I_2=\int_0^{\frac{l}{2n}}\phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt\leq\pi\omega\left(f;\frac{l}{n}\right)\to 0,\quad\frac{n}{l}\to\infty.$$

By lemma 2.2 we get

$$|I_3|\leq\frac{1}{l-\frac{2l}{n}}\int_{l-\frac{2l}{n}}^{l}|\phi_x(t)|dt\to 0,\quad\frac{n}{l}\to\infty$$

uniformly in $x\in[a,b]$.

It is clear that

$$I_1=\int_{(m-\frac{1}{2})\frac{l}{n}}^{(q-\frac{1}{2})\frac{l}{n}}\left[f(x+t)-f(x)\right]\frac{\sin\frac{n\pi t}{l}}{t}dt$$

$$+\int_{(m-\frac{1}{2})\frac{l}{n}}^{(q-\frac{1}{2})\frac{l}{n}}\left[f(x-t)-f(x)\right]\frac{\sin\frac{n\pi t}{l}}{t}dt.$$

Let us estimate

$$I_1^{\pm} := \int_{(m-\frac{1}{2})\frac{l}{n}}^{(q-\frac{1}{2})\frac{l}{n}}\left[f(x\pm t)-f(x)\right]\frac{\sin\frac{n\pi t}{l}}{t}dt.$$

For this introduce s_k and t_k:

$$s_k := k\frac{l}{n}, \quad k = m, m+1, \cdots, q-1;$$

$$t_k := \left(k - \frac{1}{2}\right)\frac{l}{n}, \quad k = m, m+1, \cdots, q.$$

It is clear that s_k and t_k are points of the interval $[(m-\frac{1}{2})\frac{l}{n}, (q-\frac{1}{2})\frac{l}{n}]$.

Let $\lambda^{n,l}(t) : [(m-\frac{1}{2})\frac{l}{n}, (q-\frac{1}{2})\frac{l}{n}] \to [(m-\frac{1}{2})\frac{l}{n}, (q-\frac{1}{2})\frac{l}{n}]$ where

$$\lambda^{n,l}(t) = \begin{cases} t_{k+1} - (t - t_k), & \text{if} \quad t \in (t_k, s_k), \quad k \text{ is odd}, \\ t, & \text{if} \quad t \in (s_k, t_{k+1}), \ k \text{ is odd}, \\ t_{k+1} - (t - t_{k+1}), & \text{if} \quad t \in (t_{k+1}, s_{k+1}), \ k \text{ is odd}, \\ t_{k+1} - (t_{k+2} - t), & \text{if} \quad t \in (s_{k+1}, t_{k+2}), \ k \text{ is odd}, \\ t, & \text{if} \quad t = t_k, s_k. \end{cases} \tag{3.5}$$

If $t \in [(m-\frac{1}{2})\frac{l}{n}, (q-\frac{1}{2})\frac{l}{n}]$ then by (3.5)

$$|t - \lambda^{n,l}(t)| \le \frac{2l}{n}. \tag{3.6}$$

It is obvious that

$$I_1^{\pm} = \int_{(m-\frac{1}{2})\frac{l}{n}}^{(q-\frac{1}{2})\frac{l}{n}} \left[f(x \pm t) - f\left(x \pm \lambda^{n,l}(t)\right)\right] \frac{\sin\frac{n\pi t}{l}}{t} dt$$

$$+ \int_{(m-\frac{1}{2})\frac{l}{n}}^{(q-\frac{1}{2})\frac{l}{n}} \left[f\left(x \pm \lambda^{n,l}(t)\right) - f(x)\right] \frac{\sin\frac{n\pi t}{l}}{t} dt =: I_{1,1}^{\pm} + I_{1,2}^{\pm}.$$

Hence,

$$I_{1,2}^{\pm} = {\sum_{k=m}^{q-2}}' \int_{t_k}^{t_{k+2}} \left[f\left(x \pm \lambda^{n,l}(t)\right) - f(x)\right] \frac{\sin\frac{n\pi t}{l}}{t} dt =: {\sum_{k=m}^{q-2}}' v_k. \tag{3.7}$$

Let us estimate the integral

$$v_k = \int_{t_k}^{t_{k+2}} \left[f\left(x \pm \lambda^{n,l}(t)\right) - f(x)\right] \frac{\sin\frac{n\pi t}{l}}{t} dt$$

$$= \int_{t_k}^{s_k} + \int_{s_k}^{t_{k+1}} + \int_{t_{k+1}}^{s_{k+1}} + \int_{s_{k+1}}^{t_{k+2}} =: v_k^{(1)} + v_k^{(2)} + v_k^{(3)} + v_k^{(4)}.$$

Since points s_k, t_k are finite, we can consider that $f(t_k) = f(t_{k+1}) = \ldots = f(s_k) = f(s_{k+1}) = \ldots$.

If in $v_k^{(1)}$ we make the change of variable $t = u + t_k = u + (k - \frac{1}{2})\frac{l}{n}$, we get

$$v_k^{(1)} = \int_0^{\frac{l}{2n}} \left[f\left(x \pm \lambda^{n,l}(u + t_k)\right) - f(x)\right] \frac{\sin\left(\frac{n\pi u}{l} + (k - \frac{1}{2})\pi\right)}{u + t_k} du.$$

Note that k is odd and $u + t_k \in [t_k, s_k]$ when $u \in [0; \frac{l}{2n}]$. By (3.5) $\lambda^{n,l}(u + t_k) = t_{k+1} - u$. Hence

$$v_k^{(1)} = \int_0^{\frac{l}{2n}} \left[f\left(x \pm (t_{k+1} - u)\right) - f(x)\right] \frac{\cos\frac{n\pi u}{l}}{t_k + u} du. \tag{3.8}$$

If in $v_k^{(2)}$ we make the change of variables $t = t_{k+1} - u = (k+\frac{1}{2})\frac{l}{n} - u$ then

$$v_k^{(2)} = -\int_{\frac{l}{2n}}^{0} \left[f\left(x \pm \lambda^{n,l}(t_{k+1}-u) \right) - f(x) \right] \frac{\sin\left(-\frac{n\pi u}{l} + (k+\frac{1}{2})\pi \right)}{t_{k+1}-u} du.$$

It is clear that k is odd and $t_{k+1}-u \in [s_k; t_{k+1}]$ when $u \in [0; \frac{l}{2n}]$. (3.5) Implies $\lambda^{n,l}(t_{k+1}-u) = t_{k+1}-u$. Hence

$$v_k^{(2)} = -\int_0^{\frac{l}{2n}} \left[f\left(x \pm (t_{k+1}-u) \right) - f(x) \right] \frac{\cos\frac{n\pi u}{l}}{t_{k+1}-u} du. \tag{3.9}$$

If in $v_k^{(3)}$ we make the change of variables $t = u + t_{k+1} = u + (k+\frac{1}{2})\frac{l}{n}$ then

$$v_k^{(3)} = \int_0^{\frac{l}{2n}} \left[f\left(x \pm \lambda^{n,l}(u+t_{k+1}) \right) - f(x) \right] \frac{\sin\left(\frac{n\pi u}{l} + (k+\frac{1}{2})\pi \right)}{u+t_{k+1}} du.$$

We note that k is odd and $t_{k+1}+u \in [t_{k+1}; s_{k+1}]$ when $u \in [0; \frac{l}{2n}]$. By (3.5) $\lambda^{n,l}(t_{k+1}+u) = t_{k+1}-u$. Hence

$$v_k^{(3)} = -\int_0^{\frac{l}{2n}} \left[f\left(x \pm (t_{k+1}-u) \right) - f(x) \right] \frac{\cos\frac{n\pi u}{l}}{t_{k+1}+u} du. \tag{3.10}$$

Now if we make the change of variable $t = t_{k+2} - u = (k+\frac{3}{2})\frac{l}{n} - u$ then

$$v_k^{(4)} = -\int_{\frac{l}{2n}}^{0} \left[f\left(x \pm \lambda^{n,l}(t_{k+2}-u) \right) - f(x) \right] \frac{\sin\left(-\frac{n\pi u}{l} + (k+\frac{3}{2})\pi \right)}{t_{k+2}-u} du.$$

It is obvious that k is odd and $t_{k+2}-u \in [s_{k+1}; t_{k+2}]$ when $u \in [0; \frac{l}{2n}]$. By (3.5) $\lambda^{n,l}(t_{k+2}-u) = t_{k+1}-u$. Hence

$$v_k^{(4)} = \int_0^{\frac{l}{2n}} \left[f\left(x \pm (t_{k+1}-u) \right) - f(x) \right] \frac{\cos\frac{n\pi u}{l}}{t_{k+2}-u} du. \tag{3.11}$$

Thus, by (3.8)-(3.11) we have

$$v_k = \int_0^{\frac{l}{2n}} \left[f\left(x \pm (t_{k+1}-u) \right) - f(x) \right] \cos\frac{n\pi u}{l}$$
$$\times \left\{ \frac{1}{t_k+u} - \frac{1}{t_{k+1}-u} - \frac{1}{t_{k+1}+u} + \frac{1}{t_{k+2}-u} \right\} du. \tag{3.12}$$

Since $0 \le u \le \frac{l}{2n} \le 1$ and $k \ge m \ge 1$, we get

$$\frac{1}{t_k+u} - \frac{1}{t_{k+1}-u} - \frac{1}{t_{k+1}+u} + \frac{1}{t_{k+2}-u} = \frac{t_k+t_{k+2}}{(t_k+u)(t_{k+2}-u)} - \frac{2t_{k+1}}{t_{k+1}^2-u^2}$$

$$= 2t_{k+1}\left(\frac{1}{\left(t_{k+1}-(\frac{l}{n}-u)\right)\left(t_{k+1}+(\frac{l}{n}-u)\right)} - \frac{1}{t_{k+1}^2-u^2} \right)$$

$$\leq \frac{2t_{k+1}\left(\frac{l}{n}\right)^2}{\left(t_{k+1}^2-\left(\frac{l}{n}\right)^2\right)\left(t_{k+1}^2-\left(\frac{l}{n}\right)^2\right)} = \frac{2(k+\frac{1}{2})\frac{l}{n}}{\left(\frac{l}{n}\right)^2\left(k+\frac{3}{2}\right)^2\left(k-\frac{1}{2}\right)^2}$$

$$\leq \frac{2}{\frac{l}{n}\left(k+\frac{3}{2}\right)\left(k-\frac{1}{2}\right)^2} = \leq \frac{1}{t_k}\cdot\frac{2}{k^2}.$$

Hence, by (3.12) we obtain

$$|v_k| \leq \frac{2}{k^2}\cdot\frac{1}{t_k}\int_0^{\frac{l}{2n}} |f\left(x\pm(t_{k+1}-u)\right)-f(x)|du,$$

$$|v_k| \leq \frac{2}{k^2}\cdot\frac{1}{t_k}\int_{s_k}^{t_{k+1}} |f\left(x\pm t\right)-f(x)|dt \leq \frac{2}{k^2}\cdot\frac{1}{t_k}\int_{t_k}^{1+t_k} |f\left(x\pm t\right)-f(x)|dt.$$

If $H>0$, by remark 2.3 there exists a number $M>0$ such that

$$\frac{1}{t_k}\int_{t_k}^{1+t_k} |f(x\pm t)-f(x)|dt \leq M$$

for every t_k and $x\in[a,b]$. Hence $|v_k|\leq \frac{2M}{k^2}$.

Since $m \geq \frac{n\Delta}{l}+\frac{1}{2}\to\infty$ as $\frac{n}{l}\to\infty$ and $\sum\frac{2M}{k^2}$ is convergent, by (3.7)

$$|I_{1,2}^{\pm}| \leq {\sum_{k=m}^{q-2}}' |v_k| \leq {\sum_{k=m}^{q-2}}' \frac{2M}{k^2}\to 0, \quad \text{as} \quad \frac{n}{l}\to\infty$$

uniformly in $x\in[a,b]$.

If $H=0$ that is f is uniformly continuous in $(-\infty,\infty)$, then for every $\varepsilon>0$ there exists $\delta_1>0$ such that $|f\left(x\pm t\right)-f(x)|<\varepsilon/\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}$ when $|t|<\delta_1$.

Let m_1 be the largest positive odd integer such that $m_1\frac{l}{n}\leq\delta_1$, hence $m_1\geq\frac{n\delta_1}{l}-2\to\infty$. It is clear that $|v_k|\leq\frac{1}{k^2}\cdot\frac{\varepsilon}{\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}}$ when $k<m_1$. Thus, we get

$$|I_{1,2}^{\pm}| \leq {\sum_{k=1}^{q-2}}'|v_k| = {\sum_{k=1}^{m_1}}'|v_k| + {\sum_{k=m_1}^{q-2}}'|v_k|$$

$$\leq \frac{\varepsilon}{\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}}\cdot\sum_{k=1}^{\infty}\frac{1}{k^2}+\sum_{k=m_1}^{\infty}\frac{2M}{k^2} = \varepsilon+o(1) \quad \text{as} \quad \frac{n}{l}\to\infty,$$

uniformly in $x\in[a,b]$.

It remains to estimate $I_{1,1}^{\pm}$. We have

$$I_{1,1}^{\pm} = \left(\int_{(m-\frac{1}{2})\frac{l}{n}}^{m\frac{l}{n}} + \int_{m\frac{l}{n}}^{q\frac{l}{n}} - \int_{(q-\frac{1}{2})\frac{l}{n}}^{q\frac{l}{n}}\right)\left[f(x\pm t)-f\left(x\pm\lambda^{n,l}(t)\right)\right]\frac{\sin\frac{n\pi t}{l}}{t}dt. \qquad (3.13)$$

For $H>0$, by (3.3), (3.4) and (3.6) imply that the absolute value of the first term of (3.13) does not exceed

$$\omega\left(f;\frac{2l}{n}\right)\frac{1}{(m-\frac{1}{2})\frac{l}{n}}\frac{l}{2n} \leq \frac{\omega\left(f;\frac{2l}{n}\right)}{2\Delta}\to 0 \quad \text{as} \quad \frac{n}{l}\to\infty.$$

If $H=0$ then $m=1$, $\Delta=0$ and for the first term of (3.13), we have

$$\left|\int_{\frac{l}{2n}}^{\frac{l}{n}}\left[f(x\pm t)-f\left(x\pm\lambda^{n,l}(t)\right)\right]\frac{\sin\frac{n\pi t}{l}}{t}dt\right|$$

$$\le\omega\left(f;\frac{2l}{n}\right)\frac{1}{\frac{l}{2n}}\frac{l}{n}=2\omega\left(f;\frac{2l}{n}\right)\to 0.$$

By (3.3), (3.4) and (3.6) implies that absolute value of the third term of (3.13) does not exceed

$$\omega\Big(f;\frac{2l}{n}\Big)\frac{1}{(q-\frac{1}{2})\frac{l}{n}}\frac{l}{n}\le\frac{\omega\left(f;\frac{2l}{n}\right)}{n-2}\to 0\quad\text{as}\quad\frac{n}{l}\to\infty.$$

It remains to estimate the mean term of (3.13):

$$J^{\pm}:=\int_{m\frac{l}{n}}^{q\frac{l}{n}}\left[f(x\pm t)-f\left(x\pm\lambda^{n,l}(t)\right)\right]\frac{\sin\frac{n\pi t}{l}}{t}dt.$$

For the simplicity we use notation: $r(t):=f(x\pm t)-f\left(x\pm\lambda^{n,l}(t)\right)$ and note that by (3.3),(3.4) and (3.6)

$$|r(t)|\le\omega\left(f;\frac{2l}{n}\right),\quad |t|\ge\Delta. \tag{3.14}$$

We have

$$J^{\pm}=\int_{m\frac{l}{n}}^{q\frac{l}{n}}r(t)\frac{\sin\frac{n\pi t}{l}}{t}dt=\sum_{k=m}^{q-1}\int_{k\frac{l}{n}}^{(k+1)\frac{l}{n}}r(t)\frac{\sin\frac{n\pi t}{l}}{t}dt.$$

By change of variables we get

$$J^{\pm}=\sum_{k=m}^{q-1}\int_{\frac{l}{n}}^{2\frac{l}{n}}r\left(u+(k-1)\frac{l}{n}\right)\frac{(-1)^{k-1}\sin\left(\frac{n\pi u}{l}\right)}{u+(k-1)\frac{l}{n}}du$$

$$=\sum_{k=m}^{q-1}\int_{l}^{2l}r\left(\frac{t}{n}+(k-1)\frac{l}{n}\right)\frac{(-1)^{k-1}\sin\left(\frac{\pi t}{l}\right)}{t+(k-1)l}dt$$

$$=\sum_{k=m}^{q-2}{}'\int_{l}^{2l}\left\{\frac{r\left(\frac{t+(k-1)l}{n}\right)}{t+(k-1)l}-\frac{r\left(\frac{t+kl}{n}\right)}{t+kl}\right\}\sin\left(\frac{\pi t}{l}\right)dt$$

(summation $\sum_{k=m}^{q-2}{}'$ is performed with respect to odd indices). A simple conversion gives:

$$J^{\pm}=\sum_{k=m}^{q-2}{}'\int_{l}^{2l}\left[r\left(\frac{t+(k-1)l}{n}\right)-r\left(\frac{t+kl}{n}\right)\right]\frac{\sin\left(\frac{\pi t}{l}\right)}{t+(k-1)l}dt$$

$$+\sum_{k=m}^{q-2}{}'\int_{l}^{2l}r\left(\frac{t+kl}{n}\right)\sin\left(\frac{\pi t}{l}\right)\left[\frac{1}{t+(k-1)l}-\frac{1}{t+kl}\right]dt=:J_1^{\pm}+J_2^{\pm}.$$

By (3.14)

$$|J_2^{\pm}|\le\sum_{k=m}^{q-2}{}'\int_{l}^{2l}\left|r\left(\frac{t+kl}{n}\right)\right|\cdot\frac{l}{(t+(k-1)l)(t+kl)}dt$$

$$\leq \sum_{k=m}^{q-2}{}' \frac{1}{k(k+1)l} \int_l^{2l} \left| r\left(\frac{t+kl}{n}\right)\right| dt \leq \omega\Big(f; \frac{2l}{n}\Big) \cdot \sum \frac{1}{k(k+1)} \to 0$$

as $\frac{n}{l} \to \infty$. Besides,

$$J_1^{\pm} = \sum_{k=m}^{q-2}{}' \int_l^{2l} \left[r\left(\frac{t+(k-1)l}{n}\right) - r\left(\frac{t+kl}{n}\right)\right]$$

$$\times \sin\left(\frac{\pi t}{l}\right) \left\{ \frac{1}{t+(k-1)l} - \frac{1}{kl}\right\} dt +$$

$$\sum_{k=m}^{q-2}{}' \int_l^{2l} \left[r\left(\frac{t+(k-1)l}{n}\right) - r\left(\frac{t+kl}{n}\right)\right] \frac{\sin\left(\frac{\pi t}{l}\right)}{kl} dt. \tag{3.15}$$

By (3.14) the absolute value of the first term of (3.15) does not exceed

$$2\omega\left(f; \frac{2l}{n}\right) \sum_{k=m}^{q-2}{}' \int_l^{2l} \frac{t-l}{(t+(k-1)l)\, kl} dt$$

$$\leq 2\omega\left(f; \frac{2l}{n}\right) \sum_{k=m}^{q-2}{}' \int_l^{2l} \frac{l}{k^2 l^2} dt \leq 2\omega\left(f; \frac{2l}{n}\right) \sum_{1}^{\infty} \frac{1}{k^2} \to 0 \quad \text{as} \quad \frac{n}{l} \to \infty.$$

Now represent the second term of (3.15) in the following way:

$$\sum_{k=m}^{q-2}{}' \int_l^{\frac{3}{2}l} \left[r\left(\frac{t+(k-1)l}{n}\right) - r\left(\frac{t+kl}{n}\right)\right] \frac{\sin\left(\frac{\pi t}{l}\right)}{kl} dt$$

$$+ \sum_{k=m}^{q-2}{}' \int_{\frac{3}{2}l}^{2l} \left[r\left(\frac{t+(k-1)l}{n}\right) - r\left(\frac{t+kl}{n}\right)\right] \frac{\sin\left(\frac{\pi t}{l}\right)}{kl} dt =: J_{1,1}^{\pm} + J_{1,2}^{\pm}.$$

Since points s_k, t_k are finite, we can consider that $f(t_k) = f(t_{k+1}) = ... = f(s_k) = f(s_{k+1}) = ...\ f(x+t_k) = f(x+t_{k+1}) = ... = f(x+s_k) = f(x+s_{k+1}) = ...$

We note that if $t \in [l; \frac{3}{2}l]$ then

$$\frac{t+(k-1)l}{n} \in [s_k; t_{k+1}]; \quad \frac{t+kl}{n} \in [s_{k+1}; t_{k+2}].$$

Thus, by (3.5)

$$\lambda^{n,l}\left(\frac{t+(k-1)l}{n}\right) = \frac{t+(k-1)l}{n}; \quad \lambda^{n,l}\left(\frac{t+kl}{n}\right) = \frac{t+(k-1)l}{n}$$

and for $t \in [l; \frac{3}{2}l]$ we get

$$r\left(\frac{t+(k-1)l}{n}\right) - r\left(\frac{t+kl}{n}\right) = f\left(x \pm \frac{t+(k-1)l}{n}\right) - f\left(x \pm \frac{t+kl}{n}\right).$$

Therefore,

$$J_{1,1}^{\pm} = \int_l^{\frac{3}{2}l} \sum_{k=m}^{q-2}{}' \left[f\left(x \pm \frac{t+(k-1)l}{n}\right) - f\left(x \pm \frac{t+kl}{n}\right)\right] \frac{\sin\left(\frac{\pi t}{l}\right)}{kl} dt,$$

$$|J_{1,1}^{\pm}| \le \int_l^{\frac{3}{2}l} \left| \sum_{k=m}^{q-2}{}' \left[f\left(x \pm \frac{t+(k-1)l}{n}\right) - f\left(x \pm \frac{t+kl}{n}\right)\right]\right| \frac{1}{kl} dt$$

$$= \frac{1}{l}\int_l^{\frac{3}{2}l} \left| \sum_{k=m}^{q-2}{}' \frac{\left[f\left(x \pm \frac{t}{n} \pm \frac{(k-1)l}{n}\right) - f\left(x \pm \frac{t}{n} \pm \frac{kl}{n}\right)\right]}{k}\right| dt$$

$$= \frac{1}{l}\int_l^{\frac{3}{2}l} \left| T_{n,l,\Delta}^{\pm}\left(x \pm \frac{t}{n}\right)\right| dt.$$

It is clear that $\frac{t}{n} \to 0$ when $t \in [l; \frac{3}{2}l]$ and $\frac{n}{l} \to \infty$. If $T_{n,l,\Delta}^{\pm}(x)$ is uniformly convergent to 0 in some interval containing $[a, b]$, then

$$J_{1,1}^{\pm} \to 0 \quad \text{as} \quad \frac{n}{l} \to \infty$$

uniformly in $x \in [a, b]$.

Using simple transformations we get

$$J_{1,2}^{\pm} = \sum_{k=m}^{q-2}{}' \int_{\frac{3}{2}l}^{2l} \left\{ f\left(x \pm \frac{t+(k-1)l}{n}\right) - f\left(x \pm \frac{t+kl}{n}\right)\right\} \frac{\sin\left(\frac{\pi t}{l}\right)}{kl} dt$$

$$-\sum_{k=m}^{q-2}{}' \int_{\frac{3}{2}l}^{2l} \left\{ f\left(x \pm \lambda^{n,l}\left(\frac{t+(k-1)l}{n}\right)\right) - f\left(x \pm \lambda^{n,l}\left(\frac{t+kl}{n}\right)\right)\right\} \frac{\sin\left(\frac{\pi t}{l}\right)}{kl} dt.$$

The first term of the last expression can be estimated as $J_{1,1}^{\pm}$ and absolute value of it does not exceed $\frac{1}{l}\int_{\frac{3}{2}l}^{2l} \left|T_{n,l,\Delta}^{\pm}(x \pm \frac{t}{n})\right| dt$.

Let us estimate

$$W^{\pm} :=$$

$$\sum_{k=m}^{q-2}{}' \int_{\frac{3}{2}l}^{2l} \left\{ f\left(x \pm \lambda^{n,l}\left(\frac{t+(k-1)l}{n}\right)\right) - f\left(x \pm \lambda^{n,l}\left(\frac{t+kl}{n}\right)\right)\right\} \frac{\sin\left(\frac{\pi t}{l}\right)}{kl} dt.$$

It is easy to see that if $t \in [\frac{3}{2}l, 2l]$ then

$$\frac{t+(k-1)l}{n} \in [t_{k+1}, s_{k+1}], \qquad \frac{t+kl}{n} \in [t_{k+2}, s_{k+2}], \tag{3.16}$$

and by (3.5)

$$\lambda^{n,l}\left(\frac{t+(k-1)l}{n}\right) = \frac{(3l-t)+(k-1)l}{n}; \quad \lambda^{n,l}\left(\frac{t+kl}{n}\right) = \frac{(3l-t)+(k+1)l}{n}.$$

Using (3.16) and the last asumptions, we get

$$W^{\pm} =$$

$$\int_{\frac{3}{2}l}^{2l} \sum_{k=m}^{q-2}{}' \left\{ f\left(x \pm \frac{(3l-t)+(k-1)l}{n}\right) - f\left(x \pm \frac{(3l-t)+kl}{n}\right)\right\} \frac{\sin\left(\frac{\pi t}{l}\right)}{kl} dt$$

$$+\int_{\frac{3}{2}l}^{2l} \sum_{k=m}^{q-2}{}' \left\{ f\left(x \pm \frac{(4l-t)+(k-1)l}{n}\right) - f\left(x \pm \frac{(4l-t)+kl}{n}\right)\right\} \frac{\sin\left(\frac{\pi t}{l}\right)}{kl} dt$$

$$=: W_1^{\pm} + W_2^{\pm}.$$

Hence

$$|W_1^{\pm}| \le \frac{1}{l}\int_{\frac{3}{2}l}^{2l} \left| T_{n,l,\Delta}^{\pm}\left(x \pm \frac{3l-t}{n}\right)\right| dt.$$

It is obvious that $\frac{3l-t}{n} \to 0$ when $t \in [\frac{3}{2}l; 2l]$ and $\frac{n}{l} \to \infty$. If $T_{n,l,\Delta}^{\pm}(x)$ uniformly converges to 0 in some interval containing $[a,b]$ then

$$W_1^{\pm} \to 0 \quad \text{as} \quad \frac{n}{l} \to \infty$$

uniformly $x \in [a,b]$.

Besides,

$$|W_2^{\pm}| \le \frac{1}{l}\int_{\frac{3}{2}l}^{2l} \left| T_{n,l,\Delta}^{\pm}\left(x \pm \frac{4l-t}{n}\right)\right| dt.$$

It is easy to see that $\frac{4l-t}{n} \to 0$ when $t \in [\frac{3}{2}l; 2l]$ and $\frac{n}{l} \to \infty$. If $T_{n,l,\Delta}^{\pm}(x)$ uniformly converges to 0 in some interval containing $[a,b]$, then

$$W_2^{\pm} \to 0 \quad \text{as} \quad \frac{n}{l} \to \infty,$$

uniformly in $[a,b]$.

Thus, the proof is completed.

Remark 3.3. It is clear that Theorem 3.2 implies Salem test.

The following two theorems are the analogues of the well-known Dini-lipschits and Dini criterions for generalized Dirichlet integrals.

Theorem 3.4 *Let f be uniformly continuous in $(-\infty;\infty)$.*
If $\omega\left(f;\frac{l}{n}\right)\log n \to 0$ when $l, \frac{n}{l} \to \infty$, then $\lim_{l,\frac{n}{l}\to\infty} S_n^l(x;f) = f(x)$ uniformly for x from every fixed $[a,b]$ $(-\infty < a \le b < \infty)$.

Proof. By Theorem 3.1

$$S_n^l(x,f) - f(x) = \frac{1}{\pi}\int_0^l \phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt + o(1). \tag{3.17}$$

Without loss of generality we can consider the case when n is odd. Then

$$|T_{n,l,0}^{\pm}(x)| \le {\sum_{k=1}^{n}}' \frac{|f\left(x \pm (k-1)\frac{l}{n}\right) - f\left(x \pm k\frac{l}{n}\right)|}{k} \le \omega\left(f;\frac{l}{n}\right)\log n \to 0.$$

whence, by (3.17) and Theorem 3.2

$$\lim_{l,\frac{n}{l}\to\infty} S_n^l(x;f) = f(x)$$

uniformly for every interval $[a,b]$ $(-\infty < a \le b < \infty)$.

Theorem 3.5. *Let f be a locally integrable function that is uniformly continuous over the intervals $(-\infty,-H]$, $[H,\infty)$ $(H>0)$ and $\omega(f;\frac{l}{n})\log l \to 0$, $l, n/l \to \infty$.*

(i) Suppose that for a fixed x

$$\int_{0+}\frac{|\phi_x(t)|}{t}dt < \infty.$$

Then $\lim\limits_{l,n/l\to\infty} S_n^l(x;f)=f(x)$.

(ii) If f is a continuous in $[a,b]$ and

$$\lim_{\tau\to 0+}\int_0^\tau \frac{|\phi_x(t)|}{t}dt=0 \quad \text{uniformly in} \quad x\in[a,b],$$

then $\lim\limits_{l,n/l\to\infty} S_n^l(x;f)=f(x)$ *uniformly in* $x\in[a,b]$.

Proof. By Theorem 3.1

$$S_n^l(x;f)-f(x)=\frac{1}{\pi}\int_0^l \phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt+o(1). \tag{3.18}$$

We suppose that n is odd. Let $\Delta\geq H+\max\{|a|+1;|b|+1\}$. Then

$$|T^{\pm}_{n,l,\Delta}(x)|\leq {\sum_{k=m}^{n}}' \frac{|f\left(x\pm(k-1)\frac{l}{n}\right)-f\left(x\pm k\frac{l}{n}\right)|}{k}\leq\omega\left(f;\frac{l}{n}\right)\sum_{k=m}^{n}\frac{1}{k}$$

$$\leq\omega\left(f;\frac{l}{n}\right)(\log n-\log m)\leq\omega\left(f;\frac{l}{n}\right)(\log n-\log\frac{n\Delta}{l})$$

$$\leq\omega\left(f;\frac{l}{n}\right)\log l\to 0$$

for every $x\in(a-1,b+1)$. Thus, by Theorem 3.2

$$\frac{1}{\pi}\int_{\delta_1}^l \phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt=o(1) \tag{3.19}$$

uniformly in $[a,b]$ for every fixed $\delta_1>0$.

(i) In this case we consider that $a=x=b$. By condition of Theorem 3.5 for every $\varepsilon>0$ there exists $\delta_1>0$ such that

$$\int_0^{\delta_1}\frac{|\phi_x(t)|}{t}dt<\varepsilon,$$

whence, (3.18) and (3.19) imply

$$\left|S_n^l(x;f)-f(x)\right|=\left|\frac{1}{\pi}\left(\int_0^{\delta_1}+\int_{\delta_1}^l\right)\phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt\right|+o(1)<\varepsilon+o(1),$$

when $l,\frac{n}{l},\to\infty$ and $\omega(f;\frac{l}{n})\log l\to 0$.

(ii) If $\varepsilon>0$ is given then there exists $\delta_1>0$ such that

$$\int_0^{\delta_1}\frac{|\phi_x(t)|}{t}dt<\varepsilon,$$

for every $x\in[a,b]$. Hence, by (3.18) and by (3.19)

$$\left|S_n^l(x;f)-f(x)\right|\leq\left|\frac{1}{\pi}\left(\int_0^{\delta_1}+\int_{\delta_1}^l\right)\phi_x(t)\frac{\sin\frac{n\pi t}{l}}{t}dt\right|+o(1)<\varepsilon+o(1)$$

uniformly in $[a,b]$.

Remark 3.6. If in Theorem 3.4

$$O\left(\frac{n}{\log n}\right) \le l \le n \tag{3.20}$$

then f will be constant. Indeed, for such l and $h \in \left(\frac{1}{\log(n+1)}, \frac{1}{\log n}\right]$ we have

$$\left|\frac{f(x+h)-f(x)}{h}\right| \le 2\omega\left(f, \frac{1}{\log n}\right)\log n$$

$$= O\left(\omega\left(f, \frac{l}{n}\right)\right)\log n \to 0, n \to \infty.$$

Therefore, $f = const.$

Under condition (3.20) a similar conclusion can be drawn for Theorem 3.5.

Various results can be obtained for classes of generalized bounded variation, which were considered in [8, 9].

Let f be a function defined on a finite closed interval $[a, b]$. Suppose p_n and $\phi(n)$ are sequences such that $p_1 \ge 1$, $p_n \uparrow \infty$, $n \to \infty$ and $\phi(1) \ge 1$, $\phi(n) \uparrow \infty$, $n \to \infty$. We say that $f \in BV(p_n \uparrow \infty, \phi, [a, b])$ if

$$V(f, p_n \uparrow \infty, \phi, [a, b])$$

$$= \sup_n \sup_\Delta \left(\sum_{i=1}^m |f(t_i) - f(t_{i-1})|^{p_n} : \rho(\Delta) \ge \frac{1}{\phi(n)}\right)^{1/p_n} < +\infty,$$

where Δ is $a = t_0 < t_1 < ... < t_m = b$ partition of the interval $[a, b]$ and $\rho(\Delta) = \min_i(t_i - t_{i-1})$.

Define

$$V(f; a, \infty) = V(f, p_n \uparrow \infty, \phi, [a, \infty)) := \sup_{b\in[a,\infty)} V(f, p_n \uparrow \infty, \phi, [a, b]);$$

$$V(f; -\infty, b) = V(f, p_n \uparrow \infty, \phi, (-\infty, b]) := \sup_{a\in(-\infty,b]} V(f, p_n \uparrow \infty, \phi, [a, b]);$$

$$V(f; \pm H, \pm\infty) := \max\left\{V(f; -\infty, -H), V(f; H, \infty)\right\}.$$

Theorem 3.7. *Let f be a locally integrable function and let it be bounded in $[a, b]$ $(-\infty < a \le b < \infty)$. Suppose that $V(f; \pm H, \pm\infty) < \infty$ for some $H \ge 0$ and*

$$\omega^H\left(f; \frac{l}{n}\right)$$

$$= o(1)\left\{p\left(\tau\left(\frac{n}{l}\right)\right)\log\left[p\left(\tau\left(\frac{n}{l}\right)\right)\right]\right\}^{-1}, \quad \tau(u) = \min_{r\in Z_+}\left\{\phi(r) \ge u\right\}. \tag{3.21}$$

Then (see (3.1) and (3.2))

$$T^{\pm}_{n,l,\Delta}(x) \to 0 \quad \text{as} \quad \frac{n}{l} \to \infty \ (l \ge 1)$$

uniformly in some finite neighborhood of $[a, b]$, where $\Delta = 0$ when $H = 0$ and $\Delta > H + \max\{|a|, |b|\}$ is any fixed number when $H > 0$.

Proof. For simpicity we use notation $\sigma_j = f(x \pm (j-1)\frac{l}{n}) - f(x \pm j\frac{l}{n})$ and n is odd. Then $T^{\pm}_{n,l,\Delta}(x) = \sum_{j=m}^{n}{}' \frac{\sigma_j}{j}$.

Let $\varepsilon_{n/l} = [\log p(\tau(n/l))]^{-1}$, $s_{n/l} = \frac{p(\tau(n/l))}{\varepsilon_{n/l}}$ and $\frac{1}{s_{n/l}} + \frac{1}{t_{n/l}} = 1$. Then by the Hölder inequality we get

$$|T^{\pm}_{n,l,\Delta}(x)| \le {\sum_{j=m}^{n}}' \frac{|\sigma_j|}{j} = {\sum_{j=m}^{n}}' |\sigma_j|^{\varepsilon_{n/l}} \frac{|\sigma_j|^{1-\varepsilon_{n/l}}}{j}$$

$$\le \left\{ {\sum_{j=m}^{n}}' \left(|\sigma_j|^{\varepsilon_{n/l}}\right)^{s_{n/l}} \right\}^{\frac{1}{s_{n/l}}} \cdot \left\{ {\sum_{j=m}^{n}}' \left(\frac{|\sigma_j|^{1-\varepsilon_{n/l}}}{j} \right)^{t_{n/l}} \right\}^{\frac{1}{t_{n/l}}}$$

$$\le \left\{ \sum_{j=m}^{n} |\sigma_j|^{p(\tau(n/l))} \right\}^{\frac{\varepsilon_{n/l}}{p(\tau(n/l))}} \omega\left(f; \frac{l}{n}\right)^{1-\varepsilon_{n/l}} \left\{ \sum_{j=m}^{n} \left(\frac{1}{j}\right)^{t_{n,l}} \right\}^{\frac{1}{t_{n,l}}}$$

$$= o(1) V(f; \pm H, \pm\infty)^{\varepsilon_{n/l}} \frac{1}{\{p(\tau(n/l)) \log[p(\tau(n/l))]\}^{1-\varepsilon_{n/l}}} \left\{ \sum_{j=1}^{n} \left(\frac{1}{j}\right)^{t_{n/l}} \right\}^{\frac{1}{t_{n/l}}}.$$

Since

$$\left\{ \sum_{j=1}^{n} \left(\frac{1}{j}\right)^{t_{n/l}} \right\}^{\frac{1}{t_{n/l}}} \le \frac{p(\tau(n/l))}{\varepsilon_{n/l}},$$

we obtain

$$|T^{\pm}_{n,l,\Delta}(x)| = o(1) V(f; \pm H, \pm\infty)^{\varepsilon_{n/l}} \frac{p(\tau(n/l))^{\varepsilon_{n/l}}}{\varepsilon_{n/l} \{\log p(\tau(n/l))\}^{1-\varepsilon_{n/l}}}. \qquad (3.22)$$

For the proof of theorem it is sufficient to show that the right side of (3.22) is bounded. This follows from choice of $\varepsilon_{n/l}$.

Corollary 3.8. If a function f satisfies conditions given in the Theorem 3.7 for $H = 0$, then $\lim_{l, \frac{n}{l} \to \infty,} S^l_n(x; f) = f(x)$ uniformly in $x \in [a, b]$ $(-\infty < a \le b < \infty)$.

Corollary 3.9. Let f be a locally integrable function. Suppose that $V(f; \pm H, \pm\infty) < \infty$ for some $H > 0$ and (3.21) is fulfilled when $\frac{n}{l} \to \infty$.

(i) Suppose that $\int_{0+} \frac{|\phi_x(t)|}{t} dt < \infty$ for a fixed x. Then

$$\lim_{l, \frac{n}{l} \to \infty} S^l_n(x; f) = f(x).$$

(ii) If f is continuous in $[a, b]$ $(-\infty < a \le b < \infty)$ and

$$\lim_{\mu \to 0+} \int_0^{\mu} \frac{|\phi_x(t)|}{t} dt = 0$$

uniformly in $x \in [a, b]$, then

$$\lim_{l, \frac{n}{l} \to \infty} S^l_n(x; f) = f(x)$$

uniformly $x \in [a, b]$.

Corollaries 3.8 and 3.9 can be proved by the similar method as respectively Theorems 3.4 and 3.5.

REFERENCES

1. Zygmund A. Trigonometric series, volumes I and II combined. *Cambridge University Press,* 2002.

2. Bary N. K. A treatise on trigonometric series. vol. I. *Macmillan, New York,* 1964.

3. Torchinsky A. Real-variable methods in harmonic analysis. *Academic Press,* 1986.

4. Salem R. Essais sur les séries trigonometries. *Actualités sci. indust., Herman, Paris,* 1940.

5. Waterman D. A generalization of Salem test. *Proc. A. M. S.,* **105** (1972), 129-133.

6. Akhobadze T. and Zviadadze Sh. On the generalization of the Salem test. *Acta Math. Hung.,* **154** (2018), 199-214.

7. Taberski R. On General Dirichlet's integrals. *Prace matyczne,* **17** (1973), 499-512.

8. Kita H. Convergence of Fourier series of a function on generalized Winer's class $BV(p(n) \uparrow \infty)$. *Acta Math. Hung.,* **57** (1991), 233-243.

9. Akhobadze T., Ivanadze K. On the classes of functions of generalized bounded variation. *Banach J. Math. Anal.,* **14** (2020), 762-783.

Received 08.10.2023; revised 02.11.2023; accepted 30.11.2023

Authors' addresses:

T. Akhobadze
I. Javakhishvili Tbilisi State University
11, University St., Tbilisi, 0186
Georgia
E-mail:takhoba@gmail.com

K. Ivanadze
I. Javakhishvili Tbilisi State University
11, University St., Tbilisi, 0186
Georgia
E-mail: kobaivanadze21@gmail.com