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# THE PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURE OF DEFINING A HOLE OF UNIFORM STRENGTH IN A POLYGONAL PLATE 

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#### Abstract

In the present work we consider the problem of statics of the linear theory of elastic mixture of finding a hole with a uniformly strong boundary in a finite isotropic plate, shaped as a convex polygon. It is assumed that projection of the displacement vector on the normal on each side of the polygon has a constant value, and projection of the stress vector on the tangent is equal to zero on the boundary hole. Assume also that a normal pressing concentrated force is applied to the middle of each side, further note that the boundary of the unknown hole is free from external stresses. The goal of the problem is to find an unknown contour under the condition that tangential normal stress takes constant value at every point of the contour.


Keywords and phrases: Elastic mixture, conformal mapping, Riemann - Hilbert problem. Kolosov - Muskhelishvili type formulas.

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## 1. Introduction

The problem of the plane theory of elasticity for infinite domains weakened by equally strong holes have been studied by many authors, particularly in [1], [9] the same problem for simple and doubly - connected domains with partially unknown boundaries are investigated in [2]. The mixed boundary value problems of the plane theory of elasticity for domain with partially unknown boundaries have been studied by R. Bantsuri [3]. Analogous problem in the case of the plane theory of elastic mixtures has been studied in [12].

In the work of R. Bantsuri and G. Kapanadze [4] the problem of statics of the plane theory of elasticity of finding a full-strength contour inside the polygon are considered. Analogous problem in the case of the plane theory of elastic mixtures has been studied in [13].

The problem of the plane theory of elasticity of finding a hole with a uniformly strong boundary in a finite plane have been studied by R. Bantsuri ([5], 4.2).

In the present paper in the case of the plane theory of elastic mixture we study the problem analogous to that solved in ([5], 4.2) For the solution of the problem the use will be made of the generalized Kolosov - Muskhelishvili formula [12] and the method developed in ([5], 4.2).

## 2. Some auxiliary formulas and operators

The homogeneous equation of statics of the theory of elastic mixtures in a complex form looks as follows [8]

$$
\begin{equation*}
\frac{\partial^{2} U}{\partial z \partial \bar{z}}+K \frac{\partial^{2} \bar{U}}{\partial \bar{z}^{2}}=0, \tag{2.1}
\end{equation*}
$$

where $z=x_{1}+i x_{2} ; \quad \bar{z}=x_{1}-i x_{2}, \frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x_{1}}-i \frac{\partial}{\partial x_{2}}\right), \frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x_{1}}+i \frac{\partial}{\partial x_{2}}\right), U=$ $\left(u_{1}+i u_{2}, u_{3}+i u_{t}\right)^{T}, u^{\prime}=\left(u_{1}, u_{2}\right)^{T}$ and $u^{\prime \prime}=\left(u_{3}, u_{4}\right)^{T}$ are partial displacements.

$$
\begin{gathered}
K=-\frac{1}{2} l m^{-1} \quad l=\left[\begin{array}{cc}
l_{4} & l_{5} \\
l_{5} & l_{6}
\end{array}\right], \quad m^{-1}=\frac{1}{\Delta_{0}}\left[\begin{array}{c}
m_{3}-m_{2} \\
-m_{2} m_{1}
\end{array}\right], \\
\Delta_{0}=m_{1} m_{3}-m_{2}^{2}, \quad m_{k}=l_{k}+\frac{1}{2} l_{3+k}, \quad k=1,2,3 ; \\
l_{1}=\frac{a_{2}}{d_{2}}, \quad l_{2}=-\frac{c}{d_{2}} ; \quad l_{3}=\frac{a_{1}}{d_{2}}, \quad a_{1}=\mu_{1}-\lambda_{5}, \quad a_{2}=\mu_{2}-\lambda_{5}, \\
c=\mu_{3}+\lambda_{5}, \quad d_{2}=a_{1} a_{2}-c^{2}, \quad l_{1}+l_{4}=\frac{b}{d_{1}}, \quad l_{2}+l_{5}=-\frac{c_{0}}{d_{1}}, \\
l_{3}+l_{6}=\frac{a}{d_{1}}, \quad d_{1}=a b-c_{0}^{2}, \quad a=a_{1}+b_{1}, \quad b=a_{2}+b_{2}, \quad c_{0}=c+d \\
b_{1}=\mu_{1}+\lambda_{1}+\lambda_{5}-\alpha_{2} \frac{\rho_{2}}{\rho}, \quad b_{2}=\mu_{2}+\lambda_{2}+\lambda_{5}+\alpha_{2} \frac{\rho_{1}}{\rho}, \quad \rho=\rho_{1}+\rho_{2}, \\
\alpha_{2}=\lambda_{3}-\lambda_{4}, \quad d=\mu_{2}+\lambda_{3}-\lambda_{5}-\alpha_{2} \frac{\rho_{1}}{\rho} \equiv \mu_{3}+\lambda_{4}-\lambda_{5}+\alpha_{2} \frac{\rho_{2}}{\rho} .
\end{gathered}
$$

Here $\mu_{1}, \mu_{2}, \mu_{3} \quad$ and $\quad \lambda_{p}, \quad p=\overline{1,5}$ are elastic modules characterizing mechanical properties of the mixture, $\rho_{1}$ and $\rho_{2}$ are particular densities. The elastic constants $\mu_{1}, \mu_{2}, \mu_{3}$ and $\lambda_{p}, \quad p=\overline{1,5}$ and particular densities $\rho_{1}$ and $\rho_{2}$ will be assumed to satisfy the conditions of inequality [6].

In [7] M. Basheleishvili obtained the following representations (Kolosov-Muskhelishvili type formulas)

$$
\begin{gather*}
2 \mu U=2 \mu\left(u_{1}+i u_{2}, u_{3}+i u_{4}\right)^{T}=A \varphi(z)+B z \overline{\varphi^{\prime}(z)}+2 \mu \overline{\psi(z)},  \tag{2.2}\\
\left.T U=\left((T u)_{2}-i(T u)_{1},(T u)_{4}-i(T u)_{3}\right)\right)^{T} \\
=\frac{\partial}{\partial S(x)}\left[(A-2 E) \varphi(z)+B z \overline{\varphi^{\prime}(z)}+2 \mu \overline{\psi(z)}\right], \tag{2.3}
\end{gather*}
$$

where $\varphi=\left(\varphi_{1}, \varphi_{2}\right)^{T}$ and $\psi=\left(\psi_{1}, \psi_{2}\right)^{T}$ are arbitrary analytic vector-functions,

$$
\begin{gathered}
A=2 \mu m, \quad \mu=\left[\begin{array}{ll}
\mu_{1} & \mu_{3} \\
\mu_{3} & \mu_{2}
\end{array}\right], \quad B=\mu l, m=\left[\begin{array}{ll}
m_{1} & m_{2} \\
m_{2} & m_{3}
\end{array}\right], \quad E=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right], \\
\frac{\partial}{S(x)}
\end{gathered}=-n_{2} \frac{\partial}{\partial x_{1}}+n_{1} \frac{\partial}{\partial x_{2}}, \quad \frac{\partial}{\partial n(x)}=n_{1} \frac{\partial}{\partial x_{1}}+n_{2} \frac{\partial}{\partial x_{2}}, ~ \$
$$

$n=\left(n_{1}, n_{2}\right)^{T}$ is the unit vector of the outer normal, $(T U)_{p}, \quad p=\overline{1,4}$, the stress components [6],

$$
\begin{array}{ll}
(T U)_{1}=r_{11}^{\prime} n_{1}+r_{21}^{\prime} n_{2}, & (T U)_{2}=r_{12}^{\prime} n_{1}+r_{22}^{\prime} n_{2}, \\
(T U)_{3}=r_{11}^{\prime \prime} n_{1}+r_{21}^{\prime \prime} n_{2}, & (T U)_{4}=r_{12}^{\prime \prime} n_{1}+r_{22}^{\prime \prime} n_{2} .
\end{array}
$$

Consider the following vectors [13]:

$$
\begin{align*}
& \tau^{(1)}=\binom{r_{11}^{\prime}}{r_{11}^{\prime \prime}}=\left[\begin{array}{cc}
a & c_{0} \\
c_{0} & b
\end{array}\right]\binom{\Theta^{\prime}}{\Theta^{\prime \prime}}-2 \frac{\partial}{\partial x_{2}} \mu\binom{u_{2}}{u_{4}},  \tag{2.4}\\
& \tau^{(2)}=\binom{r_{22}^{\prime}}{r_{22}^{\prime \prime}}=\left[\begin{array}{cc}
a & c_{0} \\
c_{0} & b
\end{array}\right]\binom{\Theta^{\prime}}{\Theta^{\prime \prime}}-2 \frac{\partial}{\partial x_{1}} \mu\binom{u_{1}}{u_{3}}, \\
& \eta^{(1)}=\binom{r_{21}^{\prime}}{r_{21}^{\prime \prime}}=-\left[\begin{array}{cc}
a_{1} & c \\
c & a_{2}
\end{array}\right]\binom{\omega^{\prime}}{\omega^{\prime \prime}}+2 \frac{\partial}{\partial x_{1}} \mu\binom{u_{2}}{u_{4}},  \tag{2.5}\\
& \eta^{(2)}=\binom{r_{12}^{\prime}}{r_{12}^{\prime \prime}}=\left[\begin{array}{cc}
a_{1} & c \\
c & a_{2}
\end{array}\right]\binom{\omega^{\prime}}{\omega^{\prime \prime}}+2 \frac{\partial}{\partial x_{2}} \mu\binom{u_{1}}{u_{3}} . \\
& \Theta^{\prime}=\operatorname{divu}^{\prime}, \quad \Theta^{\prime \prime}=\operatorname{divu}^{\prime \prime}, \quad \omega^{\prime}=\text { rotu}^{\prime}, \quad \omega^{\prime \prime}=\text { rotu" } .
\end{align*}
$$

Let $(n, s)$ be the right rectangular system where $s$ and $n$ are respectively the tangent and the normal of the curve $L$ at the point $t=t_{1}+i t_{2}$.

Assume that $n=\left(n_{1}, n_{2}\right)^{T}=(\cos \alpha, \sin \alpha)^{T}$ and $S^{0}=\left(-n_{2}, n_{1}\right)^{T}=(-\sin \alpha, \cos \alpha)^{T}$, where $\alpha$ is the angle of inclination of the normal $n$ to the $O x_{1}$-axis.

Let us introduce the vectors:

$$
\begin{array}{cc}
U_{n}=\binom{u_{1} n_{1}+u_{2} n_{2}}{u_{3} n_{1}+u_{4} n_{2}}, & U_{s}=\binom{u_{2} n_{1}-u_{1} n_{2}}{u_{4} n_{1}-u_{3} n_{2}}, \\
\sigma_{n}=\binom{(T u)_{1} n_{1}+(T u)_{2} n_{2}}{(T u)_{3} n_{1}+(T u)_{4} n_{2}}, & \sigma_{s}=\binom{(T u)_{2} n_{1}-(T u)_{1} n_{2}}{(T u)_{4} n_{1}-(T u)_{3} n_{2}} . \\
\sigma_{t}=\left(\begin{array}{cc}
{\left[r_{21}^{\prime} n_{1}-r_{11}^{\prime} n_{2},\right.} & \left.r_{22}^{\prime} n_{1}-r_{12}^{\prime} n_{2}\right]^{T} S^{0} \\
{\left[r_{21}^{\prime \prime} n_{1}-r_{11}^{\prime \prime} n_{2},\right.} & \left.r_{22}^{\prime \prime} n_{1}-r_{12}^{\prime \prime} n_{2}\right]^{T} S^{0}
\end{array}\right) . \tag{2.8}
\end{array}
$$

Let us call the vector (2.8) outer normal of the tangential normal stress in the linear theory of elastic mixture.

After elementary calculation we obtain:

$$
\begin{aligned}
& \sigma_{n}=\tau^{(1)} \cos ^{2} \alpha+\tau^{(2)} \sin ^{2} \alpha+\eta \sin \alpha \cos \alpha, \\
& \sigma_{t}=\tau^{(1)} \sin ^{2} \alpha+\tau^{(2)} \cos ^{2} \alpha-\eta \sin \alpha \cos \alpha, \\
& \sigma_{s}=\frac{1}{2}\left[\left(\tau^{(2)}-\tau^{(1)}\right) \sin 2 \alpha+\eta \cos 2 \alpha-\varepsilon^{*}\right],
\end{aligned}
$$

where $\eta=\eta^{(1)}+\eta^{(2)}, \quad \varepsilon^{*}=\eta^{(1)}-\eta^{(2)}$.
Direct calculations allow us to check on $L$ [12]

$$
\begin{gather*}
\sigma_{n}+\sigma_{t}=\tau^{(1)}+\tau^{(2)}=2(2 E-A-B) R e \varphi^{\prime}(t),  \tag{2.9}\\
\sigma_{n}+2 \mu\left(\frac{\partial U_{S}}{\partial S}+\frac{U_{n}}{\rho_{0}}\right)+i\left[\sigma_{s}-2 \mu\left(\frac{\partial U_{n}}{\partial S}-\frac{U_{S}}{\rho_{0}}\right)\right]=2 \varphi^{\prime}(t),  \tag{2.10}\\
{\left[(A-2 E) \varphi(t)+B t \overline{\varphi^{\prime}(t)}+2 \mu \overline{\psi(t)}\right]_{L}=-i \int_{L} e^{i \alpha}\left(\sigma_{n}+i \sigma_{s}\right) d s,} \tag{2.11}
\end{gather*}
$$

where $\operatorname{det}(2 E-A-B)>0, \frac{1}{\rho_{0}}$ is the curvature of $L$ at the point $t=t_{1}+i t_{2}$. Everywhere in the sequel it will be assumed that the components $U_{n}$ and $U_{s}$ are bounded [8].

Formulas (2.2), (2.3) (2.9) and (2.10) are analogous in the linear theory of elastic mixtures to those of Kolosov-Muskhelishvili [12].

## 3. Statement of the problem and the method of its solving

In the present work we consider the problem of statics of the linear theory of elastic mixture of finding a hole with a uniformly strong boundary in a finite plate.
let us consider an isotropic plate shaped as a convex polygon weakened by a curvilinear hole. Assume that the vector $U n$ (see (2.6)) on each side of the polygon has a constant value, the vector $\sigma_{S}$ (see (2.7)) on the external boundary of the plate is equal to zero, while the internal boundary is under the action of the constant normal force and the tangent stress is equal to zero. We consider two cases where 1) the values of the constant $U n$ are given, and 2) the values of the principal vector are given on either side of the external boundary of the plate.

The mechanical meaning of the first case consists in the following: an elastic washer is inserted into the hole of polygonal configuration made in a fixed rigid body. Prior to deformation the shape of the washer contour differs but little from the shape of the hole. In the second case it is assumed that the dies with rectilinear bases adjoin the sides of the plate.

We pose the following problem; find a stressed state of the body and the boundary of the hole assuming that the boundary of the hole is uniformly strong and the tangential normal stress on it takes constant value $\sigma_{t}=-K^{0}, \quad K^{0}=\left(K_{1}^{0}, K_{2}^{0}\right)^{T}=$ const .

Let on the plane of the complex variable $z=x_{1}+i x_{2}$ the plate occupy the domain $D^{0}$ bounded by the closed convex broken line $A_{1}^{0}, A_{2}^{0}, \ldots, A_{n}^{2}$ which we denote by $L_{1}$ and by the smooth closed contour $L_{2}$ lying inside $L_{1}$. To simplify the notation, the affixes of the points $A_{k}^{0}, \quad k=\overline{1, n}$, which are the vertices of the broken line are denoted by the same symbols.

It is also assumed that the point $z=0$ lies within the sought contour $L_{2}$.
Relying on the analogous Kolosov-Mushelisvhili formulas (2.9)-(2.11) the above posed problem is reduced to finding two analytic vector-functions $\varphi(z)$ and $\psi(z)$ in the domain $D^{0}$ by, the following conditions on $L=L_{1} \bigcup L_{2}$ :

$$
\begin{gather*}
\operatorname{Re} \varphi^{\prime}(t)=H, \quad t \in L_{2}, \quad H=-\frac{1}{2}(2 E-A-B)^{-1} K^{0},  \tag{3.1}\\
\operatorname{Im} \varphi^{\prime}(t)=0, \quad t \in L_{1},  \tag{3.2}\\
(A-2 E) \varphi(t)+B t \overline{\varphi^{\prime}(t)}+2 \mu \overline{\psi(t)}=\nu=\nu^{(1)}+i \nu^{(2)}, \quad t \in L_{2}, \tag{3.3}
\end{gather*}
$$

where $\nu^{(1)}=\left(\nu_{1}^{(1)}, \nu_{2}^{(1)}\right)^{T}$ and $\nu^{(2)}=\left(\nu_{1}^{(2)}, \nu_{2}^{(2)}\right)^{T}$ are arbitrary real constants vectors.
Moreover if $t \in L_{1}$ we can write

$$
\begin{gather*}
\operatorname{Re}\left[e^{-i \alpha(t)}\left(A \varphi(t)+B t \overline{\varphi^{\prime}(t)}+2 \mu \overline{\psi(t)}\right)\right]=2 \mu U_{n},  \tag{3.4}\\
\operatorname{Re}\left[e^{-i \alpha(t)}\left((A-2 E) \varphi(t)+B t \overline{\varphi^{\prime}(t)}+2 \mu \overline{\psi(t)}\right)\right]=C(t), \tag{3.5}
\end{gather*}
$$

where $\alpha(t)$ is the angle formed by the normal to $L_{1}$ at the point with the $o x_{1}$ - axis.

$$
\begin{equation*}
C(t)=\operatorname{Re}\left\{-i \int_{0}^{S} \sigma_{n}\left(t_{0}\right) \exp i\left[\alpha\left(t_{0}\right)-\alpha(t)\right] d S_{0}\right\} \tag{3.6}
\end{equation*}
$$

$\sigma_{n}(t)$ is the normal stress (see (2.7)) to $L_{1}$ at the point $t, S$ is the arc abscissa at the point $t$ counted from the point $A_{1}^{0}$ in the positive direction.

Taking into account that $\alpha(t)$ is a piecewise-constant function we obtain

$$
C(t)=\sum_{j=1}^{K} \sin \left(\alpha_{j}-\alpha_{k}\right) \int_{S_{j}}^{S_{j+1}} \sigma_{n}\left(t_{0}\right) d S_{0}
$$

for $t \in A_{1}^{0} A_{k+1}^{0}, \quad k=\overline{1, n} ; \quad A_{n+1}^{0}=A_{1}^{0}$, where $\alpha_{k}, s_{j}$ are the values of the function $\alpha(t)$ on $A_{k}^{0} A_{k+1}^{0}, \quad k=\overline{1, n}, \quad S_{j}$ is the arc abscissa of the point $A_{j}^{0}$, i.e. the length of the broken line $A_{1}^{0}, A_{2}^{0} \ldots A_{j}^{0}$. It is obvious that $C(t)$ is also a piecewise-constant vector-function.

Since in the first case $U_{n}$ is a given piecewise-constant vector-function, by virtue of formulas (3.4) and (3.5) both cases reduce to identical problems of the analytic function theory.

We will consider the second case where the values of the principal vector of external stress are given on the segments $A_{k}^{0} A_{k+1}^{0}$

$$
P^{(k)}=-\int_{S_{k}}^{S_{k+1}} \sigma_{n}(S) d S, \quad k=\overline{1, n}
$$

From the equilibrium condition we have

$$
\begin{equation*}
\sum_{k=1}^{n} P^{(k)} e^{i \alpha_{k}}=0 \tag{3.7}
\end{equation*}
$$

Now note that the condition (3.1) and (3.2) is Keldysh-Sedov problem having a solution [10]

$$
\varphi(z)=H z=-\frac{1}{2}(2 E-A-B)^{-1} K^{0} z, \quad z \in D^{0}
$$

(an arbitrary constat is assumed to be equal to zero).
Thus the boundary conditions (3.3) and (3.5) take the form

$$
\begin{gather*}
\frac{1}{2} K^{0} t+2 \mu \overline{\psi(t)}=\nu, \quad \text { on } \quad L_{2}  \tag{3.8}\\
\operatorname{Re}\left[e^{-i \alpha(t)}\left(\frac{1}{2} K^{0} t+2 \mu \overline{\psi(t)}\right)\right]=C(t), \quad \text { on } \quad L_{1}, \tag{3.9}
\end{gather*}
$$

If $t \in A_{k}^{0} A_{k+1}^{0}$, then

$$
\left(t-A_{k}^{0}\right)=i r^{0} e^{i \alpha_{k}}, r^{0}=\left|t-A_{k}^{0}\right|
$$

whence

$$
\begin{equation*}
\operatorname{Re}\left(t e^{-i \alpha(t)}\right)=\operatorname{Re}\left(A^{0}(t) e^{-i \alpha(t)}\right), \quad t \in L_{1} \tag{3.10}
\end{equation*}
$$

where $A^{0}(t)=A_{k}^{0}$ for $t \in A_{k}^{0} A_{k+1}^{0}, \quad k=\overline{1, n}$.
Let the function $z=\omega(\zeta)$ conformally map the circular ring $1<|\zeta|<R$ onto the domain $D^{0}$, where $R$ is the unknown number to be determined.

Assume that the circumference $|\zeta|=R$ is mapped onto $L_{1}$. Assume that to the vertices $A_{1}^{0}, A_{2}^{0}, A_{3}^{0} \ldots A_{n}^{0}$ there correspond the points $a_{1}^{0}, a_{2}^{0}, a_{3}^{0}, \ldots, a_{n}^{0}$ from the circumference $|\zeta|=R$.

Let $a_{k}^{0}=R e^{i \delta_{k}}, k=\overline{1, n}$, where $\delta_{k}$ are unknown numbers. Assume that $0=\delta_{1}<\delta_{2}<$ $\ldots<\delta_{n}=2 \pi$. From conditions (3.8)- (3.10) we have

$$
\begin{gather*}
\operatorname{Re}\left[e^{-i \alpha(\xi)}\left(\frac{1}{2} K^{0} \omega(\xi)+\overline{\psi_{0}(\xi)}\right)\right]=C(\xi), \quad|\xi|=R,  \tag{3.11}\\
\frac{1}{2} K^{0} \omega(\xi)+\overline{\psi_{0}(\xi)}=\nu, \quad|\xi|=1  \tag{3.12}\\
\operatorname{Re}\left[e^{-i \alpha(\xi)} \omega(\xi)\right]=\operatorname{Re}\left[e^{-i \alpha(\xi)} A^{0}(\xi)\right], \quad|\xi|=R \tag{3.13}
\end{gather*}
$$

where $\psi_{0}(\zeta)=2 \mu \psi[\omega(\zeta)], \quad 1<|\zeta|<R$. For the sake of simplicity we write $\alpha(\xi), A^{0}(\xi), C(\xi)$ instead of $\left.\alpha[\omega(\xi)], A^{0}[\omega(\xi)], C[\omega(\xi)]\right]$ respectively. These functions are defined all over the plane by the qualities

$$
\alpha(r \xi)=\alpha(\xi), \quad A^{0}(r \xi)=A^{0}(\xi), \quad C(r \xi)=C(\xi), \quad 0<r<\infty, \quad|\xi|=1
$$

Let $W(\zeta)$ be the vector-function defined by the equalities

$$
W(\zeta)=\left\{\begin{array}{lll}
\frac{1}{2} K^{0} \omega\left(\frac{\zeta}{R}\right), & \text { for } & R<|\zeta|<R^{2}  \tag{3.14}\\
\nu-\overline{\psi_{0}\left(\frac{R}{\zeta}\right),} & \text { for } & 1<|\zeta|<R
\end{array}\right.
$$

It is obvious that $W(\zeta)$ is a holomorphic vector-function in domains $1<|\zeta|<R$ and $R<|\zeta|<R^{2}$. By virtue of condition (3.12) on the circumference $W(\zeta)$ the boundary values of $|\zeta|=R$ are equal to each other from the inside and outside. Therefore $W(\zeta)$ is holomorphic in the ring $1<|\zeta|<R^{2}$.

From (3.14) we have

$$
\begin{gathered}
\frac{1}{2} K^{0} \omega(R \xi)=W\left(R^{2} \xi\right), \quad \text { for } \quad|\xi|=1 \\
\overline{\psi_{0}(R \xi)}=\nu-W(\xi), \quad \text { for } \quad|\xi|=1
\end{gathered}
$$

The substitution of the values into conditions (3.11), (3.13) gives

$$
\begin{equation*}
\operatorname{Re}\left[e^{-i \alpha(\xi)} W(\xi)\right]=f(\xi), \quad \xi \in \Gamma \tag{3.15}
\end{equation*}
$$

where $W=\left(W_{1}, W_{2}\right)^{T}, \quad f=\left(f_{1} ; f_{2}\right)^{T}$,

$$
\begin{align*}
& \Gamma=\Gamma_{1} \bigcup \Gamma_{2}, \quad \Gamma_{1}=\left\{\xi:|\xi|=R^{2}\right\}, \quad \Gamma_{2}=\{\xi: \quad|\xi|=1\}, \\
& f(\xi)=\left\{\begin{array}{l}
\frac{1}{2} K^{0} \operatorname{Re}\left[e^{-i \alpha(\xi)} A^{0}(\xi)\right], \quad \xi \in \Gamma_{1}, \\
R e \nu e^{-i \alpha(\xi)}-C(\xi)+\frac{1}{2} K^{0} R e\left[e^{-i \alpha(\xi)} A^{0}(\xi)\right], \quad \xi \in \Gamma_{2} .
\end{array}\right. \tag{3.16}
\end{align*}
$$

We have thus reduced the posed problem to the Riemann-Hilbert problem for the circular ring with piecewise-constant coefficients. All discontinuity points are nonsingular ( [11], p. 256).

Since the vector-function $W(\zeta)$ must be bounded on the domain boundary, a solution of problem (3.15) should be sought in the class of functions bounded on the boundary i.e. in the class $h_{2} n$ ( [11], p. 256).

The coefficient index of problem (3.15) corresponding to this class is equal to $2-n$ on $\Gamma_{1}$ and to - 2 on $\Gamma_{2}$.

Therefore the index of the Riemann-Hilbert problem (3.15) corresponding to the class $h_{2 n}$ is equal to $-n$.

Let us represent the boundary condition (3.15) in the form

$$
\begin{align*}
& W(\xi)+e^{2 i \alpha(\xi)} \overline{W(\xi)}=2 f^{(1)}(\xi) e^{i \alpha(\xi)}, \quad \text { on } \quad \Gamma_{1} \\
& W(\xi)+e^{2 i \alpha(\xi)} \overline{W(\xi)}=2 f^{(2)}(\xi) e^{i \alpha(\xi)}, \quad \text { on } \quad \Gamma_{2} \tag{3.17}
\end{align*}
$$

where $f^{(1)}(\xi)$ and $f^{(2)}(\xi)$ are the values of the vector-functions $f(\xi)$ on $\Gamma_{1}$ and $\Gamma_{2}$ respectively.
Taking into account the results cited in ([5],4.2; (4.2.50)) we obtain

$$
\begin{equation*}
W(z)=\frac{\aleph(z) X\left(\frac{R^{2} z}{\xi}\right)}{\pi i} \int_{\Gamma} \frac{K_{\lambda}\left(\frac{R^{4} z}{\xi}\right) f(\xi) e^{i \alpha(\xi)} d \xi}{\xi \aleph(\xi) X\left(R^{2} \xi\right)}, \quad 1<|z|<R^{2} \tag{3.18}
\end{equation*}
$$

where $f$ is the given vector-function defined by (3.16), and ([5], 4.2),

$$
\begin{gather*}
K_{\lambda}(z)=\frac{R^{4}}{R^{4}-z}+\frac{1}{\lambda} \frac{1}{1-z}+\lambda \sum_{n \geq 1} \frac{1}{R^{4 n}-\lambda}\left(\frac{z}{R^{4}}\right)^{n} \\
+\frac{1}{\lambda} \sum_{n \leq-1} \frac{R^{4} z^{n}}{R^{4 n}-\lambda}+ \begin{cases}\frac{\lambda}{1-\lambda}, & \text { for } \lambda \neq 1, \\
0, & \text { for } \quad \lambda=1,\end{cases}  \tag{3.19}\\
\aleph(z)=z \exp \left(i \beta+\int_{\Gamma_{2}} \frac{\ln \xi^{2} e^{-2 i \alpha(\xi)}}{\xi-z} d \xi\right), \quad|z|>1,  \tag{3.20}\\
\beta=-\frac{1}{4 \pi} \int_{0}^{2 \pi} \arg \left(\xi^{2} e^{2 i \alpha(\xi)}\right) d \Im \quad \Im=\arg \xi, \\
X(z)=T_{n}(z) \exp \left(\frac{1}{2 \pi i} \int_{\Gamma_{2}} K_{1}\left(\frac{z}{\xi}\right) \ln \frac{G(\xi) T_{n}(\xi)}{\lambda T_{n}\left(R^{4} \xi\right)}\right) \frac{d \xi}{\xi} .  \tag{3.21}\\
G(\xi)=\lambda \frac{X\left(R^{4} \xi\right)}{X(\xi)}, \quad|\xi|=1,  \tag{3.22}\\
T_{n}(z)=\prod_{k=1}^{n}\left(z-R^{2} z_{k}\right)^{-1} z^{\left[\frac{n}{2}\right]} e^{\frac{i \Im_{0} n}{2}},  \tag{3.23}\\
z_{k}=\exp \left(i \Im_{0}+\frac{2 \pi(k-1)}{n} i\right), \quad k=\overline{1, n}
\end{gather*}
$$

is a fixed number $0 \leq \Im_{0} \leq 2 \pi$,

$$
\begin{gather*}
\lambda=\exp \left(\frac{1}{2 \pi} \int_{\Gamma_{2}} \frac{1}{\xi} \ln \frac{G(\xi) T_{n}(\xi)}{T_{n}\left(R^{4} \xi\right)} d \xi\right),  \tag{3.24}\\
|\lambda|=\left\{\begin{array}{llll}
1, & \text { for } & \text { even } & n \\
R^{2}, & \text { for } & \text { odd } & n .
\end{array}\right.
\end{gather*}
$$

Now note that, since $X\left(R^{2} z\right)$ (see (3.20-(3.24)) has simple poles at the points $z=z_{k}$ for the vector-function $W(z)$ to be bounded it is necessary and sufficient that the conditions ([5], 4.2; (4.2.51))

$$
\begin{equation*}
\int_{\Gamma} K_{\lambda}\left(\frac{R^{4} z}{\xi}\right) \frac{f(\xi) e^{i \alpha(\xi)}}{\xi \aleph(\xi) X\left(R^{2} \xi\right)} d \xi=0, \quad k=\overline{1, n} \tag{3.25}
\end{equation*}
$$

be fulfilled.
Let us write the function $K_{\lambda}\left(\frac{R^{4} z}{\xi}\right)$ in the form

$$
K_{\lambda}\left(\frac{R^{4} z}{\xi}\right)=\frac{\xi}{\xi-z}+K_{\lambda}^{0}\left(\frac{R^{4} z}{\xi}\right), \quad 1<|z|<R^{2} .
$$

then by virtue of (3.18) we have

$$
\begin{align*}
W(z)= & \frac{\aleph(z) X\left(R^{2} z\right)}{\pi i}\left[\int_{\Gamma} \frac{f(\xi) e^{i \alpha(\xi)} d \xi}{\xi X\left(R^{2} \xi\right) \aleph(\xi)(\xi-z)}\right. \\
& \left.+\int_{\Gamma} \frac{f(\xi) K_{\lambda}^{0}\left(\frac{R^{4} Z}{\xi}\right) e^{i \alpha(\xi)} d \xi}{\xi X\left(R^{2} \xi\right) \aleph(\xi)} d \xi\right] \tag{3.26}
\end{align*}
$$

The second summand in the right -hand part of equality (3.26) is a holomorphic vectorfunction in the ring $D^{*}\left(1<|z|<R^{2}\right)$ and continuous one in the closed ring $\overline{D^{*}}$. The first summand is a Cauchy type integral whose density is a Holder-continuous vector-function on each open arc $\left(R a_{k}^{0}, R a_{k+1}^{0}\right),\left(R^{-1} a_{k}^{0}, R^{-1} a_{k+1}^{0}\right), k=\overline{1, n}$. Therefore according to the Plemelj - Privalov theorem (see e. g. [11]) the vector-function $W(z)$ is continuously extendable on these open arcs and its boundary value satisfies the Holder condition on them. Applying now the results of N.I. Muskhelishvili's monograph [11, $\S 26]$, we see that $W(z)$ is a continuous extension on $\Gamma$ and its boundary value is a Holder-continuous vector-function on $\Gamma$.

Now note that (3.25) is a vectorial system of $n$ equations with respect to $n+3$ real unknowns $K^{0}=\left(K_{1}^{0}, K_{2}^{0}\right)^{T}, \nu^{(1)}=\left(\nu_{1}^{(1)}, \nu_{2}^{(1)}\right)^{T}, \nu^{(2)}=\left(\nu_{1}^{(2)}, \nu_{(2)}^{(2)}\right)^{T}, R, \delta_{k} k=\overline{2, n}, \quad 0<$ $\delta_{k}<2 \pi$. To each solution of system (3.25) if it is solvable we can assign by formula (3.18) the unique solution of the Riemann-Hilbert problem (3.15). Hence solutions by formula (3.14) are defined by the function $\omega$ and vector-function $\psi_{0}$

$$
\begin{array}{cc}
\omega(\zeta)=\frac{2}{\left|K^{0}\right|^{2}} K^{0} W(R \zeta), & 1<|\zeta|<R \\
\psi_{0}(\zeta)=\bar{\nu}-\overline{W\left(\frac{R}{\bar{\zeta}}\right),} & 1<|\zeta|<R \tag{3.28}
\end{array}
$$

Since $\omega^{\prime}(\zeta)$ is shown to be different from zero in the domain of its definition $z=\omega(\zeta)$ conformally maps a circular ring $1<|\zeta|<R$ onto the domain $D^{0}$ and $t=\omega(\xi)$ whereas $\omega(\xi)=\frac{2 K^{0}}{\left|K^{0}\right|^{2}} W(R \xi)$ is the equation of the sought contour.

To show one important application we will prove that the system of algebraic equations (3.25) is always solvable and find the solution in an explicit form.

Let $L_{1}$ be the boundary of a regular polygon. Assume that the die with rectilinear base adjoins each side of the polygon.

Assume that a normal pressing concentrated force , $-P ;\left(P=\left(P_{1}, P_{2}\right)^{T}\right)$ is applied to the middle of each die. The origin is supposed to lie at the centre of the polygon $A_{1}^{0}, A_{2}^{0} \ldots A_{n}^{0}$ and the $o x_{1}$ axis to be directed normally to the side $A_{1}^{0}, A_{2}^{0}$. Then

$$
A_{k}^{0}=r^{0} \exp \left(\frac{\pi}{n}(2 k-3)\right), \quad \alpha_{k}=\frac{2 \pi}{n}(k-1), \quad k=\overline{1, n} .
$$

By the symmetry property it can be assumed that

$$
a_{k}^{0}=R e^{\frac{2 \pi}{n}(k-1) i}, \quad k=\overline{1, n} .
$$

This assumption is justified if system (3.25) is solvable with respect to the unknowns $K^{0}, \nu^{(1)}, \nu^{(2)}, R$.

Let us show that if one of conditions (3.25) is fulfilled then all other conditions are fulfilled too.

First we give some equalities whose validity is easy to verify ([5], 4.2)

$$
\begin{gathered}
T_{n}\left(z e^{\frac{2 \pi i}{n}}\right)=\left\{\begin{array}{l}
-T_{n}(z) \text { if } \mathrm{n} \text { is even, } \\
e^{-\frac{\pi i}{n}} T_{n}(z) \text { if } \mathrm{n} \text { is odd, }
\end{array}\right. \\
\alpha\left(\xi e^{\frac{2 \pi i}{n}}\right)=\left\{\begin{array}{l}
\alpha(\xi)+\frac{2 \pi}{n} \text { if } \mathrm{n} \text { is even, } \xi \in a_{k}^{0} a_{k+1}^{0}, \quad 1 \leq k \leq n-1, \\
e^{-\frac{\pi i}{n}} T_{n}(z) \text { if } \mathrm{n} \text { is odd, } \xi \in a_{n}^{0} a_{1}^{0},
\end{array}\right. \\
\left.\ln \left(e^{-2 i \alpha\left(\xi_{0}\right)} \xi_{0}^{2}\right)\right|_{\xi_{0}=\xi e^{\frac{2 \pi i}{n}}=\ln \left(e^{-2 i \alpha(\xi)} \xi^{2}\right),} \\
X\left(z e^{\frac{2 \pi i}{n}}\right)=e^{\frac{2 \pi i}{n} \aleph(z), \quad G\left(\xi e^{\frac{2 \pi i}{n}}\right)=G(\xi), \quad A^{0}\left(\xi e^{\frac{2 \pi i}{n}}\right)=e^{\frac{2 \pi i}{n}} A^{0}(\xi) .}
\end{gathered}
$$

By means of these equalities we easily conclude that the function $X(z)$ satisfies the condition

$$
X\left(z e^{\frac{2 \pi i}{n}}\right)=\left\{\begin{array}{l}
-X(z), \text { if } \mathrm{n} \text { is even } \\
e^{-\frac{\pi i}{n}} T_{n}(z) \text { if } \mathrm{n} \text { is odd. }
\end{array}\right.
$$

In the case $f^{(1)}(\xi)$ is a constants vector

$$
f^{(1)}(\xi)=\frac{1}{2} K^{0} r^{0} \cos \frac{\pi}{n} .
$$

Let us now show that the constant vectors $\nu^{(1)}=\left(\nu_{1}^{(1)} ; \nu_{2}^{(1)}\right)^{T}$ and $\nu^{(2)}=\left(\nu_{1}^{(2)} ; \nu_{2}^{(2)}\right)^{T}$ can be chosen so that the vector-function $f^{(2)}$ would also be a constant

In the considered case: ([5], 4.2)

$$
C(\xi)=P \sum_{q=1}^{k-1} \sin \frac{2 \pi}{n} q=\frac{P}{2 \sin \frac{\pi}{n}}\left(\cos \frac{\pi}{n}-\cos (2 k-1) \frac{\pi}{n}\right),
$$

$$
\xi \in a_{0}^{k} a_{k+1}^{0}, \quad k=\overline{1, n}
$$

By virtue of (3.16)

$$
\begin{gathered}
f^{(2)}(\xi)=\nu^{(1)} \cos \alpha+\nu^{(2)} \sin \alpha-C(\xi)+f^{(1)}(\xi) \\
\xi \in a_{k}^{0} a_{k+1}^{0}, \quad k=\overline{1, n}
\end{gathered}
$$

Therefore if $\xi \in a_{k}^{0} a_{k+1}^{0}$, then

$$
\begin{aligned}
f^{(2)}(\xi)=\nu^{(1)} \cos \frac{2 \pi}{n}(k-1) & +\nu^{(2)} \sin \frac{2 \pi}{n}(k-1)-\frac{1}{2} P \operatorname{ctg} \frac{\pi}{n}+\frac{1}{2} P \cos \frac{2 \pi}{n}(k-1) \operatorname{ctg} \frac{\pi}{n} \\
& -\frac{1}{2} P \sin \frac{2 \pi}{n}(k-1)+f^{(1)}(\xi)
\end{aligned}
$$

If we now take

$$
\nu^{(1)}=-\frac{1}{2} \operatorname{Pctg} \frac{\pi}{n}, \quad \nu^{(2)}=\frac{1}{2} P .
$$

Then we obtain

$$
f^{(2)}(\xi)=\frac{1}{2}\left(K^{0} r^{0} \cos \frac{\pi}{n}-\operatorname{Pctg} \frac{\pi}{n}\right)
$$

Thus $f^{(2)}(\xi)$ is a constants.
If we introduce the notation

$$
\begin{aligned}
& D(\zeta)=\int_{|\xi|=1} K_{\lambda}\left(\frac{R^{2} \zeta}{\xi}\right) \frac{f^{(1)}(\xi) e^{i \alpha(\xi)} d \xi}{\xi X\left(R^{4} \xi\right) \aleph\left(R^{2} \xi\right)} \\
& \quad-\int_{|\xi|=1} K_{\lambda}\left(\frac{R^{4} \zeta}{\xi}\right) \frac{f^{(2)}(\xi) e^{i \alpha(\xi)} d \xi}{\xi X\left(R^{2} \xi\right) \aleph(\xi)}
\end{aligned}
$$

then condition (3.25) take the form

$$
\begin{equation*}
D\left(\zeta_{k}\right)=0, \quad k=\overline{1, n} \tag{3.29}
\end{equation*}
$$

By virtue of the above equalities we readily obtain

$$
D\left(\zeta e^{\frac{2 \pi i}{n}}\right)=\left\{\begin{array}{l}
-D(\zeta) \text { if } \mathrm{n} \quad \text { is even } \\
-e^{\frac{\pi i}{n}} D(\zeta) \text { if } \mathrm{n} \text { is odd }
\end{array}\right.
$$

Hence it follows that $D\left(\zeta_{1}\right)=0$, then $D\left(\zeta_{k}\right)=0, \quad k=\overline{2, n}$.
Therefore system (3.29) reduces to one equation with two unknowns

$$
\begin{gathered}
K^{0} r^{0} \int_{|\xi|=1} K_{\lambda}\left(\frac{R^{2} \zeta_{1}}{\xi}\right) \frac{e^{i \alpha(\xi)} d \xi}{\xi X\left(R^{4} \xi\right) \aleph\left(R^{2} \xi\right)} \\
=\left(K^{0} r^{0}+\frac{P}{\sin \frac{\pi}{n}}\right) \int_{|\xi|=1} K_{\lambda}\left(\frac{R^{2} \zeta_{1}}{\xi}\right) \frac{e^{i \alpha(\xi)} d \xi}{\xi X\left(R^{2} \xi\right) \aleph(\xi)} .
\end{gathered}
$$

Hence we obtain

$$
K^{0}=-\frac{P \gamma(R)}{r^{0}[\delta(R)-\gamma(R)] \sin \frac{\pi}{n}}
$$

where

$$
\left\{\begin{array}{l}
\delta(R)=\int_{|\xi|=1} K_{\lambda}\left(\frac{R^{2} e^{i \Im_{0}}}{\xi}\right) \frac{e^{i \alpha(\xi)} d \xi}{\xi X\left(R^{4} \xi\right) \aleph\left(R^{2} \xi\right)}  \tag{3.30}\\
\gamma(R)=\int_{|\xi|=1} K_{\lambda}\left(\frac{R^{4} e^{i \Im_{0}}}{\xi}\right) \frac{e^{i \alpha(\xi) d \xi}}{\xi X\left(R^{2} \xi\right) \aleph(\xi)}
\end{array}\right.
$$

Using formula (3.30) and assuming $R$ to be given we define the tangential normal stress value on the sought contour. Giving $R$ various values we obtain a table of relationship between $K^{0}$ and $R$ i. e. the position of a uniformly strong contour can be defined by the given values of $K^{0}$.

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