Seminar of I. Vekua Institute of Applied Mathematics
REPORTS, Vol. 48, 2022

BELL POLYNOMIALS AND 2ND KIND HYPERGEOMETRIC BERNOULLI NUMBERS

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Abstract. After showing a recursive computation of the 2nd kind hypergeometric Bernoulli numbers, we exploit the Blissard problem to derive a connection of these Bernoulli-type numbers with the Bell polynomials.

Keywords and phrases: Hypergeometric Bernoulli polynomials and numbers, Bell's polynomials, Blissard's problem.

AMS subject classification (2010): 11B83; 11B73, 33C15.

1. Introduction

Bernoulli numbers, together with Stirling numbers, are perhaps the most important objects of Number theory because they intervene in the most disparate fields and are also closely related to Riemann's zeta-function and even with Fermat's last theorem. D.E. Smith [28], noted that among all the number sequences "*there is hardly a species so important and so generally applicable as the Bernoulli numbers*". In fact, the literature concerning them is extremely large, as can be seen in [8].

Several extensions of these mathematical entities appeared in the literature, see e.g. the articles [2, 4, 9, 19, 21, 22, 23], where the main generalizations are recalled.

The most important are the articles by F.T. Howard [13], B. Kurt [15, 16], and M. Miloud and M. Tiachachat [17, 18], the latter in relation to r-associated Stirling numbers [3, 6, 25].

All these extensions are obtained by replacing the denominator $e^x - 1$ by the rth partial sum the McLaurin expansion of the exponential, that is the truncated exponential $e^x - T_r(x)$, where $T_r(x)$ is a polynomial of degree $r-1$.

A. Hassen and H.D. Nguyen [12], extended the definition of generalized Bernoulli polynomials to non-integer values of r, (see also [1, 11, 14]), and H.L. Geleta and A. Hassen [10] shown a connection with a fractional hypergeometric version of the Riemann zeta function.

More general extensions have been considered in [24], raising the denominator $e^x - T_r(x)$ to the integer power k. The corresponding polynomials, denoted by $B_n^{[r-1,\,k]}(t)$, and numbers (which are their values at the origin) are naturally linked to the r-associated Stirling numbers of the second kind $S(n,k;r)$. Therefore the extension of the $B_n^{[r-1,\,k]}(t)$ to non integer values of r can be used to define the rational-type Stirling numbers. Some tables of these numbers were reported in [24].

In this article we first give a recursive method for computing the 2nd kind hypergeometric Bernoulli numbers $B_n^{[r-1,1]}$. This method avoids the definition of these numbers by using partition of integers (as it is done in [24]), since the use of partitions is a computationally more expensive method. Then, by using the Blissard problem, we derive a connection of these Bernoulli-type numbers with the Bell polynomials.

A Table of the $B_n^{[r-1,1]}$ numbers, for rational values of r, derived by the second author by using the Mathematica© computer algebra program, is reported in the Appendix.

2. The Bell polynomials and Blissard problem

Considering the composite function $\Phi(t) := f(g(t))$ of (sufficiently smooth) component functions $x = g(t)$ and $y = f(x)$, the nth derivative of $\Phi(t)$, putting

$$\Phi_m := D_t^m \Phi(t), \qquad f_h := D_x^h f(x)|_{x=g(t)}, \qquad g_k := D_t^k g(t)\,,$$

is expressed by

$$\Phi_n = Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n),$$

where Y_n denotes the nth Bell polynomial.
The first few Bell polynomials are:

$$\begin{aligned}
&Y_1(f_1, g_1) = f_1 g_1\\
&Y_2(f_1, g_1; f_2, g_2) = f_1 g_2 + f_2 g_1^2\\
&Y_3(f_1, g_1; f_2, g_2; f_3, g_3) = f_1 g_3 + f_2(3g_2 g_1) + f_3 g_1^3\\
&\ldots
\end{aligned} \tag{1}$$

More general values can be found in [25], p. 49.
The Bell polynomials [5] are given by:

$$Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n) = \sum_{k=1}^{n} B_{n,k}(g_1, g_2, \ldots, g_{n-k+1})\, f_k\,. \tag{2}$$

where the $B_{n,h}$ are called Bell polynomials of the second kind and satisfy the recursion [5]:

$$B_{n,k}(g_1, g_2, \ldots, g_{n-k+1}) = \sum_{h=0}^{n-k} \binom{n-1}{h} B_{n-h-1,k-1}(g_1, g_2, \ldots, g_{n-h-k+1})\, g_{h+1}\,. \tag{3}$$

The $B_{n,k}$ functions for any $k = 1, 2, \ldots, n$ are polynomials in the $g_1, g_2, \ldots, g_n$ variables homogeneous of degree k and *isobaric* of weight n (i.e. they are linear combinations of monomials $g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}$ whose weight is constantly given by $k_1 + 2k_2 + \ldots + nk_n = n$).
Therefore they satisfy the equations

$$B_{n,k}(\alpha\beta\, g_1, \alpha\beta^2 g_2, \ldots, \alpha\beta^{n-k+1} g_{n-k+1}) = \alpha^k \beta^n B_{n,k}(g_1, g_2, \ldots, g_{n-k+1})\,, \tag{4}$$

and

$$Y_n(f_1,\beta\, g_1; f_2,\beta^2 g_2;\ldots; f_n,\beta^n g_n) = \beta^n\, Y_n(f_1,g_1;f_2,g_2;\ldots;f_n,g_n)\,. \tag{5}$$

2.1 The Blissard problem

Consider the formal exponential

$$e^{at} = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!}\,. \tag{6}$$

associated with the umbral sequence $a = \{a_k\}$, where

$$a^k := a_k, \qquad \forall k \geq 0, \qquad a_0 := 1, \tag{7}$$

According to Blissard [25], the solution $b = \{b_n\}$, of the umbral equation

$$e^{at} e^{bt} = 1 \tag{8}$$

is expressed by the Bell polynomials $Y_n(f_1,g_1;f_2,g_2;\ldots;f_n,g_n)$ as follows

$$\begin{cases} b_0 := 1, \\ b_n = Y_n(-1!,a_1;2!,a_2;-3!,a_3;\ldots;(-1)^n n!,a_n), \quad (\forall\, n>0)\,. \end{cases} \tag{9}$$

At present, the Blissard symbolic method is called the *umbral calculus*, a term coined by J.J. Sylvester.

The modern version of the umbral calculus is due to G.C. Rota and S. Roman [26, 27]. An extensive bibliography of this subject can be found in [7].

3. Basic definitions

In what follows, dealing with hypergeometric functions, and for typographical convenience, we use for the rising factorial the Pochhammer symbol according to the notation:

$$(x)_n = \begin{cases} x(x+1)\cdots(x+n-1) = \dfrac{\Gamma(x+n)}{\Gamma(x)}\,, & n \geq 1\,, \\ 1\,, & n = 0\,. \end{cases} \tag{10}$$

As we never use in this article the falling factorial, this will not be misleading.
The Stirling numbers of the second kind are defined by

$$S(n,k) = \frac{1}{k!} \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} m^n\,. \tag{11}$$

We put, for brevity:

$$T_r(x) := \sum_{\ell=0}^{r-1} \frac{x^\ell}{\ell!} \,, \tag{12}$$

that is the rth partial sum of the exponential series, which is a polynomial of degree $r-1$. Then, the r-associate Stirling numbers of the second kind $S(n, k; r)$ are defined by

$$\left(\sum_{\ell=r}^{\infty} \frac{x^\ell}{\ell!}\right)^k = (e^x - T_r(x))^k = k! \sum_{n=rk}^{\infty} S(n, k; r) \frac{x^n}{n!} \,. \tag{13}$$

Of course, $S(n, k; 1) = S(n, k)$. Note that

$$e^x - T_r(x) = \frac{x^r}{r!} \, {}_1F_1(1, r+1, x) = \sum_{n=0}^{\infty} \frac{1}{(r+1)_n} \, x^n \,. \tag{14}$$

4. Hypergeometric Bernoulli polynomials

In what follows we use the definition by Booth and Hassen [1] for the generalized Bernoulli polynomials, which is different from that introduced by Kurt [1] and in a previous article by Natalini and Bernardini [19].
The hypergeometric Bernoulli polynomials are defined by the generating function:

$$\frac{\frac{x^r}{r!} \, e^{tx}}{e^x - T_r(x)} = \frac{e^{tx}}{{}_1F_1(1, r+1, x)} = \sum_{n=0}^{\infty} B_n^{[r-1,\,1]}(t) \, \frac{x^n}{n!} \,. \tag{15}$$

Remark 1. Note that, denoting by $\mathcal{B}_n^{[r-1]}(t)$ the polynomials introduced in [19], (according to our notation the variables x and t are interchanged), they are linked to those in equation (7) by means of the equation

$$r! \, B_n^{[r-1,\,1]}(t) = \mathcal{B}_n^{[r-1]}(t) \tag{16}$$

and similarly, for the generalized Bernoulli polynomials considered in [15], which are denoted here by $\mathcal{B}_n^{[r-1,\,k]}(t)$, we have the relation:

$$(r!)^k \, B_n^{[r-1,\,k]}(t) = \mathcal{B}_n^{[r-1,\,k]}(t) \,. \tag{17}$$

4.1 The exponential generating function

Since

$$e^x - T_r(x) = \sum_{n=r}^{\infty} \frac{x^n}{n!} = x^r \sum_{n=0}^{\infty} \frac{1}{(n+r)!} \, x^n \,,$$

and

$$\frac{\frac{x^r}{r!}\ e^{tx}}{e^x - T_r(x)} = \frac{e^{tx}}{r! \sum_{n=0}^{\infty} \frac{1}{(n+r)!}\ x^n}\,, \tag{18}$$

we have the exponential generating function of the hypergeometric Bernoulli polynomials $B_n^{[r-1,\,1]}(t)$

$$\frac{e^{tx}}{\Gamma(r+1) \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+r+1)}\ x^n} = \sum_{n=0}^{\infty} B_n^{[r-1,\,1]}(t)\, \frac{x^n}{n!}\ . \tag{19}$$

Remark 2. In what follows we give a method to derive explicit expressions for the hypergeometric Bernoulli numbers of the second kind $B_n^{[r-1,\,1]}$. This method is different from that reported in [24], where, for general k, the $S(n+r,k;r)$ and the hypergeometric Bernoulli numbers of the second kind $B_n^{[r-1,\,k]}$ have been considered. In fact in [24], Theorems 1 and 2, the expression of these numbers are given in terms of partitions, as in the case of Faà di Bruno Formula for Bell polynomials.

4.2 Recursive computation of the 2nd kind hypergeometric Bernoulli numbers

In (19), putting $t = 0$, gives the exponential generating function [29, 30] of the generalized hypergeometric Bernoulli numbers $B_n^{[r-1,\,1]} := B_n^{[r-1,\,1]}(0)$:

$$\sum_{n=0}^{\infty} B_n^{[r-1,\,1]}\, \frac{x^n}{n!} = \frac{1}{\Gamma(r+1) \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+r+1)}\ x^n}\,, \tag{20}$$

which is valid even for non integer (in particular for fractional) values of the parameter r.

From equation (20) we find:

$$\sum_{n=0}^{\infty} \sum_{h=0}^{n} \binom{n}{h} B_{n-h}^{[r-1,\,1]}\ \frac{h!\,\Gamma(r+1)}{\Gamma(h+r+1)}\ \frac{x^n}{n!} = 1,$$

and therefore

$$B_0^{[r-1,\,1]} = 1\,,$$

and for $n = 1, 2, 3, \ldots$ we find the $B_n^{[r-1,\,1]}$ numbers solving by recursion the triangular system:

$$\sum_{h=0}^{n} \binom{n}{h} B_{n-h}^{[r-1,\,1]}\ \frac{h!\,\Gamma(r+1)}{\Gamma(h+r+1)} = 0\,.$$

5. Representation in terms of Bell polynomials

According to the Blissard problem [25], the reciprocal of a Taylor series can be expressed in terms of Bell polynomials. In fact, consider the sequences $a := \{a_k\} = (1, a_1, a_2, a_3, \dots)$, and $b := \{b_k\} = (b_0, b_1, b_2, b_3, \dots)$, and the function:

$$\frac{1}{\displaystyle\sum_{h=0}^{\infty} a_h \frac{x^h}{h!}} \qquad (t \geq 0)\,. \tag{21}$$

Using the umbral formalism (that is, letting $a_k \equiv a^k$ and $b_k \equiv b^k$), the solution of the equation,

$$\frac{1}{\displaystyle\sum_{n=0}^{\infty} \frac{a^n t^n}{n!}} = \sum_{n=0}^{\infty} \frac{b^n t^n}{n!}\,, \qquad \text{i.e.} \qquad \exp[a\,t]\exp[b\,t] = 1\,, \tag{22}$$

is given by

$$\begin{cases} b_0 := 1, \\ \\ b_n = Y_n(-1!, a_1; 2!, a_2; -3!, a_3; \dots; (-1)^n n!, a_n), \quad (\forall\, n > 0), \end{cases} \tag{23}$$

where Y_n is the nth Bell polynomial [25].
The Bell polynomials are usually written in the form (2). By using this equation, the function (21) can be rewritten as

$$\frac{1}{\displaystyle\sum_{n=0}^{\infty} \frac{a_n t^n}{n!}} = 1 + \sum_{n=1}^{\infty}\sum_{h=1}^{n} (-1)^h h!\, B_{n,h}(a_1, a_2, \dots, a_{n-h+1})\, \frac{t^n}{n!}\,. \tag{24}$$

It is convenient to introduce the definition

$$C_n(a) := \sum_{h=1}^{n} (-1)^h h!\, B_{n,h}(a_1, a_2, \dots, a_{n-h+1})\,, \quad C_0(a) := 1\,, \tag{25}$$

so that equation (24) becomes

$$\frac{1}{\displaystyle\sum_{n=0}^{\infty} \frac{a_n t^n}{n!}} = \sum_{n=0}^{\infty} C_n(a)\, \frac{t^n}{n!}\,. \tag{26}$$

Applying this result to equation (20), with

$$a_n = \frac{n!\,\Gamma(r+1)}{\Gamma(n+r+1)}\,,$$

we find our the result

Theorem 1. *The 2nd kind hypergeometric Bernoulli numbers* $B_n^{[r-1,\,1]}$ *are expressed in terms of Bell polynomials by the equation*

$$B_n^{[r-1,\,1]} = C_n\left(\frac{n!\,\Gamma(r+1)}{\Gamma(n+r+1)}\right),$$

that is

$$\begin{aligned} B_n^{[r-1,\,1]} &= \sum_{h=1}^{n}(-1)^h h!\,B_{n,h}\left(\frac{\Gamma(r+1)}{\Gamma(r+2)},\frac{2!\,\Gamma(r+1)}{\Gamma(r+3)},\dots,\frac{(n-h+1)!\,\Gamma(r+1)}{\Gamma(n-h+r+2)}\right) \\ &= \sum_{h=1}^{n}(-1)^h h!\,B_{n,h}\left(\frac{1}{(r+1)},\frac{2!}{(r+1)_2},\dots,\frac{(n-h+1)!}{(r+1)_{n-h+1}}\right). \end{aligned} \tag{27}$$

Many representation formulas in terms of Bell's polynomials are reported in [20].

6. Conclusion

Starting from some recent generalizations of Bernoulli polynomials we have introduced in [24] a generalized hypergeometric versions of these polynomials and of the corresponding r-associated Stirling numbers.

In the same article we have introduced values of these mathematical entities for rational values of r, thus expanding the original definitions. In this way the combinatorial meaning was lost [5], but, as in the case of the Gamma function, a significant extension was obtained. In this article we have computed the hypergeometric Bernoulli numbers of the second kind $B_n^{[r-1,\,1]}$ by recursion, avoiding the general definition, introduced in [24], making use of partitions. Then, exploiting the Blissard problem and Bell's polynomials, we have shown an expression of these Bernoulli-type numbers in terms of Bell polynomials.

APPENDIX

Examples of hypergeometric Bernoulli numbers of the second kind for fractional values of r are reported in the following table.

$B_n^{[r-1,1]}$

	$r = 1/2$	$r = 3/2$	$r = 5/2$
$n = 0$	1	1	1
$n = 1$	- 2/3	- 2/5	-2/7
$n = 2$	16/45	16/175	16/441
$n = 3$	-32/315	32/2625	32/3773
$n = 4$	-256/4725	-12032/1010625	-7424/9270261
$n = 5$	512/6237	-91648/13138125	-147968/64891827
$n = 6$	47104/14189175	6649856/1379503125	-218281984/254830204629
$n = 7$	-94208/868725	17502208/2393015625	3723309056/4841773887951
$n = 8$	10289152/140970375	-334267875328/122534365078125	4641498791936/3965412814231869
$n = 9$	12720668672/68746552875	-423312621568/36039519140625	10506370678784/86700569061785679
$n = 10$	-846506491904/2268636244875	-463452541288448/1282322130542578125	-12538170178011136/18907393330012492305

Figure 1: Table of $B_n^{[r-1,1]}$, $r = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$; $n = 0, 1, 2, \ldots, 10$.

Compliance with ethical standards

Funding: This research received no external funding.

Conflicts of interest: The authors declare no conflict of interests.

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Received 20.06.2022; accepted 19.09.2022

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