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ON A METHOD FOR SOLVING AN EQUATION OF VIBRATIONS OF A VISCOUS-ELASTIC BEAM

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Abstract. In the paper, a new solution of an integro-differential equation of vibrations of linear visco-elasticity for arbitrary kernels at small viscosity was constructed in the form of a series by the method of Laplace integral transformation and its convergence was proved.It was shown that the first term of this series is an appropriate solution of the indicated equation obtained by the well known averaging method. The originals of two terms of the series were constructed and the influence of the second term on the solution for a specific kernel was estimated.It was obtained that at low frequencies the influence of subsequent terms is insignificant and they increase with increasing the frequency and the amplitude of all members of the series decrease over time by exponential law, the phases are shifted.

Keywords and phrases: Viscoelasticity, kernel, vibrations, integro-differential equation, rheology.

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Introduction

Development of modern technology and industry requires a wide application of polymer composite and other materials with pronounced rheological properties. The study of physical and mechanical properties of these materials and analysis of their application in industrial structures have shown the need to use the methods of theory of viscoelasticity in strength analysis of appropriate constructions.Nonstationary dynamic problems that have important practical applications in many fields of modern technology are especially difficult. The problems on vibrations of viscoelastic systems and nonstationary wave problems should be distinguished among dynamic problems of viscoelasticity. This fact led to the development of numerical [1,2,3], approximate [4,5] , asymptotic [6,7] and other methods of solution. One of these methods is the averaging method developed by A.A.Ilyushin and his staff [7,8] in application to integro-differential equations [9,10,11]. İn the papers [9,10,12] the problems on vibrations of visco-elastic systems in which terminal solutions lead to the solution of integro-differential equations and are implemented by the averaging method or by the freezing method [10,11,13], and in the papers [13,14,15] by the continuation method.Hence it is seen that the questions on studying the problems of vibrations of viscoelastic systems remain insuficiently studied, therefore they are relevant and important problems for practical application.

Statement of the problem

In the paper we consider a problem on lateral vibration of a viscoelastic beam of constant cross section and whose ends are hingely fixed.

Assume that the axis OX is directed along the longitudinal symmetry axis of the beam at rest and the vibrations of the beam can be described by the function $w(x,t)$ that characterizes at the moment of time t the lateral deviation of the point that has abscissa x in the equilibrium position.

Let the beam, starting from the moment of time t commits forced lateral vibrations in a resistanceless sphere.

A mathematical problem on forced lateral vibrations of a viscoelastic beam in resistanceless medium can be described by an integro-differential equation of the following form :

$$EJ\left[\frac{\partial^4 w(x,t)}{\partial x^4} - \int_0^t \Gamma(t-\tau)\frac{\partial^4 w(x,t)}{\partial x^4}d\tau\right] + \rho F\frac{\partial^2 w(x,t)}{\partial t^2} = q(x,t), \tag{1}$$

where $w(x,t)$ is the lateral displacement, ρ is the density of the material of the beam, F is the beam's cross-section considered to be constant, $q(x,t)$ is a lateral load, J is the inertia moment of the cross-section of the beam with respect to neutral axis of the section perpendicular to the plane of vibrations, $\Gamma(t) = -\frac{dR(t)}{dt}$, $R(t)$ is the relaxation function of the material of the beam and contains a small parameter in its representation.

In order to determine a unique solution of the equation, at first two conditions should be taken into account: the first to take into account the initial condition, the second to take into account the nature of fastening the ends of the beam.
We take the initial conditions that cause vibrations in the form:

$$w(x,t) = u_0; \quad \frac{\partial w(x,t)}{\partial x} = v_0 \ \text{ for } \ t = 0, 0 < x < \ell. \tag{2}$$

These conditions determine the deviation of the beam and speed at the initial moment of time $t = 0$. Since the both end of the beam were hingely fixed, then the following boundary conditions are fulfilled:

$$w(x,t) = 0 \ \text{ for } \ x = 0 \ \text{ and } \ x = \ell, \tag{3}$$

$$\frac{\partial^2 w(x,t)}{\partial x^2} = 0, x = 0 \ \text{ and } \ x = \ell \ \ 0 \le t \le T. \tag{4}$$

Condition (3) means that the ends of the viscoelastic beam have no displacements at any moment of time t. Conditions (4) show that bending moments acting in the sections $x = 0$ and $x = \ell$ of the beam equal zero and we assume that boundary value problem (1)-(4) has a solution.

We will look for the solution satisfying homogeneous boundary conditions (3) and (4) in the form

$$w(x,t) = \sum_{k=1}^{\infty} u_k(x) z_k(t). \tag{5}$$

Having substituted the solution (6) in equation (1) and after some calculations we get the following system of equations:

$$u_k^{IV}(x) - \lambda_k^4 u_k(x) = 0, \tag{6}$$

$$\begin{aligned} &u_k(0) = 0; \ u_k''(0) = 0 \ \text{ for } \ x = 0, \\ &u_k(\ell) = 0; \ u_k''(\ell) = 0 \ \text{ for } \ x = \ell. \end{aligned} \tag{7}$$

The functions $u_k(x)$ determine the main forms of vibrations and we will consider them orthogonal and somehow normalized. Since the boundary value problem (6) and (7) has the eigenvalues $\lambda_k = \frac{\pi k}{\ell}$; $k = 1, 2 \ldots$, then their corresponding eigenfunctions $u_k(x)$ do not depend on the magnitude characterizing viscous resistance of the beam, and they will be determined by the formula

$$u_k(x) = \sin \frac{\pi k}{\ell} x.$$

The functions $z_k(t)$ satisfy the integro-differential equation

$$z_k''(t) + \varepsilon \omega_k^2 z_k(t) - \varepsilon \omega_k^2 \int_0^t \Gamma(t-\varepsilon) z_k(\tau) d\tau = \omega_s^2 q_k(t) \tag{8}$$

with the initial conditions

$$z_k(0) = u_0^{(k)}; \quad z_k'(0) = v_0^{(k)}, t = 0, \tag{9}$$

where $\omega_k^2 = \frac{EJ}{\rho F} \lambda_k^4$, λ_k^4 forms a spectrum of fundamental number of the problem

$$q_k(t) = \frac{1}{\omega_k^4 \rho F} \int_0^\ell q(x,t) u_k(x) dx.$$

$\varepsilon > 0 -$ is some small parameter.

If $q_k(t)$ is some periodic function of time t, then due to the linearity of the problem, it is possible to individually look for the solution corresponding to each of Fourier-components of this function and the sum of contributions made by each of the components is the desired function $z_k(t)$.

Thus, the problem is reduced to solving integro-differential equation (8) under initial conditions (9).

The solution of the integro-differential equation

Applying the Laplace transform to equation (8) allowing for (9) and omitting indices for simplicity of notation we obtain:

$$\bar{z}(p) = \frac{pu_0 + v_0}{p^2 + \omega^2 - \varepsilon \omega^2 \bar{\Gamma}(p)} + \frac{\bar{q}(p)}{p^2 + \omega^2 - \varepsilon \omega^2 \bar{\Gamma}(p)}, \tag{10}$$

where p is a Laplace transform operator, $\bar{z}(p)$ and $\bar{\Gamma}(p)$ are the Laplace images of the function of the same name $z(t)$ and $\bar{\Gamma}(t)$ respectively.

Here, in equation (10) the first summand corresponds to the free vibration and we denote it by

$$\bar{\varphi}(p) = \frac{pu_0 + v_0}{p^2 + \omega^2 - \varepsilon \omega^2 \bar{\Gamma}(p)}. \tag{11}$$

The second summand characterizes the forced vibration of the beam

$$\bar{g}(p) = \frac{\bar{q}(p)}{p^2 + \omega^2 - \varepsilon \omega^2 \bar{\Gamma}(p)}. \tag{12}$$

At first we consider free vibrations of the beam, then $q(t) = 0$. Let us consider the inequality

$$\left| \frac{\varepsilon \omega^2 \bar{\Gamma}(p)}{p^2 + \omega^2} \right| < \varepsilon \tag{13}$$

and determine the limits of its validity depending on the time change.

Note that for small values of time t the parameter p is rather large and since we consider the materials with instant elasticity, the image of the relaxation kernel $\bar{\Gamma}(p)$ with increasing the parameter p tends to zero, therefore in this case inequality (13) remains valid for an arbitrary ε. For other values of time we will use inequality [7]

$$0 \le \varepsilon \int_0^t \Gamma(\tau) d\tau << 1, \quad \varepsilon\Gamma(t) \ge 0$$

established by A.A. Ilyushin, that is valid for any time t. It follows from the fact that rigid polymers have little viscous resistance compared to the main one, elastic. Therefore, the considered inequality will be valid for any time t.

Then we can expand formula (11) in an absolutely convergent series:

$$\bar{\varphi}(p) = \frac{pu_0 + v_0}{p^2 + \omega^2} \sum_{n=0}^{\infty} \left(\frac{\varepsilon\omega^2 \bar{\Gamma}(p)}{p^2 + \omega^2} \right)^n . \tag{14}$$

Applying the Laplace inverse transform to the expression $\frac{\varepsilon\omega^2\bar{\Gamma}(p)}{p^2+\omega^2}$ we find

$$L^{-1}\left\{ \frac{\varepsilon\lambda^2\bar{\Gamma}(p)}{p^2 + \lambda^2} \right\} = \varepsilon\lambda \int_0^t \Gamma(\tau) \sin\lambda\,(t-\tau)\, d\tau$$

$$= \varepsilon\lambda \sin\lambda t \int_0^t \Gamma(\tau) \cos\lambda\tau d\tau - \varepsilon\lambda\cos\lambda t \int_0^t \Gamma(\tau)\sin\lambda\tau d\tau$$

$$= \varepsilon\lambda \sin\lambda t \int_0^\infty \Gamma(\tau) \cos\lambda\tau d\tau - \varepsilon\lambda\cos\lambda t \int_0^\infty \Gamma(\tau)\sin\lambda\tau d\tau$$

$$-\varepsilon\lambda \sin\lambda t \int_t^\infty \Gamma(\tau) \cos\lambda\tau d\tau - \varepsilon\lambda\cos\lambda t \int_t^\infty \Gamma(\tau)\sin\lambda\tau d\tau$$

$$L^{-1}\left\{ \frac{\varepsilon\omega^2\bar{\Gamma}(p)}{p^2 + \omega^2} \right\} = \varepsilon\omega\, \Gamma_c \sin\omega t - \varepsilon\omega\, \Gamma_s \cos\omega t - \varepsilon\omega M(t), \tag{15}$$

where L^{-1} is Laplace's inverse transform operator.

$$\Gamma_c = \int_0^\infty \Gamma(\tau)\cos\omega\tau d\tau; \quad \Gamma_s = \int_0^\infty \Gamma(\tau)\sin\omega\tau\, d\tau,$$

$$M(t) = \int_t^\infty \Gamma(\tau)\sin\omega(t-\tau) d\tau.$$

Here the continuity of the function $\Gamma(t)$ in the domain $0 \le t < \infty$ is taken into account. Passing in the right hand side of formula (15) to the Laplace image, we find:

$$\frac{\varepsilon\omega^2\bar{\Gamma}(p)}{p^2+\omega^2} = \frac{\varepsilon\omega^2\Gamma_c - \varepsilon\omega\, p\Gamma_s - \varepsilon\omega(p^2+\omega^2)\bar{M}(p)}{p^2+\omega^2}$$

.

Taking into account the last formula in (14), we obtain

$$\bar{\varphi}(p) = \frac{pu_0 + v_0}{\bar{\alpha}(p) - \varepsilon\omega^2\bar{\beta}(p)}, \tag{16}$$

where

$$\bar{\alpha}(p) = \Big(p + \frac{1}{2}\varepsilon\omega\,\Gamma_s\Big)^2 + \omega^2\Big(1 - \frac{1}{2}\varepsilon\Gamma_c\Big)^2,$$

$$\bar{\beta}(p) = \bar{\Gamma}(p) + \frac{p}{\omega}\Gamma_s + \Gamma_c + \frac{1}{4}\varepsilon(\Gamma_s^2 + \Gamma_c^2).$$

Here it is taken into account that the value $\left|\varepsilon\omega^2\bar{\beta}(p)\right|$ will be as small as how small will be $|M(t)|$ for rather large values of time.

Similarly to the above, for the same values of time t and consequently of the parameter p we can show the validity of the inequality

$$\left|\frac{\varepsilon\omega^2\bar{\beta}(p)}{\bar{\alpha}(p)}\right| < 1.$$

Taking this inequality into account, we expand formula (16) in the following absolutely convergent series:

$$\bar{\varphi}(p) = \frac{pu_0 + v_0}{\bar{\alpha}(p)}\Big[1 + \varepsilon\omega^2\frac{\bar{\beta}(p)}{\bar{\alpha}(p)} + \varepsilon^2\omega^4\frac{\bar{\beta}^2(p)}{\bar{\alpha}^2(p)} + ...\Big]. \tag{17}$$

Thus, we proved that formula (14) is equivalent to the formula (17), is equivalent to the formula in more convenient in the sense of conversion and application. Note that if in the denominator (16) or (17) we neglect the term $\varepsilon\omega^2\bar{\beta}(p)$, then we obtain the image of the solution of equation (8) with appropriate initial conditions that is obtained by the averaging method.

The original of the first term of the series (17) is of the form:

$$\begin{aligned}\varphi_1(t) = \exp\Big(-\tfrac{1}{2}\varepsilon\omega\Gamma_s t\Big)[u_0\cos\omega\Big(1 - \tfrac{1}{2}\varepsilon\Gamma_c\Big)t \\ +\frac{v_0 - \frac{1}{2}\varepsilon\Gamma_s\omega}{\omega(1 - \frac{1}{2}\varepsilon\Gamma_c)}\cdot\sin\omega(1 - \frac{1}{2}\varepsilon\Gamma_c)t].\end{aligned} \tag{18}$$

This is the well-known solution to the problem (8)-(9) obtained by the averaging method [10, 11, 12].

To find the original of the second approximation we represent it in the form:

$$\varphi_2(t) = \varepsilon\omega^2\varphi_1(t) * L^{-1}\left\{\frac{\bar{\beta}(p)}{\bar{\alpha}(p)}\right\}. \tag{19}$$

Here the asterisk means the convolution of functions

$$f(t) * \psi(t) = \int_0^t f(t-\tau)\psi(\tau)d\tau.$$

To restore the function $L^{-1}\left\{\frac{\bar{\beta}(p)}{\bar{\alpha}(p)}\right\}$ we represent the ratio $\frac{\bar{\beta}(p)}{\bar{\alpha}(p)}$ in the following form

$$\frac{\bar{\beta}(p)}{\bar{\alpha}(p)} = \frac{\bar{\Gamma}(p)}{\bar{\alpha}(p)} + \frac{\Gamma_s}{\omega}\frac{p+m}{\bar{\alpha}(p)},$$

where

$$m = \frac{\Gamma_c}{\Gamma_s}\omega + \frac{\varepsilon\omega}{4\Gamma_s}(\Gamma_s^2 + \Gamma_c^2)$$

then denoting

$$\bar{\psi}(p) = \frac{\bar{\beta}(p)}{\bar{\alpha}(p)}$$

hence we find

$$\psi(t) = \Gamma(t) * \exp\Big(-\frac{1}{2}\varepsilon\omega\,\Gamma_s t\Big)\frac{1}{\omega\Big(1-\frac{1}{2}\varepsilon E\Big)}\cdot\sin\omega\Big(1-\frac{1}{2}\varepsilon\Gamma_c\Big)t$$
$$+A\exp\Big(-\frac{1}{2}\varepsilon\omega\,\Gamma_s t\Big)\sin\Big[\omega\Big(1-\frac{1}{2}\varepsilon\Gamma_c\Big)t+\theta\Big],$$

where

$$\theta = arctg\frac{\omega\Big(1-\frac{1}{2}\varepsilon\Gamma_c\Big)}{d-\frac{1}{2}\varepsilon\omega\,\Gamma_s},$$

$$A = \frac{\Gamma_s}{\omega}\sqrt{1+\frac{\Big(m-\frac{1}{2}\varepsilon\omega\Gamma_s\Big)^2}{\omega^2\Big(1-\frac{1}{2}\varepsilon\Gamma_c\Big)^2}}.$$

In this case the second approximation is found in the form:

$$\varphi_2(t) = \varepsilon\omega^2\varphi_1(t) * \psi(t) \tag{20}$$

Restoration of originals of the next approximations of series (17) is not difficult.

Equation (10) shows that at forced vibrations there appears the term $\bar{g}(t)$, determined by equation (12). The original of these expression is obtained by convoluting the function $q(t)$ with the expression $[p^2+\omega^2-\varepsilon\omega\,\bar{\Gamma}(p)]^{-1}$.

Represent the denominator of equation (12) in the form of the series

$$\frac{1}{p^2+\omega^2-\varepsilon\omega\bar{\Gamma}(p)} = \sum_{m=0}^{\infty}\frac{(\varepsilon\omega\bar{\Gamma}(p))^m}{(p^2+\omega^2)^{m+1}}. \tag{21}$$

and introduce the following notation

$$L^{-1}\left[\frac{\bar{\Gamma}(p)}{p^2+\omega^2}\right] = \tfrac{1}{\omega}\int_0^t \Gamma(t-\tau)\sin\omega\tau d\tau = F_0(t),$$
$$L^{-1}\left[\frac{\omega^2\bar{\Gamma}(p)}{(p^2+\omega^2)^2}\right] = \omega\int_0^t F_0(t-\tau)\sin\omega\tau d\tau = F_1(t),$$
....
$$L^{-1}\left[\frac{(\omega^2\bar{\Gamma}(p))^m}{(p^2+\omega^2)^{m+1}}\right] = \omega\int_0^t F_{m-1}(t-\tau)\sin\omega\tau d\tau = F_m(t).$$

This time expression (21) corresponds to the original

$$L^{-1}\left[\frac{1}{p^2+\omega^2-\varepsilon\omega\bar{\Gamma}(p)}\right] = \frac{1}{\omega}\sin\omega t+\varepsilon F_1(t)+\varepsilon^2F_1(t)+...+\varepsilon^mF_m(t).$$

Then the original of the function $\bar{g}(p)$ is determined by the expression

$$g(t) = \frac{1}{\omega}\int_0^t \sin\omega(t-\tau)q(\tau)d\tau+\varepsilon\int_0^t F_1(t-\tau)q(\tau)d\tau+...+\varepsilon^m\int_0^t F_m(t-\tau)q(\tau)d\tau. \tag{22}$$

Thus, at first approximation for the forced vibrations, the function $z_1(t)$ is obtained by summing the last expression with (18). It we take into account the second approximation, then we find the function $z_2(t)$ by summing the expression (18), (20) and (22).

Analysis of the obtained solutions for a specific kernel

To estimate the influence of the second term $\varphi_2(t)$ of series (17) on the solution

$$z_1(t) = \varphi_1(t) + \varphi_2(t)$$

we consider the Rzhanitsin kernel represented in the form

$$\Gamma(t) = \varepsilon t^{\alpha-1} e^{-\beta t},$$

where $0 < \alpha < 1$, β is a constant, $\varepsilon > 0$ is some small parameter, and the values of the functions $\varphi_1(t)$ and $\varphi_2(t)$ are calculated for the following values of the parameters $\alpha, \beta, \varepsilon, \lambda, u_0$ and v_0. $\alpha = 0,12,\ \beta = 0,07,\ \varepsilon = 0,9,\ \lambda = 10,\ \lambda = 100,\ u_0 = 1,\ v_0 = 0$.

As a result, it was obtained that accounting of subsequent terms of series (17) improves the accuracy of the solution. Since for small values of the frequency the error is small, with increasing the frequency it increases. For $\lambda = 100$the amplitude of the second term of the series for some values of time constitutes 15-20% of the amplitude of the first term and the amplitudes of all terms of the series decrease exponentially over time and the phases are shifted.

Conclusions

The solution of an integro-differential equation of vibrations of viscoelastic systems is constructed in the form of a series. It is shown that the first term of this series is the solution of the indicated equation obtained by the averaging method, and the subsequent terms give clarification to this solution.

The solution obtained by the averaging method corresponds to the fact that in formula (15) the term $\varepsilon\omega M(t)$ is neglected and this is valid for rather large values of time. Consequently, for small values of time, the averaging method gives big errors.

Numerical calculation shows that for lower frequencies the error is insignificant, but with increasing frequency it increases.

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