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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF n-th ORDER EMDEN-FOWLER TYPE DIFFERENCE EQUATIONS WITH DEVIATING ARGUMENT

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Abstract. The following difference equation is considered

$$\Delta^{(n)}u(k)+p(k)\,\big|u(\sigma(k))\big|^{\lambda}\operatorname{sign}u(\sigma(k))=0,$$

where $n\geq 2$, $\lambda>1$, $p:\mathbf{N}\to\mathbf{R}$, $\sigma:\mathbf{N}\to\mathbf{N}$ and $\lim\limits_{k\to+\infty}\sigma(k)=+\infty$. Here $\Delta^{(0)}u(k)=u(k)$, $\Delta^{(1)}u(k)=u(k+1)-u(k)$, $\Delta^{(i)}=\Delta^{(1)}\circ\Delta^{(i-1)}$ $(i=1,\dots,n)$.
Sufficient conditions of new type are established for oscillation of solutions.

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1. Introduction

This work deals with oscillatory properties of solutions of Emden-Fowler type difference equation

$$\Delta^{(n)}u(k)+p(k)\big|u(\sigma(k))\big|^{\lambda}\operatorname{sign}u(\sigma(k))=0,\tag{1.1}$$

where $n\geq 2$, $p:\mathbf{N}\to\mathbf{R}$, $\sigma:\mathbf{N}\to\mathbf{N}$ and

$$\lambda>1,\qquad\lim_{k\to+\infty}\sigma(k)=\infty\quad\text{for}\quad k\in\mathbf{N}.\tag{1.2}$$

Here $\Delta^{(0)}u(k)=u(k)$, $\Delta^{(1)}u(k)=u(k+1)-u(k)$, $\Delta^{(i)}=\Delta^{(1)}\circ\Delta^{(i-1)}$ $(i=1,\dots,n)$. It will always be assumed that the conditions

$$p(k)\geq 0\quad\text{for}\quad k\in\mathbf{N},\tag{1.3}$$

or

$$p(k)\leq 0\quad\text{for}\quad k\in\mathbf{N}\tag{1.4}$$

are fulfilled.

The following notation will be used throughout the work:

Let $k_0\in\mathbf{N}$. By $\mathbf{N}^{+}_{\mathbf{k_0}}$ $(\mathbf{N}^{-}_{\mathbf{k_0}})$ we denote the set of natural numbers $\mathbf{N}^{+}_{\mathbf{k_0}}=\{\mathbf{k_0},\mathbf{k_0}+\mathbf{1},\dots\}$ $(\mathbf{N}^{-}_{\mathbf{k_0}}=\{\mathbf{1},\mathbf{2},\dots,\mathbf{k_0}\})$.

Definition 1.1. Let $k_0\in\mathbf{N}$ and $k_*=\inf\{\min(k,\sigma(k)):k\in\mathbf{N}_{\mathbf{k_0}}\}$. We will call a function $u:\mathbf{N}_{\mathbf{k_*}}\to\mathbf{R}$ a proper solution of equation (1.1), if it satisfies (1.1) on $\mathbf{N}^{+}_{\mathbf{k_0}}$ and

$$\sup\big\{|u(i)|:i\in\mathbf{N}^{+}_{\mathbf{k}}\big\}>\mathbf{0}\quad\text{for any}\quad\mathbf{k}\in\mathbf{N}^{+}_{\mathbf{k_0}}.$$

Definition 1.2. We say that a proper solution $u:\mathbf{N}^{+}_{\mathbf{k_0}}\to\mathbf{R}$ of equation (1.1) is oscillatory, if for any $k\in\mathbf{N}^{+}_{\mathbf{k_0}}$ there exist $k_1;k_2\in\mathbf{N}^{+}_{\mathbf{k}}$ such that $u(k_1)\,u(k_2)\leq 0$. Otherwise the solution is called nonoscillatory.

Definition 1.3. We say that equation (1.1) has Property **A** if any of its proper solutions is oscillatory when n is even and either is oscillatory or satisfies

$$\left|\Delta^{(i)}u(k)\right| \downarrow 0 \quad \text{as} \quad k \uparrow +\infty, \quad k \in \mathbf{N} \quad (\mathbf{i} = \mathbf{0}, \dots, \mathbf{n-1}), \tag{1.5}$$

when n is odd.

Definition 1.4. We say that equation (1.1) has Property **B** if any of its proper solutions is either oscillatory or satisfies (1.5) or

$$\left|\Delta^{(i)}u(k)\right| \uparrow +\infty \quad \text{as} \quad k \uparrow +\infty, \quad k \in \mathbf{N} \quad (\mathbf{i} = \mathbf{0}, \dots, \mathbf{n-1}) \tag{1.6}$$

when n is even and is either oscillatory or satisfies (1.6) when n is odd.

Sufficient conditions higher order Emden-Fowler type difference equation to have property **A** and **B**, when $0 < \lambda < 1$ and $\sigma(k) \geq k + 1$, can be found in [13]. Some results analogous to those of the paper are given without proofs in [12]. The problem of establishing sufficient condition for the oscillation of all solutions to the second order linear and nonlinear difference equations is considered in [16–18]. Analogous results for linear ordinary and nonlinear functional differential equations can be found in [1–13].

Lemma 1.1 ([13]). *Let $u : \mathbf{N} \to \mathbf{R}$, $m; s \in \mathbf{N}$. Then*

$$\begin{aligned} \Delta^{(i)}u(k) = {} & \sum_{j=i}^{m-1} \frac{\Delta^{(j)}u(s)}{(j-i)!} \prod_{r=1}^{j-i}(k-s-r+1) \\ & + \frac{1}{(m-i-1)!} \sum_{j=s}^{k} \prod_{r=1}^{j-i-1} (k-j-r+1)\Delta^{(m)}u(j-1), \\ & i = 0, \dots, m-1 \quad \textit{for} \quad k \in \mathbf{N}_{\mathbf{s}}^{+}, \end{aligned} \tag{1.7}$$

where

$$\Delta^{(m)}u(s-1) = 0, \quad \prod_{r=1}^{0}(k-s-r+1) = 1$$

and

$$\begin{aligned} \Delta^{(i)}u(k) = {} & \sum_{j=i}^{m-1} \frac{\Delta^{(j)}u(s)}{(j-i)!} \prod_{r=1}^{j-i}(k-s-r+1) \\ & - \frac{1}{(m-i-1)!} \sum_{j=k}^{s} \prod_{r=1}^{m-i-1} (k-j-r+1)\Delta^{(m)}u(j), \\ & i = 0, \dots, m-1 \quad \textit{for} \quad k \in \mathbf{N}_{\mathbf{s}}^{-}, \end{aligned} \tag{1.8}$$

where

$$\Delta^{(m)}u(s) = 0, \quad \prod_{r=1}^{0}(k-s-r+1) = 1.$$

Lemma 1.2 ([13]). *Let $u : \mathbf{N} \to \mathbf{R}$, $m; s \in \mathbf{N}$. Then the equality*

$$\begin{aligned}
&\sum_{i=s}^{k} i^{m-j-1}\Delta^{(m)}u(i) \\
&= \sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(k+1)\Delta^{(m-i-1)}(s+i+1-m)^{m-j-1} \\
&- \sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(s+1)\Delta^{(m-i-1)}(s+i+1-m)^{m-j-1} \quad \text{for} \quad k \in \mathbf{N}_\mathbf{s}^+
\end{aligned} \tag{1.9}$$

holds, where

$$\Delta^{(m)}u(s) = 0, \tag{1.10}$$

and

$$\begin{aligned}
&- \sum_{i=k}^{s}(i+1)^{m-j-1}\Delta^{(m)}u(i+1) \\
&= \sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(k+1)\Delta^{(m-i-1)}(k+i+1-m)^{m-j-1} \\
&- \sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(s+1)\Delta^{(m-i-1)}(s+i+1-m)^{m-j-1} \quad \text{for} \quad k \in \mathbf{N}_\mathbf{s}^-,
\end{aligned} \tag{1.11}$$

where $\Delta^{(m)}u(s+1) = 0$.

2. On some classes of nonoscillatory discrete functions

Lemma 2.1. *Let $n \geq 2$, $k_0 \in \mathbf{N}$, $u : \mathbf{N}_{\mathbf{k_0}}^+ \to \mathbf{R}$ and $u(k) > 0$, $\Delta^{(n)}u(k) \leq 0$ ($\Delta^{(n)}u(k) \geq 0$) for $k \in \mathbf{N}_{\mathbf{k_0}}^+$, $\Delta^{(n)}u(k) \not\equiv 0$ for any $s \in \mathbf{N}_{\mathbf{k_0}}^+$ and $k \in \mathbf{N}_\mathbf{s}^+$.Then there exist $k_1 \in \mathbf{N}_{\mathbf{k_0}}^+$ and $\ell \in \{0, \dots, n\}$ such that $\ell + n$ is odd ($\ell + n$ is even) and*

$$\begin{aligned}
\Delta^{(i)}u(k) > 0 \quad &\text{for} \quad k \in \mathbf{N}_{\mathbf{k_1}}^+ \quad (\mathbf{i} = \mathbf{0}, \dots, \ell), \\
(-1)^{i+\ell}\Delta^{(1)}u(k) > 0 \quad &\text{for} \quad k \in \mathbf{N}_{\mathbf{k_1}}^+ \quad (\mathbf{i} = \ell, \dots, \mathbf{n-1}), \\
(-1)^{n-\ell}\Delta^{(n)}u(k) \geq 0 \quad &\text{for} \quad k \in \mathbf{N}_{\mathbf{k_1}}^+.
\end{aligned} \tag{2.1}$$

The lemma follows immediately from the fact that, if $u(k) > 0$ and $\Delta^{(2)}u(k) \leq 0$ for $k \in \mathbf{N}_{\mathbf{k_0}}^+$, then there exists $k_1 \in \mathbf{N}_{\mathbf{k_0}}^+$, such that $\Delta^{(1)}u(k) > 0$ for $k \in \mathbf{N}_{\mathbf{k_1}}^+$.

Lemma 2.2 ([13]). *Let $u : \mathbf{N} \to \mathbf{R}$, $k_0; n \in \mathbf{N}$ and*

$$\begin{aligned}
(-1)^i\Delta^{(i)}u(k) > 0 \quad &(i = 0, \dots, n-1), \\
(-1)^n\Delta^{(n)}u(k) \geq 0 \quad &\text{for} \quad k \in \mathbf{N}_{\mathbf{k_0}}^+.
\end{aligned} \tag{2.2}$$

Then

$$\sum_{k=1}^{+\infty} k^{n-1}\big|\Delta^{(n)}u(k)\big| < +\infty, \tag{2.3}$$

$$\big|\Delta^{(i)}u(k)\big| \geq \frac{1}{(n-i-1)!}\sum_{j=k}^{+\infty}\prod_{r=1}^{n-i-1}(j-k-r-1)\big|\Delta^{(n)}u(j)\big| \tag{2.4}$$

$$\textit{for} \quad k \in \mathbf{N}^{+}_{\mathbf{k_0}}, \quad (\mathbf{i} = \mathbf{0},\dots,\mathbf{n-1}).$$

Lemma 2.3. *Let $u : \mathbf{N} \to \mathbf{R}$ and let for some $k_0 \in \mathbf{N}$ and $\ell \in \{1,\dots,n-1\}$, (2.1) be fulfilled. Then*

$$\sum_{k=1}^{+\infty} k^{n-\ell-1}\big|\Delta^{(n)}u(k)\big| < +\infty \tag{2.5}$$

and there exists $k_1 \in \mathbf{N}^{+}_{\mathbf{k_0}}$ such that

$$\big|\Delta^{(i)}u(k)\big| \geq \frac{1}{(n-i-1)!}\sum_{j=k}^{+\infty}\prod_{r=1}^{n-i-1}(j+r-k-1)\big|\Delta^{(n)}u(j)\big| \tag{2.6}$$

$$\textit{for} \quad k \in \mathbf{N}^{+}_{\mathbf{k_1}}, \quad (\mathbf{i} = \ell,\dots,\mathbf{n-1}),$$

$$\Delta^{(i)}u(k) \geq \Delta^{(i)}u(k_1) + \frac{1}{(\ell-i-1)!(n-\ell-1)!}\sum_{s=k_1}^{k-1}\prod_{r=1}^{\ell-i-1}\big(k+r-(1+s)\big)$$
$$\times \sum_{j=k}^{+\infty}\prod_{r=1}^{n-\ell-1}\big(j+r-(1+s)\big)\big)\big|\Delta^{(n)}u(j)\big| \tag{2.7}$$

$$\textit{for} \quad k \in \mathbf{N}^{+}_{\mathbf{k_1+1}} \quad (\mathbf{j} = \mathbf{0},\dots,\ell-\mathbf{1}).$$

If in addition

$$\sum_{k=1}^{+\infty} k^{n-\ell}\big|\Delta^{(n)}u(k)\big| = +\infty, \tag{2.8}$$

then

$$\frac{\Delta^{(\ell-i)}u(k)}{\prod\limits_{r=0}^{i-1}(k-r)}\downarrow, \qquad \frac{\Delta^{(\ell-i)}u(k)}{\prod\limits_{r=1}^{i-1}(k-r)}\uparrow \tag{2.9}$$

for large k,

$$u(k) \geq \frac{1+o(1)}{\ell!}\,k^{\ell-1}\Delta^{(\ell-1)}u(k) \tag{2.10}$$

and

$$\Delta^{(\ell-1)}u(k) \geq \frac{k}{(n-\ell-1)!}\sum_{i=k}^{+\infty} i^{n-\ell-1}\big|\Delta^{(n)}u(k)\big|$$
$$+\frac{1}{(n-\ell-1)!}\sum_{i=k_1}^{k} i^{n-\ell}\big|\Delta^{(n)}u(i)\big| \quad \textit{for} \quad k \in \mathbf{N}^{+}_{\mathbf{k_1}}. \tag{2.11}$$

The proof of Lemma 2.3 in a slightly different way when $\Delta^{(n)}u(k) \le 0$ ($\Delta^{(n)}u(k) \ge 0$) is given in [13] ([14]).

So below we present the complete proof of Lemma 2.3.

Proof of Lemma 2.3. Let $s; k \in \mathbf{N}^+_{\mathbf{k_1}}$ and $s < k$. Assume that (1.10) is fulfilled. By vitrue of (2.1), from equality (1.9) with $j = \ell$ and $m = n$ we have

$$\begin{aligned}
&\sum_{i=s}^{k}(-1)^{n+\ell} i^{n-\ell-1}\Delta^{(n)}u(i)\\
&= \sum_{i=\ell}^{n-1}(-1)^{\ell+i}\Delta^{(i)}u(s+1)\Delta^{(n-i-1)}(s+i+1-n)^{n-\ell-1}\\
&- \sum_{i=\ell}^{n-1}(-1)^{\ell+i}\Delta^{(i)}u(k+1)\Delta^{(n-i-1)}(k+i+1-n)^{n-\ell-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{i=s}^{k} i^{n-\ell-1}\big|\Delta^{(n)}u(i)\big| &\le \sum_{i=\ell}^{n-1}\big|\Delta^{(i)}u(s+1)\big|\Delta^{(n-i-1)}(s+i+1-n)^{n-\ell-1}\\
&- \sum_{i=\ell}^{n-1}(-1)^{\ell+i}\Delta^{(i)}u(k+1)\Delta^{(n-i-1)}(k+i+1-n)^{n-\ell-1} \quad \text{for} \ \ k \in \mathbf{N}^+_{\mathbf{s}}.
\end{aligned}$$

From the last inequality, with $k \to +\infty$, we obtain (2.5). The equality (1.11) also implies the inequality

$$\begin{aligned}
&\sum_{i=\ell}^{n-1}\big|\Delta^{(i)}u(k+1)\big|\Delta^{(n-i-1)}(k+i+1-n)^{n-\ell-1}\\
&\qquad \ge \sum_{i=k}^{+\infty} i^{n-\ell-1}\big|\Delta^{(n)}u(i+1)\big| \quad \text{for} \ \ k \in \mathbf{N}^+_{\mathbf{k_1}}. \qquad (2.12)
\end{aligned}$$

On account of (2.1) and (2.5), from (1.7) we obtain (2.6).

Analogously, equality (1.7) with $s = k$, and $m = \ell$, gives

$$\Delta^{(i)}u(k) \ge \Delta^{(i)}u(k_1) + \frac{1}{(\ell-i-1)!}\sum_{i=k_1}^{k}\prod_{r=1}^{\ell-i-1}(k-j+r-1)\Delta^{(\ell)}u(j-1)$$
$$(i = 0, \dots, \ell-1) \quad \text{for} \quad k \in \mathbf{N}^+_{\mathbf{k_1}}.$$

Hence, by (2.6) we obtain (2.7). Using (2.1), from (1.9) with $j = \ell - 1$ and $m = n$, for $s = k_1$, we have

$$\begin{aligned}
\Delta^{(\ell-1)}u(k) &\ge \frac{1}{(n-\ell)!}\sum_{i=k_1}^{k} s^{n-\ell}\big|\Delta^{(n)}u(i)\big|\\
&+ \frac{1}{(n-\ell)!}\sum_{i=\ell}^{n-1}\big|\Delta^{(i)}u(k+1)\big|\Delta^{(n-i-1)}(k+i+1-n)^{n-\ell}\\
&+ \frac{1}{(n-\ell)!}\sum_{i=\ell-1}^{n-1}(-1)^{n+i-1}\Delta^{(i)}u(k_1+1)\Delta^{(n-i-1)}(k_1+i+1-n)^{n-\ell}.
\end{aligned}$$

Therefore, according to (2.8), there exist $k^* > k_1$ such that

$$\Delta^{(\ell-1)}u(k+1) \geq \frac{1}{(n-\ell)!}\sum_{i=k^*}^{k} i^{n-\ell}\big|\Delta^{(n)}u(i)\big| + \frac{1}{(n-\ell)!}\sum_{i=\ell}^{n-1}\big|\Delta^{(i)}u(k+1)\big|\Delta^{(n-i-1)}(k+i+1-n)^{n-\ell}$$
$$\text{for} \quad k \in \mathbf{N}^+_{\mathbf{k^*}}.$$

From the last inequality by (2.8) we have

$$\Delta^{(\ell-1)}u(k+1) - (k+\ell-1-n)\Delta^{(\ell)}u(k+1) \to +\infty \quad \text{for} \quad k \to +\infty, \tag{2.13}$$

and by (2.12) the inequality (2.11) holds.

Let $k_0 \in \mathbf{N}$ and for any $k \in \mathbf{N}^+_{\mathbf{k_0}}$ and $i \in \{1,\dots,\ell\}$ put

$$\rho_i(k) = i\Delta^{(\ell-n)}u(k) - (k+1-i)\Delta^{(\ell-n+1)}u(k), \tag{2.14}$$
$$\gamma_i(k) = (k-i)\Delta^{(\ell-n+1)}u(k) - (1-i)\Delta^{(\ell-i)}u(k). \tag{2.15}$$

Applying (2.13) and L'Hôpital's rule, we have

$$\lim_{k\to+\infty}\frac{\Delta^{(\ell-i)}u(k)}{\prod\limits_{i=1}^{i-1}(k-j)} = +\infty \quad (j=1,\dots,\ell). \tag{2.16}$$

(Here it is meant that $\prod\limits_{j=1}^{0}(k-j)=1$).

Since

$$\Delta^{(1)}\left(\frac{\Delta^{(\ell-i)}u(k)}{\prod\limits_{i=1}^{i-1}(k-j)}\right) = \frac{\gamma_i(k)}{\prod\limits_{j=0}^{i-1}(k-j-1)}\,,$$

by (2.16) there exist $k_\ell > \cdots > k_1$ such that $\gamma_i(k_i) > 0$ $(i=1,\dots,\ell)$. Therefore, by (2.13) $\rho_i(k) \to +\infty$ as $k \to +\infty$, $\Delta^{(1)}\rho_{i+1}(k) = \rho_i(k)$, $\Delta^{(1)}\gamma_{i+1}(k) = \gamma_i(k)$ and $\gamma_1(k) = (k-1)\Delta^{(\ell)}u(k) > 0$ for $k \in \mathbf{N}^+_{\mathbf{k}_\ell}$, we find that $\rho_i(k) \to +\infty$ as $k \to +\infty$ and $\gamma_i(k) > 0$ for $k \in \mathbf{N}^+_{\mathbf{k_i}}$ $(i=1,\dots,\ell)$. This fact along with (2.13)–(2.16) proves (2.9).

On the other hand, since $\rho_i(k) \to +\infty$ $(i=1,\dots,\ell)$, by (2.14), for large k we have

$$i\Delta^{(\ell-i)}u(k) > (k+1-i)\Delta^{(\ell-i+1)}u(k) \quad (i=1,\dots,\ell),$$

which implies (2.10).

3. Necessary conditions for the existence of solutions of type (2.1)

The results of this section play an important role in establishing sufficient conditions for equations (1.1) to have Properties **A** and **B**.

Let $k_0 \in \mathbf{N}$ and $\ell \in \{1,\dots,n-1\}$. By U_{ℓ,k_0} we denote the set of all solutions of equation (1.1) satisfying condition (2.1).

Theorem 3.1. *Let conditions* (1.2), (1.3) ((1.4)) *be fulfilled,* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd* ($\ell + n$ *even*) *and*

$$\sum_{k=1}^{+\infty} k^{n-\ell}\big(\sigma(k)\big)^{\lambda(\ell-1)}\big|p(k)\big| = +\infty. \tag{3.1}$$

If, moreover, for some $k_0 \in \mathbf{N}$, $U_{\ell,k_0} \neq \varnothing$, *then for any* $\varepsilon \in (0, \lambda]$ *we have*

$$\sum_{k=1}^{+\infty} k^{n-\ell}\big(\sigma(k)\big)^{\lambda(\ell-1)}\Big(\frac{\widetilde{\sigma}(k)}{k}\Big)^{1+\varepsilon}\big|p(k)\big| < +\infty, \tag{3.2}$$

where

$$\widetilde{\sigma}(k) = \begin{cases} \sigma(k), & \sigma(k) \le k, \\ k, & \sigma(k) > k. \end{cases} \tag{3.3}$$

Proof. Let $k_0 \in \mathbf{N}$, $\ell \in \{1, \dots, n-1\}$, $\ell + n$ be odd ($\ell + n$ be even) and $U_{\ell,k_0} \neq \varnothing$. By definition of the set U_{ℓ,k_0}, equation (1.1) has a proper solution $u \in U_{\ell,k_0}$ satisfying the condition (2.1). By (1.1), (2.1) and (3.1) it is clear that condition (2.8) holds. Thus, by Lemma 2.3, (2.5)–(2.11) are fulfilled and by (1.1) and (2.10), from (2.11) we get

$$\begin{aligned} \Delta^{(\ell-1)}u(k) \ge{}& \frac{k}{2\ell!(\ell-1)!}\sum_{i=k}^{+\infty} i^{n-\ell-1}\big(\sigma(i)\big)^{\lambda(\ell-1)}\big(\Delta^{(\ell-1)}u\big(\sigma(i)\big)\big)^{\lambda}\big|p(i)\big| \\ &+ \frac{1}{2\ell!(n-\ell)!}\sum_{i=k_*}^{k} i^{n-\ell}\big(\sigma(i)\big)^{\lambda(\ell-1)}\big(\Delta^{(\ell-1)}\big(\sigma(i)\big)\big)^{\lambda}\big|p(i)\big| \\ &\qquad\qquad \text{for } \ k \in \mathbf{N}^{+}_{\mathbf{k}_*}, \end{aligned} \tag{3.4}$$

where k_* is a sufficiently large natural number. Therefore, from (3.4) we have

$$\begin{aligned} \Delta^{(\ell-1)}u(k) \ge{}& \frac{1}{2\ell!(n-\ell)!}\sum_{i=k_*}^{k} i^{n-\ell}\big(\sigma(i)\big)^{\lambda(\ell-1)}\Big(\Delta^{(\ell-1)}\big(u(\widetilde{\sigma}(i))\big)\Big)^{1+\varepsilon}\big|p(i)\big| \\ ={}& \frac{1}{2\ell!(n-\ell)!}\sum_{i=k_*}^{k} i^{n-\ell}\big(\sigma(i)\big)^{\lambda(\ell-1)}\big(\widetilde{\sigma}(i)\big)^{1+\varepsilon} \\ &\times\Big(\frac{\Delta^{(\ell-1)}u(\widetilde{\sigma}(i))}{\widetilde{\sigma}(i)}\Big)^{1+\varepsilon}|p(i)|, \quad \text{where} \quad \varepsilon \in (0, \lambda). \end{aligned} \tag{3.5}$$

Since $\dfrac{\Delta^{(\ell-1)}u(k)}{k}\downarrow$, from (3.5) we get

$$\begin{aligned} \Delta^{(\ell-1)}u(k) \ge{}& \frac{1}{2\ell!(n-\ell)!}\sum_{i=k_*}^{k} i^{n-\ell}\big(\sigma(i)\big)^{\lambda(\ell-1)}\Big(\frac{\widetilde{\sigma}(i)}{i}\Big)^{1+\varepsilon}|p(i)| \\ &\times\big(\Delta^{(\ell-1)}k(i)\big)^{1+\varepsilon} \quad \text{for} \quad k \in \mathbf{N}^{+}_{\mathbf{k}_*}. \end{aligned} \tag{3.6}$$

By (3.1), there exists $k_1 \in \mathbf{N}^{+}_{\mathbf{k_n}}$ such that

$$\sum_{i=k_*}^{k_1} i^{n-\ell}\big|p(i)\big|\big(\sigma(i)\big)^{\lambda(\ell-1)}\Big(\frac{\widetilde{\sigma}(i)}{i}\Big)^{1+\varepsilon}\big(\Delta^{(\ell-1)}u(i)\big)^{1+\varepsilon} > 0.$$

Therefore, from (3.6) we get

$$\frac{\Delta^{(\ell-1)}u(k)}{\sum\limits_{i=k_*}^{k} i^{n-\ell}\big|p(i)\big|\big(\sigma(i)\big)^{\lambda(\ell-1)}\Big(\dfrac{\widetilde{\sigma}(i)}{i}\Big)^{1+\varepsilon}\big(\Delta^{(\ell-1)}_{u(\varepsilon)}\big)^{1+\varepsilon}} \geq \frac{1}{2\ell!(n-\ell)!}$$

$$\text{for} \quad k \in \mathbf{N}^{+}_{\mathbf{k_1}}.$$

From the last inequality we have

$$\frac{\big(\Delta^{(\ell-1)}u(k)\big)^{1+\varepsilon}}{\Big(\sum\limits_{i=k_*}^{k} i^{n-\ell}\big|p(i)\big|\big(\sigma(i)\big)^{\lambda(\ell-1)}\Big(\dfrac{\widetilde{\sigma}(i)}{i}\Big)^{1+\varepsilon}\big(\Delta^{(\ell-1)}u(\varepsilon)\big)^{1+\varepsilon}\Big)^{1+\varepsilon}} \geq \frac{1}{\big(2\ell!(n-\ell)!\big)^{1+\varepsilon}}$$

$$\text{for} \quad k \in \mathbf{N}^{+}_{\mathbf{k_1}}.$$

Therefore

$$\sum_{s=k_1}^{k} \frac{\big(\Delta^{(\ell-1)}u(s)\big)^{1+\varepsilon} s^{n-\ell}\big|p(s)\big|\big(\sigma(s)\big)^{\lambda(\ell-1)}\Big(\dfrac{\widetilde{\sigma}(s)}{s}\Big)^{1+\varepsilon}}{\Big(\sum\limits_{i=k_*}^{s} i^{n-\ell}\big|p(i)\big|\big(\sigma(i)\big)^{\lambda(\ell-1)}\Big(\dfrac{\widetilde{\sigma}(i)}{i}\Big)^{1+\varepsilon}\Big(\Delta^{(\ell-1)}u(i)\Big)^{1+\varepsilon}\Big)^{1+\varepsilon}}$$

$$\geq \frac{1}{(2\ell!(n-\ell)!)^{1+\varepsilon}} \sum_{s=k_1}^{k} s^{n-\ell}\big|p(s)\big|\big(\sigma(s)\big)^{\lambda(\ell-1)}\Big(\frac{\widetilde{\sigma}(s)}{s}\Big)^{1+\varepsilon}, \tag{3.7}$$

$$\text{for} \quad k \in \mathbf{N}^{+}_{\mathbf{k_1}}.$$

Denote

$$a_s = \sum_{i=k_*}^{s} i^{n-\ell}\big|p(i)\big|\big(\sigma(i)\big)^{\lambda(\ell-1)}\Big(\frac{\sigma(i)}{i}\Big)^{1+\varepsilon}\big(\Delta^{(\ell-1)}u(i)\big)^{1+\varepsilon}.$$

From (3.7) we get

$$\sum_{s=k_1}^{k} \frac{a_s - a_{s-1}}{(a_s)^{1+\varepsilon}} \geq \frac{1}{\big(2\ell!(n-\ell)!\big)^{1+\varepsilon}} \sum_{s=k_1}^{k} s^{n-\ell}\big|p(s)\big|\big(\sigma(s)\big)^{\lambda(\ell-1)}\Big(\frac{\widetilde{\sigma}(s)}{s}\Big)^{1+\varepsilon} \tag{3.8}$$

$$\text{for} \quad k \in \mathbf{N}^{+}_{\mathbf{k_1}}.$$

Since

$$\sum_{s=k_1}^{k} (a_s)^{-1-\varepsilon}(a_s - a_{s-1}) = \sum_{s=k_1}^{k} (a_s)^{-1-\varepsilon} \int_{a_{s-1}}^{a_s} dt$$

$$\leq \sum_{s=k_1}^{k} \int_{a_{s-1}}^{a_s} t^{-1-\varepsilon} dt = \int_{a_{s-1}}^{a_k} t^{-1-\varepsilon} dt = \frac{a^{\varepsilon}_{k_1-1}}{\varepsilon} - \frac{a_k^{-\varepsilon}}{\varepsilon} \leq \frac{1}{\varepsilon a_{k_1-1}}, \quad k \in \mathbf{N}^{+}_{\mathbf{k_1}},$$

from (3.8) we get

$$\sum_{s=k_1}^{k} s^{n-\ell}\big|p(s)\big|\big(\sigma(s)\big)^{\lambda(\ell-1)}\Big(\frac{\widetilde{\sigma}(s)}{s}\Big)^{1+\varepsilon} \leq \frac{(2\ell!(n-\ell)!)^{1+\varepsilon}}{\varepsilon a_{k_1-1}}, \quad k \in \mathbf{N}^{+}_{\mathbf{k_1}}.$$

Thus from the last inequality

$$\sum_{s=k_1}^{+\infty} s^{n-\ell}\big|p(s)\big|\big(\sigma(s)\big)^{\lambda(\ell-1)}\Big(\frac{\widetilde{\sigma}(s)}{s}\Big)^{1+\varepsilon} < +\infty,$$

which proves the validity of the theorem.

4. Sufficient conditions of nonexistence of solutions of type (2.1)

Theorem 4.1. *Let conditions* (1.2), (1.3) ((1.4)) *be fulfilled,* $\ell \in \{1,\dots,n-1\}$ $\ell+n$ *by odd* ($\ell+n$ *by even*) *and* (3.1) *hold. If, moreover, for some* $\varepsilon \in (0,\lambda]$

$$\sum_{k=1}^{+\infty} k^{n-\ell}\big(\sigma(k)\big)^{\lambda(\ell-1)}\Big(\frac{\widetilde{\sigma}(k)}{k}\Big)^{1+\varepsilon}\big|p(k)\big| = +\infty, \tag{4.1}$$

then for any $k_0 \in \mathbf{N}$, $U_{\ell,k_0} = \varnothing$, *where* $\widetilde{\sigma}$ *is defined by* (3.3).

Proof. Assume the contrary. Let there exist $k_0 \in \mathbf{N}$ such that $U_{\ell,k_0} \neq \varnothing$. Thus equation (1.1) has a proper solution $u : \mathbf{N}^+_{\mathbf{k_0}} \to (\mathbf{0}, +\infty)$, satisfying (2.1).

Since conditions of Theorem 3.1 are fulfilled, (3.2) holds for any $\varepsilon \in (0,\lambda]$, which contradicts (4.1). The obtained contradiction proves the validity of the theorem.

From this theorem if $\sigma(k) \le k$, immediately follow

Corollary 4.1. Let conditions (1.2), (1.3) ((1.4)) be fulfilled, $\ell \in \{1,\dots,n-1\}$, $\ell+n$ be odd ($\ell+n$ be even) and (3.1) hold. Then for any $k_0 \in \mathbf{N}$, $U_{\ell,k_0} = \varnothing$.

5. Difference equations with Property A

Theorem 5.1 Let conditions (1.2), (1.3) be fulfilled and for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ be odd, let for some $\varepsilon \in (0,\lambda)$ (4.1) hold as well and when n is odd

$$\sum_{k=1}^{+\infty} k^{n-1}p(k) = +\infty. \tag{5.1}$$

Then equation (1.1) has Property **A**.

Proof. Let equation (1.1) have a proper nonoscillatory solution $u : \mathbf{N}^+_{\mathbf{k_0}} \to (\mathbf{0}, +\infty)$ (the case $u(k) < 0$ is similar). Then by (1.1), (1.3) and Lemma 2.1 there exists $\ell \in \{0,\dots,n-1\}$ such that $\ell+n$ odd and condition (2.1) holds. Since the conditions of the Theorem 4.1 are fulfilled, for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ is odd, we have $\ell \notin \{1,\dots,n-1\}$. Therefore n is odd and $\ell = 0$. Then we will show that condition (1.5) hold.

If that is not the case, there exist $c > 0$ such that $u(k) \ge c$ for sufficiently large k. According to (2.1), with $\ell = 0$, from (1.1) we have

$$\sum_{i=k_0}^{k} i^{n-1}\Delta^{(n)}u(i) + c^{\lambda}\sum_{i=k_0}^{k} i^{n-1}p(i) \le 0, \quad \text{for} \quad k \in \mathbf{N}^+_{\mathbf{k_0}}, \tag{5.2}$$

where $k_0 \in \mathbf{N}$ is a sufficiently large natural number.

On the other hand, by the identity

$$\sum_{i=k_0}^{k} i^{n-1}\Delta^{(n)}u(i) = k^{n-1}\Delta^{(n-1)}u(k+1) - (k_0-1)^{n-1}\Delta^{(n-1)}u(k_0) - \sum_{i=k_0}^{k}\Delta^{(n-1)}u(i)\Delta(i-1)^{n-1}$$

it is easy to show that

$$\sum_{i=k_0}^{k} i^{n-1}\Delta^{(n)}u(i) = \sum_{i=0}^{n-1}(-1)^i\Delta^{(i)}(k-j)^{n-1}\Delta^{(n-i-1)}u(k+1) - \sum_{i=0}^{n-1}(-1)^i(k_0-i-1)^{n-i-1}\Delta^{(n-i-1)}u(k_0).$$

Since $(-1)^i\Delta^{(i)}u(k)\geq 0$, from (5.2) we have

$$c^{\lambda}\sum_{i=k_0}^{k} i^{n-1}p(i) \leq \sum_{i=0}^{n-1}(k_0-i-1)^{n-i-1}\big|\Delta^{(n-i-1)}u(k_0)\big|.$$

Therefore

$$\sum_{i=1}^{+\infty} i^{n-1}p(i) < +\infty,$$

which contradicts condition (5.1). Therefore, equation (1.1) has Property **A**.

Theorem 5.2. *Let conditions* (1.2), (1.3) *be fulfilled and*

$$\liminf_{k\to+\infty}\frac{\sigma^{\lambda}(k)}{k} > 0. \tag{5.3}$$

Then for some $\varepsilon\in(0,\lambda)$ *the condition*

$$\sum_{k=1}^{+\infty} k^{n-2-\varepsilon}\big(\widetilde{\sigma}(k)\big)^{1+\varepsilon}p(k) = +\infty, \tag{5.4}$$

for even n *and the condition* (5.1) *and*

$$\sum_{k=1}^{+\infty} k^{n-3-\varepsilon}\sigma^{\lambda}(k)\big(\widetilde{\sigma}(k)\big)^{1+\varepsilon}p(k) = +\infty, \tag{5.5}$$

for odd n *is sufficient for equation* (1.1) *to have Property* **A**.

Proof. It is obvious that, according to (5.1)–(5.4) for any $\ell\in\{1,\dots,n-1\}$, where $\ell+n$ is odd, all conditions of Theorem 5.1 are fulfilled, which proves the validity of the theorem.

Theorem 5.3. *Let conditions* (1.2), (1.3) *be fulfilled and*

$$\limsup_{k\to+\infty}\frac{\sigma^{\lambda}(k)}{k}<+\infty. \tag{5.6}$$

Then for equation (1.1) *to have Property* **A** *it is sufficient that for some* $\varepsilon\in(0,\lambda)$

$$\sum_{k=1}^{+\infty}k^{-\varepsilon}\big(\sigma(k)\big)^{\lambda(n-2)}\big(\widetilde{\sigma}(k)\big)^{1+\varepsilon}p(k)=+\infty. \tag{5.7}$$

Proof. It is obvious that by (5.5) and (5.6) all conditions of Theorem 5.1 are fulfilled, which proves the validity of the theorem.

6. Difference equations with Property B

Theorem 6.1. *Let conditions* (1.2), (1.4) *be fulfilled for any* $\ell\in\{1,\dots,n-2\}$ *with* $\ell+n$ *even, let as well as for some* $\varepsilon\in(0,\lambda)$, (4.1) *hold and*

$$\sum_{k=1}^{+\infty}\big(\sigma(k)\big)^{\lambda(n-1)}\big|p(k)\big|=+\infty. \tag{6.1}$$

If moreover, for even n

$$\sum_{k=1}^{+\infty}k^{n-1}\big|p(k)\big|=+\infty, \tag{6.2}$$

then equation (1.1) *has Property* **B**.

Proof. Let equation (1.1) have a proper nonoscillatory solution $u:\mathbf{N_{k_0}}\to(\mathbf{0},+\infty)$. By (1.1), (1.2) and Lemma 2.1, there exists $\ell\in\{0,\dots,n\}$ such that $\ell+n$ is even and condition (2.1) holds. Since the conditions of Theorem 4.1 are fulfilled, for any $\ell\in\{1,\dots,n-2\}$ with $\ell+n$ is even, we have $\ell\notin\{1,\dots,n-2\}$. Therefore $\ell=n$, or $\ell=0$ and n is even.

Assume that $\ell=n$. To complete the proof, it suffices to show that (1.4) is valid. From (2.1) with $\ell=n$, we have $u\big(\sigma(k)\big)\ge c\big(\sigma(k)\big)^{n-1}$ for $k\in\mathbf{N}^+_{\mathbf{k_1}}$, where $c>0$ and $k_1\in\mathbf{N}^+_{\mathbf{k_0}}$ is a sufficiently large natural number. Therefore, by (1.2), (6.1) and (2.1), when $p=n$, from (1.1) we get

$$\Delta^{(n-1)}u(k)\ge\Delta^{(n-1)}(k_1)+c^{\lambda}\sum_{i=1}^{k}\big|p(i)\big|\big(\sigma(i)\big)^{\lambda(n-1)}\to+\infty \quad \text{for}\quad k\to+\infty.$$

Now assume that n is even and $\ell=0$. In is case, analogously of Theorem 5.1, we show that conditions (1.3) hold. Therefore, equation (1.1) has Property **B**.

Theorem 6.2. *Let conditions* (1.2), (1.4), (5.2) *be fulfilled and for some* $\varepsilon\in(0,\lambda)$ *condition* (5.3) *for odd* n *and* (5.4) *for even* n *are fulfilled. If, moreover* (6.1) *is fulfilled, then equation* (1.1) *has Property* **B**.

Proof. It is obvious that by (5.3) and (5.4) all conditions of Theorem 6.1 are fulfilled which proves the validity of the theorem.

Theorem 6.3. *Let conditions* (1.2)*,* (1.4)*,* (6.1) *and for even* n (6.2) *be fulfilled. If moreover* (5.5) *and for some* $\varepsilon \in (0, \lambda)$

$$\sum_{k=1}^{+\infty} k^{n-1}\big(\sigma(k)\big)^{n-3}\big(\widetilde{\sigma}(k)\big)^{1+\varepsilon} = +\infty. \tag{6.3}$$

are fulfilled, then equation (1.1) *has Property* **B**.

Proof. According to (5.5) and (6.3) it obvious that for any $\ell \in \{1, \dots, n-2\}$, for some $\varepsilon \in (0, \lambda)$ conditions (4.1) are fulfilled. That is, all conditions of Theorem 6.1 are fulfilled, which proves the validity of the theorem.

REFERENCES

1. Kondrat'ev V. A. Oscillatory properties of solutions of the equation $y^{(n)} + p(x)y = 0$ (Russian). *Trudy Moskov. Mat. Obshch.*, **10** (1961), 419-436.

2. Kiguradze I. T. and Chanturia T. A. Asymptotic properties of solution of nonautonomous ordinary differential equations. Translated from the 1985 Russian original. *Mathematics and its Applications* (Sevent Session), 89, *Kluver Academic Publishers Group, Dordracht*, 1993.

3. Koplatadze R. G. and Chanturia T. A. On oscillatory properties of differential equations with a deviating argument (Russian). *Izdat. Tbilis. Univ., Tbilisi*, 1977.

4. Koplatadze R. On oscillatory properties of solutions of functional differential equations. *Mem. Differential Equations Math. Phys.*, **3** (1994), 3-179.

5. Graef J., Koplatadze R. anf Kvinikadze G. Nonlinear functional differential equations with Properties A and B. *J. Math. Anal. Appl.*, **306**, 1 (2005), 136-160.

6. Koplatadze R. Quasi-linear functional differential equations with Property A. *J. Math. Anal. Appl.*, **330**, 1 (2007), 483-510.

7. Koplatadze R. and Litsin E. Oscillation criteria for higher order "almost linear functional differential equation. *Funct. Differ. Equ.*, **16**, 3 (2009), 387-434.

8. Domoshnitski A. and Koplatadze R. On asymptotic behavior of solutions of generalized Enmden-Fowler differential equations with delay argument. *Abstract and Applied Analysis* 2014, Art. ID 168425, 13 pp.

9. Domoshnitski A. and Koplatadze R. On hight order generalized Emden-Fowler differential equation with delay argument. *Reports in J. Math. Sci.* (N,Y), **220**, 4 (2017), 161-189; *Nonlinear Oscillations* **18**, 4 (2015), 507-526.

10. Koplatadze R. Almost linear functional differential equations with proper-ties A and B. Trans. *A. Razmadze Math. Inst.* **170**, 2 (2016), 215-242.

11. Berezansly L., Domoshnitski A. and Koplatadze R. Oscillation, nonoscillation, stability and asymptotic properties for second and higher order functional differential equations. *Taylor and Francs Group*, 2020.

12. Koplatadze R. and Khachidze N. Nonlinear difference equations with properties A and B. *Functional differential equations*, **25**, 1-2 (2018), 91-95.

13. Koplatadze R. and Khachidze N. Asymptotic behavior of solutions of n-th order Emden-Fowler difference equations with advanced argument. *J. Contemporary Mathematical Analysis of Sciences*, **55**, 4 (2021), 201–213.

14. Koplatadze R. On higher order nonlinear difference equations with property B. *Report of Enlarged Session of the Seminar of I. Vekua Institute of Applied Mathemat.*, **47** (2021), 42-56.

15. Koplatadze R., Kvinikadze G. and Stavroulakis I. Oscillation of second order linear difference equations with deviating arguments. *Adv. Math. Sci. Appl.* **12**, 1 (2002), 217-226.

16. Koplatadze R. and Pinelas S. Oscillation of nonlinear difference equations with delayed argument. *Commun. Appl. Anal.*, **16**, 1 (2012), 8795.

17. Koplatadze R. and Pinelas S. On oscillation of solutions of second order nonlinear difference equations. *translated from Nelnn Koliv.*, **17**, 2 (2014), 248-267; *J. Math. Sci.* (N.Y.) **208**, 5 (2015), 571-592.

18. Koplatadze R. and Pinelas S. Oscillation criteria for first order linear difference equations with several delay arguments. *Nonlin. Oscill.*, **17**, 2 (2014), 248-267; *J. Math. Sci.* (N.Y) **208**, 5 (2015), 571-592.

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