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THE DIRICHLET BVP OF THERMOELASTIC DIFFUSION THEORY WITH MICROTEMPERATURES AND MICROCONCENTRATIONS FOR A SPHERE

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Abstract. The main purpose of this work is to construct an explicit solution of the Dirichlet boundary value problem of thermoelastic diffusion theory with microtemperatures and microconcentrations for a sphere. The obtained solution of the considered problem is represented as absolutely and uniformly convergent series

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1. Introduction

The present paper considers the 3D linear thermoelastic diffusion theory with microtemperatures and microconcentrations. The theory of thermoelasticity with microtemperatures is a good model to explain the thermal conduction in microelements. However, the research confirmed that the field of diffusion in solids cannot be ignored. The processes of heat and mass diffusion, also basic problems of thermoelastic theory, play an important role in many engineering applications, such as satellite problems, aircraft landing on water or land, the oil extraction, etc. Therefore, to study basic problems of thermoelasticity for materials with diffusion, microtemperatures and microconcentrations have considerable attention.

Below, we will consider a few works, which give the main results and bibliographical data. The theoretical works in the field of thermodiffusion theory, was establish by Nowacki [1] and developed later by Sherief et.al. [2]. The linear theory of thermoelasticity for materials with the classical displacement and temperature fields, possess microtemperatures, was established by Grot [3]. He extended the thermodynamics of a continuum with microstructure so that the point of generic microelements are assumed to have different temperatures. He supposed that the inverse of the microelement temperature is a linear function of microcoordinates. Ieşan and Quintanilla in [4] have developed the linear theory of thermoelastic materials with microtemperatures,in which the particles are subjected to classical displacement, temperature fields and mass diffusion fields and whose microelements possess micro-temperatures and micro-concentrations. The Clausius-Duhem inequality is modified to include microtemperatures. The first-order energy equations are added to the balance laws of a continuum with microstructure, have formulated the boundary value problems and presented an uniqueness result, by Bazarra et al. in [5] proposed dynamical problem for thermoelastic body with diffusion whose microelements are assumed to possess microtemperatures and microconcentrations.In [6] by Aouadi et al. is considered with a nonlinear theory of thermodinamics for elastic materials in which particles are subjected to classical displacement, temperature and mass diffusion fields and whose microelements posses microtemperatures and microconcentrations. The equations of the linear theory are also obtained. This work represents a first step to provide a consistent theory of thermoelastic duffusion materials with microtemperatures and microconcentrations. It is shown that there exist coupling between temperature, chemical potential, microtemperatures and microconcentrations even for isotropic bodies. (see references therein). Many researchers have studied the problems of thermoelasticity theory for isotropic elastic bodies with microstructures, by applying different methods such as an analytical, numerical and the complex variable technique to investigate the two-dimensional and three-dimensional boundary value problems of the theory of thermoelasticity (see for example [7]-[30] and reference therein).

In the present paper an explicit solution of the Neumann type BVP for an isotropic space with a spherical cavity with diffusion, microtemperatures and microconcentrations is presented. The obtained solution of the considered BVP is represented as absolutely and uniformly convergent series.

2. Basic equations and boundary value problem

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean 3D space E^3. Let us assume that the isotropic elastic ball $D^+ \in E^3$, with center at the origin, be bounded by the spherical surface S of radius R.

Let us assume that the domain D^+ is composed of isotropic thermoelastic materials with diffusion, microtemperatures and microconcentrations.

The basic homogeneous system of equilibrium equations for isotropic and thermoelastic body with diffusion, microtemperatures and microconcentrations may be written as [5],[6]:

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\ \text{graddiv}\ \mathbf{u} - \gamma_1 \text{grad}\ \theta - \gamma_2\ \text{grad}\ P = 0, \tag{1}$$

$$k^*\Delta\theta + k_1^* \text{div}\ \mathbf{T} = 0, \tag{2}$$

$$h^*\Delta P + h_1 \text{div}\ \mathbf{C} = 0, \tag{3}$$

$$k_6\Delta\mathbf{T} + (k_4+k_5)\ \text{grad div}\ \mathbf{T} - k_3\ \text{grad}\ \theta - k_2\mathbf{T} = 0, \tag{4}$$

$$h_6\Delta h_5)\ \text{graddiv}\ \mathbf{C} - h_3\ \text{grad}\ P - h_2\mathbf{C} = 0, \tag{5}$$

where $\mathbf{u} := (u_1, u_2, u_3)^\top$ denotes the displacement vector in a solid, $\lambda, \mu, k_j, h_j, k^* k_1^*$, represent material constants, $\mathbf{T} := (T_1, T_2, T_3)^\top$ and $\mathbf{C} = (C_1, C_2, C_3)^\top$. T_j and C_j are called microtemperatures and microconcentrations, respectively. P is the particle chemical potential, Δ is the 3D Laplace operator. Throughout this paper the superscript $^\top$ stands for the transpose operation.

We assume that the following conditions are fulfilled:

$$\mu > 0, \quad k^* > 0, \quad k_4 + k_5 > 0, \quad k_6 - k_5 > 0, \quad \frac{4kk_2}{T_0} - \left(\frac{k_1}{T_0} + k_3\right)^2 > 0,$$

$$h^* > 0, \quad k_6 > 0, \quad h_6 > 0, \quad k_2 > 0, \quad h_2 > 0, \quad 4hh_2 - (h_1 + h_3)^2 > 0,$$

$$3\lambda + 2\mu > 0, \quad h_4 + h_5 > 0, \quad 2k_4 + k_4 + k_6 > 0, \quad k_6 + k_5 > 0.$$

Definition. A vector-function $\mathbf{U} = (\mathbf{u}, \theta, P, \mathbf{T}, \mathbf{C})$ defined in the domain D^+ is called regular if

$$\mathbf{U} \in C^2(D^+) \cap C^1(\overline{D^+})$$

For the equations (1)-(5) we consider the following BVP.

Problem 1. Find a regular solution $\mathbf{U}(x)$ to the equations (1)-(5) in D^+, satisfying the following boundary conditions on S :

$$\mathbf{u}^+ = \mathbf{f}^+(\mathbf{z}), \quad \theta^+ = f_4^+(\mathbf{z}), \quad P^+ = f_5^+(\mathbf{z}),$$

$$\mathbf{T}^+ = \mathbf{F}^+(\mathbf{z}), \quad \mathbf{C}^+ = \mathbf{\Phi}^+(\mathbf{z}), \quad \mathbf{z} \in S,$$

where the vector-functions $\mathbf{f}(f_1, f_2, f_3)$, $\mathbf{F}(F_1, F_2, F_3)$, $\mathbf{\Phi}(\Phi_1, \Phi_2, \Phi_3)$ and the functions $\mathbf{f}$, f_j $(j = 4, 5)$, are given functions on S.

The following assertion holds .

Theorem 1. *The Problem 1 has one regular solution in* D^+.

Theorem 1 can be proven similar to the uniqueness theorem in [12](see Appendix in [12]).

3. Preliminaries-auxiliary results

The following theorem holds:

Theorem 2. *The regular solutions of equations (2),(4), admit in the domain* D^+ *a representation* (for details see in [8])

$$\begin{cases} \mathbf{T}(\mathbf{x}) = -\text{grad}\left(\dfrac{k_3}{k_2}\vartheta(\mathbf{x}) + \dfrac{k^*}{k_1^*}\vartheta_1(\mathbf{x})\right) + c\,\text{rot}\boldsymbol{\varphi}^{\mathbf{3}}(\mathbf{x}), \\ \theta(\mathbf{x}) = \vartheta(\mathbf{x}) + \vartheta_1(\mathbf{x}), \quad \mathbf{x} \in D^-, \quad \mathbf{x} \in D^-, \end{cases} \tag{6}$$

where

$$\begin{cases} \Delta\vartheta = 0, \qquad (\Delta - s_1^2)\vartheta_1 = 0, \quad (\Delta - s_2^2)\boldsymbol{\varphi}^{\mathbf{3}} = 0, \qquad \text{div}\boldsymbol{\varphi}^{\mathbf{3}} = 0, \\ s_1^2 = \dfrac{kk_2 - k_1k_3}{kk_7} > 0, \quad s_2^2 = \dfrac{k_2}{k_6} > 0, \quad c = -\dfrac{k_6}{k_2}, \quad \text{div}\mathbf{T} = -\dfrac{k^*}{k_1^*}s_1^2\vartheta_1, \\ \boldsymbol{\varphi}^{\mathbf{3}}(\mathbf{x}) = [\mathbf{x}\cdot\nabla]\varphi_3(\mathbf{x}) + \text{rot}[\mathbf{x}.\nabla]\varphi_4(\mathbf{x}), \quad (\Delta - s_2^2)\varphi_j = 0, \qquad j = 3, 4. \end{cases} \tag{7}$$

In addition, if

$$\int\limits_{S(0,a_1)} \varphi_j ds = 0, \quad j = 3, 4,$$

where $S(0, a_1) \subset D^+$ *is an arbitrary spherical surface with radius* a_1, *between the vector* $(\boldsymbol{T}, \theta)$ *and the functions* ϑ, ϑ_1, φ_j, $j = 3, 4$, *there exist one-to-one correspondence.*

Remark. The solutions of Eqs. (2) and (4) can be rewritten in the following form

$$\begin{cases} \mathbf{T}(\mathbf{x}) = -\text{grad}\left(\dfrac{k_3}{k_2}\vartheta(\mathbf{x}) + \dfrac{k^*}{k_1^*}\vartheta_1(\mathbf{x})\right) + [\mathbf{x}\cdot\nabla]\varphi_4(\mathbf{x}) + c\,\text{rot}[\mathbf{x}\cdot\nabla]\varphi_3(\mathbf{x}), \\ \theta(\mathbf{x}) = \vartheta(\mathbf{x}) + \vartheta_1(\mathbf{x}), \quad \mathbf{x} \in D^+. \end{cases} \tag{8}$$

Quite similarly, it is not difficult to prove the following assertion:

Theorem 3. *The regular solutions of equations (3),(5), admit in the domain D^+ a representation*

$$\begin{cases} \mathbf{C}(\mathbf{x}) = -\left(\frac{h_3}{h_2}\,\mathrm{grad}p(\mathbf{x}) + \frac{h^*}{h_1}\,\mathrm{grad}p_1(\mathbf{x})\right) + c_1\,\mathrm{rot}\boldsymbol{\varphi}^{\mathbf{4}}(\mathbf{x}) \\ = -\left(\frac{h_3}{h_2}\,\mathrm{grad}p(\mathbf{x}) + \frac{h^*}{h_1}\,\mathrm{grad}p_1(\mathbf{x})\right) + [\mathbf{x}\cdot\nabla]\varphi_6(\mathbf{x}) + c_1\,\mathrm{rot}[\mathbf{x}\cdot\nabla]\varphi_5(\mathbf{x}), \\ P(\mathbf{x}) = p(\mathbf{x}) + p_1(\mathbf{x}), \quad \mathbf{x}\in D^+, \end{cases} \tag{9}$$

where

$$\begin{cases} \Delta p = 0, \qquad (\Delta - \upsilon_1^2)p_1 = 0, \quad (\Delta - \upsilon_2^2)\boldsymbol{\varphi}^{\mathbf{4}} = 0, \qquad \mathrm{div}\boldsymbol{\varphi}^{\mathbf{4}} = 0, \\ \upsilon_1^2 = \frac{h^*h_2 - h_1h_3}{h^*h_7} > 0, \quad \upsilon_2^2 = \frac{h_2}{h_6} > 0, \quad c_1 = -\frac{h_6}{h_2}, \quad \mathrm{div}\mathbf{C} = -\frac{h^*}{h_1}\nu_1^2 p_1, \\ \boldsymbol{\varphi}^{\mathbf{4}}(\mathbf{x}) = [\mathbf{x}\cdot\nabla]\varphi_5(\mathbf{x}) + \mathrm{rot}[\mathbf{x}.\nabla]\varphi_6(\mathbf{x}), \quad (\Delta - \nu_2^2)\varphi_j = 0, \qquad j = 5, 6. \end{cases} \tag{10}$$

In addition, if

$$\int_{S(0,a_1)} \varphi_j ds = 0, \quad j = 5, 6,$$

where $S(0,a_1) \subset D^+$ is an arbitrary spherical surface with radius a_1, between the vector $(\boldsymbol{C}, P)$ and the functions p, p_1, φ_j, $j = 5, 6$, there exist one-to-one correspondence.

Theorem 4. *The regular solution $\boldsymbol{U} = (\boldsymbol{u}, \theta, P, \boldsymbol{T}, \boldsymbol{C})$ of equations (1)- (5) admits in the domain of regularity a representation*

$$\begin{cases} \mathbf{u} = \boldsymbol{\Psi} + \mathrm{grad}\left[-\frac{\lambda+\mu}{\mu}\psi_0 + \frac{\gamma_1}{\mu}\vartheta_0 + \frac{\gamma_2}{\mu}p_0 + \frac{\gamma_1}{\mu_0 s_1^2}\vartheta_1 + \frac{\gamma_2}{\mu_0\nu_1^2}p_1\right], \\ \mathbf{T}(\mathbf{x}) = -\mathrm{grad}\left(\frac{k_3}{k_2}\vartheta(\mathbf{x}) + \frac{k^*}{k_1^*}\vartheta_1(\mathbf{x})\right) + [\mathbf{x}\cdot\nabla]\varphi_4(\mathbf{x}) + c\,\mathrm{rot}[\mathbf{x}\cdot\nabla]\varphi_3(\mathbf{x}), \\ \mathbf{C}(\mathbf{x}) = -\mathrm{grad}\left(\frac{h_3}{h_2}p(\mathbf{x}) + \mathrm{grad}\frac{h^*}{h_1}p_1(\mathbf{x})\right) + [\mathbf{x}\cdot\nabla]\varphi_6(\mathbf{x}) + c_1\,\mathrm{rot}[\mathbf{x}\cdot\nabla]\varphi_5(\mathbf{x}), \\ \theta(\mathbf{x}) = \vartheta(\mathbf{x}) + \vartheta_1(\mathbf{x}), \quad P(\mathbf{x}) = p(\mathbf{x}) + p_1(\mathbf{x}). \end{cases} \tag{11}$$

where the functions ψ_0, ϑ_0 and p_0 are chosen such that

$$\Delta\psi_0 = \psi, \quad \Delta\vartheta_0 = \vartheta, \quad \Delta p_0 = p, \quad \Delta\psi = 0, \quad \mu_0 = \lambda + 2\mu, .$$

Herein it is assumed that, the functions $\boldsymbol{\Psi}$, $\mathrm{div}\boldsymbol{\Psi}$, ψ, ϑ, p, ϑ_1, p_1 and $\mathrm{div}\mathbf{u}$ are interrelated by the following relations

$$\Delta\boldsymbol{\Psi} = 0, \quad \mathrm{div}\boldsymbol{\Psi} = \frac{\mu_0}{\mu}\psi - \frac{\gamma_1}{\mu}\vartheta - \frac{\gamma_2}{\mu}p, \quad \mathrm{div}\mathbf{u} = \psi + \frac{\gamma_1}{\mu_0}\vartheta_1 + \frac{\gamma_2}{\mu_0}p_1.$$

It is obvious that the representation of a solution of $\mathbf{u}$ contains a harmonic, bi-harmonic, and a meta-harmonic functions, while the representations of θ, P, $\mathbf{T}$ and $\mathbf{C}$ contain only a harmonic and a meta-harmonic functions.

Let us introduce the spherical coordinates equalities:

$$x_1 = \rho\sin\xi\cos\eta, \quad x_2 = \rho\sin\xi\sin\eta, \quad x_3 = \rho\cos\xi, \quad x\in D^+,$$

$$y_1 = R\sin\xi_0\cos\eta_0, \quad y_2 = R\sin\xi_0\sin\eta_0, \quad y_3 = R\cos\xi_0, \quad y\in S,$$

$$|\mathbf{x}| = \rho = \sqrt{x_1^2+x_2^2+x_3^2}, \quad 0\le\xi\le\pi, \quad 0\le\eta\le 2\pi.$$

The scalar product and the vector product of the two vectors $\mathbf{g}$ and $\mathbf{q}$ are denoted by $(\mathbf{g}\cdot\mathbf{q}) = \sum\limits_{k=0}^{3} g_k q_k$ and $[\mathbf{g}\cdot\mathbf{q}]$, respectively. The operator $\dfrac{\partial}{\partial S_k(x)}$ is defined as follows:

$$[\mathbf{x}\cdot\nabla]_k = \frac{\partial}{\partial S_k(x)}, \qquad k=1,2,3, \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right).$$

The following identities are true

$$\lambda \mathrm{div}\mathbf{u} - \gamma_1\theta - \gamma_2 P = \lambda\psi - \gamma_1\vartheta - \gamma_2 p - \frac{2\mu}{\mu_0}(\gamma_1\vartheta_1 + \gamma_2 p_1),$$

$$\mu_0 \mathrm{div}\mathbf{u} - \gamma_1\theta - \gamma_2 P = \mu_0\psi - \gamma_1\vartheta - \gamma_2 p.$$

Below we use the following identities:[29]

$$\begin{cases} (\mathbf{x}\cdot\mathrm{grad}) = \rho\dfrac{\partial}{\partial\rho}, \quad \dfrac{\partial}{\partial S_k}\dfrac{\partial}{\partial x_k} = \dfrac{\partial}{\partial x_k}\dfrac{\partial}{\partial S_k}, \quad \sum\limits_{k=1}^{3}\dfrac{\partial}{\partial S_k}\dfrac{\partial}{\partial x_k} = 0, \\ (\mathbf{x}\cdot\mathrm{rot}\mathbf{g}) = \sum\limits_{k=1}^{3}\dfrac{\partial}{\partial S_k} g_k, \quad \sum\limits_{k=1}^{3}\dfrac{\partial}{\partial S_k}(\mathrm{rot}[\mathbf{x}\cdot\nabla]\mathbf{h})_k = 0. \end{cases} \tag{12}$$

If g_m is the spherical harmonic, then

$$\sum_{k=0}^{3}\frac{\partial^2 g_m(\mathbf{x})}{\partial S_k^2(\mathbf{x})} = -m(m+1)g_m(\mathbf{x}).$$

For convenience of writing down let us introduce the following functions:

$$\begin{cases} (\mathbf{x}\cdot\mathbf{f})^+ = q_1^+, \quad (\mathrm{div}\mathbf{u})^+ = q_2^+, \quad (\sum\limits_{k=1}^{3}\dfrac{\partial u_k}{\partial S_k(\mathbf{z})})^+ = q_3^+, \\ (\mathbf{x}\cdot\mathbf{F})^+ = q_4^+, \quad (\mathrm{div}\mathbf{T})_k^+ = q_5^+, \quad \sum\limits_{k=1}^{3}\left(\dfrac{\partial T_k}{\partial S_k(\mathbf{z})}\right)^+ = q_6^+, \\ (\mathbf{x}\cdot\mathbf{\Phi})^+ = q_7^+, \quad (\mathrm{div}\mathbf{C})_k^+ = q_8^+, \quad \sum\limits_{k=1}^{3}\left(\dfrac{\partial C_k}{\partial S_k(\mathbf{z})}\right)^+ = q_9^+. \end{cases} \tag{13}$$

Let us assume that the functions q_k, $\quad k=1,2..,9$, be representable in the form of the series:

$$q_k(\mathbf{y}) = \sum_{n=0}^{\infty} q_{kn}(\xi_0,\eta_0),$$

where $q_{kn} \quad k=1,2,..,11$ are the spherical harmonics of order n :

$$q_{kn} = \frac{2n+1}{4\pi R^2}\iint\limits_S P_n(\cos\gamma)q_k(\mathbf{y})dS_y,$$

P_n is a Legender polynomial of the n-th order, γ is an angle formed by the radius-vectors Ox and Oy,

$$\cos\gamma = \frac{1}{|\mathbf{x}||\mathbf{y}|}\sum_{k=1}^{3} x_k y_k.$$

4. Explicit solution of Problem 1

In this section are present a method of construction an explicit solution to the Problem 1 in details, which may be employed in the study of the other basic BVPs.

Taking into account the identity $(\mathbf{x}\cdot \mathrm{grad}) = \rho\dfrac{\partial}{\partial\rho}$, from (11) we get

$$(\mathbf{x}\cdot\mathbf{u}) = (\mathbf{x}\cdot\mathbf{\Psi})^{+} + R\frac{\partial X}{\partial\rho},$$

$$X = -\frac{\lambda+\mu}{\mu}\psi_0 + \frac{\gamma_1}{\mu}\vartheta_0 + \frac{\gamma_2}{\mu}p_0 + \frac{\gamma_1}{\mu_0 s_1^2}\vartheta_1 + \frac{\gamma_2}{\mu_0\nu_1^2}p_1.$$

For the function $(\mathbf{x}\cdot\mathbf{\Psi})$ we shall have

$$\Delta(\mathbf{x}\cdot\mathbf{\Psi}) = 2\mathrm{div}\mathbf{\Psi} = 2\left(\frac{\mu_0}{\mu}\psi - \frac{\gamma_1}{\mu}\vartheta - \frac{\gamma_2}{\mu}p\right),$$

the solution of which has the form

$$(\mathbf{x}\cdot\mathbf{\Psi}) = \Omega + 2\left(\frac{\mu_0}{\mu}\psi_0 - \frac{\gamma_1}{\mu}\vartheta_0 - \frac{\gamma_2}{\mu}p_0\right), \tag{14}$$

where Ω is an arbitrary harmonic function $\Delta\Omega = 0$. Thus we obtain

$$(\mathbf{x}\cdot\mathbf{u}) = \Omega + 2\left(\frac{\mu_0}{\mu}\psi_0 - \frac{\gamma_1}{\mu}\vartheta_0 - \frac{\gamma_2}{\mu}p_0\right)^{+} + R\frac{\partial X}{\partial\rho},$$

$$(\mathbf{x}\cdot\mathbf{T}) = -\rho\frac{\partial}{\partial\rho}\left(\frac{k_3}{k_2}\vartheta(\mathbf{x}) + \frac{k^*}{k_1^*}\vartheta_1(\mathbf{x})\right) + c\sum_{k=1}^{3}\frac{\partial^2\varphi_3}{\partial S_k^2(x)},$$

$$(\mathbf{x}\cdot\mathbf{C}) = -\rho\frac{\partial}{\partial\rho}\left(\frac{h_3}{h_2}p(\mathbf{x}) + \frac{h^*}{h_1}p_1(\mathbf{x})\right) + c_1\sum_{k=1}^{3}\frac{\partial^2\varphi_5}{\partial S_k^2(x)},$$

$$\mathrm{div}\mathbf{u} = \psi + \frac{\gamma_1}{\mu_0}\vartheta_1 + \frac{\gamma_2}{\mu_0}p_1, \quad \mathrm{div}\mathbf{T} = -\frac{k^*}{k_1^*}s_1^2\vartheta_1,$$

$$\sum_{k=1}^{3}\frac{\partial u_k}{\partial S_k(\mathbf{z})} = \sum_{k=1}^{3}\frac{\partial \Psi_k}{\partial S_k(\mathbf{z})}, \quad \sum_{k=1}^{3}\frac{\partial T_k}{\partial S_k(x)} = \sum_{k=1}^{3}\frac{\partial^2\varphi_4}{\partial S_k^2(x)}$$

$$\mathrm{div}\mathbf{C} = -\frac{h^*}{h_1}\upsilon_1^2 p_1, \quad \sum_{k=1}^{3}\frac{\partial C_k}{\partial S_k(x)} = \sum_{k=1}^{3}\frac{\partial^2\varphi_6}{\partial S_k^2(x)},$$

$$\theta = \vartheta_1 + \vartheta, \quad P = p + p_1.$$

Passing to the limit as $\rho \to R$, for determining the unknown values, we obtain the following systems of algebraic equations

$$\begin{cases} \Omega^+ + 2\left(\frac{\mu_0}{\mu}\psi_0 - \frac{\gamma_1}{\mu}\vartheta_0 - \frac{\gamma_2}{\mu}p_0\right)^+ + \left(R\frac{\partial X}{\partial \rho}\right)^+ = q_1^+, \\ -R\frac{\partial}{\partial \rho}\left(\frac{k_3}{k_2}\vartheta(\mathbf{x}) + \frac{k^*}{k_1^*}\vartheta_1(\mathbf{x})\right)^+ + c\left[\sum_{k=1}^{3}\frac{\partial^2\varphi_3}{\partial S_k^2(x)}\right]^+ = q_4^+, \\ -R\frac{\partial}{\partial \rho}\left(\frac{h_3}{h_2}p(\mathbf{x}) + \frac{h^*}{h_1}p_1(\mathbf{x})\right)^+ + c_1\left[\sum_{k=1}^{3}\frac{\partial^2\varphi_5}{\partial S_k^2(x)}\right]^+ = q_7^+, \end{cases} \tag{15}$$

$$\begin{cases} \psi + \frac{\gamma_1}{\mu_0}\vartheta_1 + \frac{\gamma_2}{\mu_0}p_1 = q_2^+, \quad -\frac{k^*}{k_1^*}s_1^2\vartheta_1 = q_5^+, \quad \sum_{k=1}^{3}\frac{\partial \Psi_k}{\partial S_k(\mathbf{z})} = q_3^+, \\ \sum_{k=1}^{3}\frac{\partial^2\varphi_4}{\partial S_k^2(x)} = q_6^+, \quad -\frac{h^*}{h_1}\nu_1^2 p_1 = q_8^+, \quad \sum_{k=1}^{3}\frac{\partial^2\varphi_6}{\partial S_k^2(x)} = q_9^+, \\ \vartheta_1 + \vartheta = f_4^+, \quad p + p_1 = f_5^+, \quad \rho = R. \end{cases} \tag{16}$$

Let the functions $\vartheta,\ p,\ \varphi_j(\mathbf{x}),\quad j = 3,4,5,6,\ \vartheta_1,\ p_1,\quad \sum_{k=1}^{3}\frac{\partial \Psi_k}{\partial S_k(\mathbf{z})}$ be sought in the form

$$\begin{cases} \vartheta = \sum_{m=0}^{\infty}\frac{\rho^m}{R^m}Y_m, \quad \vartheta_1 = \sum_{m=0}^{\infty}\phi_m(is_1\rho)Y_{1m}, \quad p = \sum_{m=0}^{\infty}\frac{\rho^m}{R^m}Z_m, \\ p_1 = \sum_{m=0}^{\infty}\phi_m(i\nu_1\rho)Z_{1m}, \quad \Omega = \sum_{m=0}^{\infty}\frac{\rho^m}{R^m}Y_{2m}, \quad \psi = \sum_{m=0}^{\infty}\frac{\rho^m}{R^m}Y_{3m}, \\ \varphi_j = \sum_{m=0}^{\infty}\phi_m(is_2\rho)Z_{jm}, \quad j = 3,4, \quad \varphi_j = \sum_{m=0}^{\infty}\phi_m(i\nu_2\rho)Z_{jm}, \quad j = 5,6, \\ \sum_{k=1}^{3}\frac{\partial \Psi_k}{\partial S_k(\mathbf{z})} = \sum_{m=0}^{\infty}\frac{\rho^m}{R^m}Y_{4m}, \end{cases} \tag{17}$$

where

$$\phi_m(is_k\rho) = \frac{\sqrt{R}J_{m+\frac{1}{2}}(is_k\rho)}{\sqrt{\rho}J_{m+\frac{1}{2}}(is_kR)}, \quad \phi_m(i\upsilon_k\rho) = \frac{\sqrt{R}J_{m+\frac{1}{2}}(i\upsilon_k\rho)}{\sqrt{\rho}J_{m+\frac{1}{2}}(i\upsilon_kR)}, \quad k = 1,2,$$

$Y_m(\theta,\varphi)$, $\quad Z_m(\theta,\varphi)$,... are the spherical harmonics of order m.

Taking into account (17), we can write the particular solutions of equations $\Delta\vartheta_0 = \vartheta$, $\Delta\psi_0 = \psi$ and $\Delta p_0 = p$ in the following form

$$\begin{cases} \psi_0 = \dfrac{\rho^2}{2}\displaystyle\sum_{m=0}^{\infty}\dfrac{\rho^m}{R^m(3+2m)}Y_{3m}, \\ \vartheta_0 = \dfrac{\rho^2}{2}\displaystyle\sum_{m=0}^{\infty}\dfrac{\rho^m}{R^m(3+2m)}Y_{m}, \\ p_0 = \dfrac{\rho^2}{2}\displaystyle\sum_{m=0}^{\infty}\dfrac{\rho^m}{R^m(3+2m)}Z_{m}. \end{cases} \tag{18}$$

Remark. The conditions $\int\limits_{S(0,a_1)} \varphi_j ds = 0 \quad j = 3,4,5,6$ in fact mean that

$$q_{90} = q_{60} = 0, \qquad G_{2n} = 0, \quad G_{3n} = 0, \qquad Z_{30} = Z_{50} = 0.$$

Using (17) in (15) and (16), we obtain the system of equations

$$\begin{cases} Y_{3n} + \dfrac{\gamma_1}{\mu_0}Y_{1n} + \dfrac{\gamma_2}{\mu_0}Z_{1n} = q^+_{2n}, \quad -\dfrac{k^*}{k^*_1}s_1^2 Y_{1n} = q^+_{5n}, \\ -\dfrac{h^*}{h_1}\upsilon_1^2 Z_{1n} = q^+_{8n}, \quad -n(n+1)Z_{4n} = q^+_{6n}, \quad Y_{4n} = q^+_{3n}, \\ -n(n+1)Z_{6n} = q^+_{9n}, \quad Y_n + Y_{1n} = f^+_{4n}, \quad Z_n + Z_{1n} = f^+_{5n}. \end{cases} \tag{19}$$

$$\begin{cases} \Omega^+ = -2\left(\dfrac{\mu_0}{\mu}\psi_0 - \dfrac{\gamma_1}{\mu}\vartheta_0 + \dfrac{\gamma_2}{\mu}p_0\right)^+ - R\dfrac{\partial X}{\partial\rho} - q_1^+ = G_1, \\ -c\left[\displaystyle\sum_{k=1}^{3}\dfrac{\partial^2\varphi_3}{\partial S_k^2(x)}\right]^+ = R\dfrac{\partial}{\partial\rho}\left(\dfrac{k_3}{k_2}\vartheta(\mathbf{x}) + \dfrac{k^*}{k^*_1}\vartheta_1(\mathbf{x})\right)^+ + q_4^+ = G_2, \\ -c_1\left[\displaystyle\sum_{k=1}^{3}\dfrac{\partial^2\varphi_5}{\partial S_k^2(x)}\right]^+ = R\dfrac{\partial}{\partial\rho}\left(\dfrac{h_3}{h_2}p(\mathbf{x}) + \dfrac{h^*}{h_1}p_1(\mathbf{x})\right)^+ + q_7^+ = G_3. \end{cases} \tag{20}$$

Solving these systems, we get

$$\begin{cases} Y_{1n} = -\dfrac{q^+_{5n}k^*_1}{k^* s_1^2}, \quad Z_{1n} = -\dfrac{q^+_{8n}h_1}{h^*\nu_1^2}, \quad Y_{4n} = q^+_{3n}. \\ Z_{4n} = -\dfrac{q^+_{6n}}{n(n+1)}, \quad Z_{6n} = -\dfrac{q^+_{9n}}{n(n+1)}, \quad Z_{3n} = \dfrac{G_{2n}}{cn(n+1)}, \\ Z_{5n} = \dfrac{G_{3n}}{c_1 n(n+1)}, \quad Y_n = f_{4n} + \dfrac{q^+_{5n}k^*_1}{k^* s_1^2}, \quad Z_n = f_{5n} + \dfrac{q^+_{8n}h_1}{h^*\nu_1^2}, \\ Y_{3n} = q^+_{2n} - \dfrac{\gamma_1}{\mu_0}\dfrac{q^+_{5n}k^*_1}{k^* s_1^2} - \dfrac{\gamma_2}{\mu_0}\dfrac{q^+_{8n}h_1}{h^*\nu_1^2}, \quad Y_{2n} = G_{1n}. \end{cases} \tag{21}$$

Substituting in (17) the values (19),(20) and (21), we obtain

$$\begin{cases} \vartheta = \sum\limits_{n=0}^{\infty} \dfrac{\rho^n}{R^n}\left(f_{4n} + \dfrac{q_{5n}^{+} k_1^*}{k^* s_1^2}\right), \quad \vartheta_1 = -\dfrac{k_1^*}{k^* s_1^2}\sum\limits_{n=0}^{\infty} \phi_n(is_1\rho) q_{5n}^{+}, \\ p = \sum\limits_{n=0}^{\infty} \dfrac{\rho^n}{R^n}\left(f_{5n} + \dfrac{q_{8m}^{+} h_1}{h^* \nu_1^2}\right), \quad p_1 = -\dfrac{h_1}{h^* \nu_1^2}\sum\limits_{n=0}^{\infty} \phi_n(i\nu_1\rho) q_{8n}^{+}, \\ \Omega = \sum\limits_{n=0}^{\infty} \dfrac{\rho^n}{R^n} G_{1n}, \quad \psi = \sum\limits_{n=0}^{\infty} \dfrac{\rho^n}{R^n}\left(q_{2n}^{+} - \dfrac{\gamma_1}{\mu_0}\dfrac{q_{5n}^{+} k_1^*}{k^* s_1^2} - \dfrac{\gamma_2}{\mu_0}\dfrac{q_{8n}^{+} h_1}{h^* \nu_1^2}\right), \\ \varphi_3 = \sum\limits_{n=1}^{\infty} \phi_n(is_2\rho)\dfrac{G_{2n}}{cn(n+1)}, \quad \varphi_4 = -\sum\limits_{n=1}^{\infty} \phi_n(is_2\rho)\dfrac{q_{6n}^{+}}{n(n+1)}, \\ \varphi_5 = \sum\limits_{n=1}^{\infty} \phi_n(i\nu_2\rho)\dfrac{G_{3n}}{c_1 n(n+1)}, \quad \varphi_6 = -\sum\limits_{n=1}^{\infty} \phi_m(i\nu_2\rho)\dfrac{q_{9n}^{+}}{n(n+1)}, \\ \sum\limits_{k=1}^{3} \dfrac{\partial \Psi_k}{\partial S_k(\mathbf{z})} = \sum\limits_{n=0}^{\infty} \dfrac{\rho^n}{R^n} q_{3n}^{+}, \end{cases} \tag{22}$$

where G_{jn}, f_{4n}, f_{5n} are the spherical harmonics of order n.

Remark. The relations (22) may be written in the form

$$\begin{cases} \vartheta = \dfrac{R}{4\pi}\int\limits_0^{\pi}\int\limits_0^{2\pi} \dfrac{R^2-\rho^2}{r^3}\left[f_4^{+}(\eta_0,\xi_0) + \dfrac{k_1^*}{k^* s_1^2} q_5^{+}(\eta_0,\xi_0)\right]\sin\eta_0 d\eta_0 d\xi_0, \\ p = \dfrac{R}{4\pi}\int\limits_0^{\pi}\int\limits_0^{2\pi} \dfrac{R^2-\rho^2}{r^3}\left[f_5^{+}(\eta_0,\xi_0) + \dfrac{h_1}{h^* \nu_1^2} q_8^{+}(\eta_0,\xi_0)\right]\sin\eta_0 d\eta_0 d\xi_0, \\ \Omega = \dfrac{R}{4\pi}\int\limits_0^{\pi}\int\limits_0^{2\pi} \dfrac{R^2-\rho^2}{r^3} G_1(\eta_0,\xi_0)\sin\eta_0 d\eta_0 d\xi_0, \\ \psi = \dfrac{R}{4\pi}\int\limits_0^{\pi}\int\limits_0^{2\pi} \dfrac{R^2-\rho^2}{r^3}\left[q_2^{+} - \dfrac{\gamma-1k_1^*}{\mu_0 k^* s_1^2} q_5^{+} - \dfrac{\gamma_2 h_1}{\mu_0 h^* \nu_1^2} q_8^{+}\right]\sin\eta_0 d\eta_0 d\xi_0, \\ \sum\limits_{k=1}^{3} \dfrac{\partial \Psi_k}{\partial S_k(\mathbf{z})} = \dfrac{R}{4\pi}\int\limits_0^{\pi}\int\limits_0^{2\pi} \dfrac{R^2-\rho^2}{r^3} q_3^{+}(\eta_0,\xi_0)\sin\eta_0 d\eta_0 d\xi_0, \end{cases} \tag{23}$$

$$
\begin{cases}
\vartheta_1 = -\dfrac{k_1^*}{k^* s_1^2}\sum\limits_{n=0}^{\infty}\phi_n(is_1\rho)\overset{+}{q}_{5n}, \quad p_1 = -\dfrac{h_1}{h^*\nu_1^2}\sum\limits_{n=0}^{\infty}\phi_m(i\nu_1\rho)\overset{+}{q}_{8n}, \\[2ex]
\varphi_3 = -\sum\limits_{n=1}^{\infty}\phi_n(is_2\rho)\dfrac{G_{2n}}{cn(n+1)}, \quad \varphi_5 = \sum\limits_{n=1}^{\infty}\phi_n(i\nu_2\rho)\dfrac{G_{3n}}{c_1 n(n+1)}, \\[2ex]
\varphi_4 = -\sum\limits_{n=1}^{\infty}\phi_m(is_2\rho)\dfrac{\overset{+}{q}_{6n}}{n(n+1)}, \quad \varphi_6 = \sum\limits_{n=1}^{\infty}\phi_m(i\nu_2\rho)\dfrac{\overset{+}{q}_{9n}}{n(n+1)},
\end{cases} \tag{24}
$$

$$
\begin{cases}
\psi_0 = \dfrac{\rho^2}{2}\sum\limits_{n=0}^{\infty}\dfrac{\rho^n}{R^n(3+2n)}\left(\overset{+}{q}_{2n} - \dfrac{\gamma_1}{\mu_0}\dfrac{\overset{+}{q}_{5n}k_1^*}{k^* s_1^2} - \dfrac{\gamma_2}{\mu_0}\dfrac{\overset{+}{q}_{8n}h_1}{h^*\nu_1^2}\right), \\[2ex]
\vartheta_0 = \dfrac{\rho^2}{2}\sum\limits_{n=0}^{\infty}\dfrac{\rho^n}{R^n(3+2n)}\left(f_{4n} + \dfrac{\overset{+}{q}_{5n}k_1^*}{k^* s_1^2}\right), \\[2ex]
p_0 = \dfrac{\rho^2}{2}\sum\limits_{n=0}^{\infty}\dfrac{\rho^n}{R^n(3+2n)}\left(f_{5n} + \dfrac{\overset{+}{q}_{8n}h_1}{h^*\nu_1^2}\right).
\end{cases} \tag{25}
$$

$$
r^2 = \rho^2 + R^2 - 2R\rho\cos\gamma, \quad \cos\gamma = \sin\xi\sin\xi_0\cos(\eta-\eta_0) + \cos\xi\cos\xi_0.
$$

Thus, the considered problem has been solved completely.

5. Conclusions

In this paper the linear theory of thermoelasticity for materials with diffusion, microtemperatures and microconcentrations is considered. The following results are obtained:

1. The Dirichlet type boundary value problems of thermoelastic diffusion theory with microtemperatures and microconcentrations for a sphere is solved explicitly.

2. The explicit solution of the considered BVP is presented as absolutely and uniformly convergent series.

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