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ON THE INVESTIGATION OF AN ANALYTICAL SOLUTION OF A CERTAIN DIRICHLET GENERALIZED HARMONIC PROBLEM

Zakradze M., Kublashvili M., Tabagari Z.

Abstract. The present paper is devoted to the analysis of an explicit analytic solution of the Dirichlet generalized harmonic problem for a finite right circular axisymmetric cylindrical ring. We intend to use it for testing. For construction of the mentioned solution, the following methods are applied: separation of variables, particular solutions and heuristic method. Since the heuristic method does not guarantee finding the best solution, because of this, properties of the noted solution were investigated. It is shown that the above-mentioned problem can be used in the role of a test with the help of the given analytic solution.

Keywords and phrases: Dirichlet generalized harmonic problem, cylindrical ring, analytical solution.

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1. Introduction

It is known (see e. g., [1-4]) that in practical stationary problems (connected with electric, thermal and other static fields) there are cases when it is necessary to consider the Dirichlet generalized harmonic problem.

Problem A. Function $g(y)$ is given on the boundary S of the finite domain D and is continuous everywhere, except a finite number of curves $l_1, l_2, ..., l_n$, which represent discontinuity curves of the first kind for the function $g(y)$. It is required to find a function $u(x) \equiv u(x_1, x_2, x_3) \in C^2(D) \bigcap C(\overline{D} \setminus \bigcup\limits_{k=1}^{n} l_k)$ satisfying the conditions:

$$\Delta u(x) = 0, \qquad x \in D \subset R^3, \tag{1.1}$$

$$u(y) = g(y), \qquad y \in S, \quad y \overline{\in} l_k \subset S \ \ (k = \overline{1, n}), \tag{1.2}$$

$$|u(y)| < c, \qquad\qquad y \in \overline{D}, \tag{1.3}$$

where $\Delta = \sum\limits_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, c is a real constant, and S is a closed piecewise smooth surface .

It is known (see [5, 6]) that Problem (1.1),(1.2),(1.3) has a unique solution, depending continuously on the data, and for a generalized solution $u(x)$ the generalized extremum principal is valid:

$$\min_{x \in S} u(x) < \underset{x \in D}{u(x)} < \max_{x \in S} u(x),$$

where for $x \in S$ it is assumed that $x \overline{\in} l_k (k = \overline{1, n})$.

On the basis of (1.3), in general, the values of $u(y)$ are not defined on the curves l_k. For example, if Problem A concerns the determination of the thermal(or the electric) field,then $u(y) = 0$ when $y \in l_k$, respectively. In this case, in the physical sense the curves l_k are non-conductors (or dielectrics).

Remark 1. If there is an emptiness inside the surface S then we have the generalized problem with respect to the closed shell.

In general it is known (see e.g.,[1, 5, 7]) that the methods used to obtain an approximate solution to ordinary boundary problems are less suitable(or not suitable at all) for solving generalized boundary problems of type A. In particular, the convergence of the approximate process is very slow in the neighborhood of boundary singularities and, consequently, the accuracy of the approximate solution of the generalized problem is very low.

In the literature, simplified, or so called "solvable" generalized problems (problems "whose" solutions can be constructed by series, whose terms are represented by special functions) are considered and some methods, namely, separation of variables, particular solutions and heuristic method are mainly applied for their solving, therefore the accuracy of the solutions is rather low. In the mentioned problems, the boundary conditions are mainly constants, and in the general case, the analytic form of the "exact" solution is so difficult in the sense of numerical implementation, that it only has theoretical significance. Therefore, construction of high accuracy and effectively realizable computational schemes for approximate solution of 3D Dirichlet generalized harmonic problems(whose application is possible to a wide class of domains) have both theoretical and practical importance.

2. The investigation of an analytical solution of a certain Dirichlet generalized harmonic problem

Let the domain D be a right circular axisymmetric cylindrical ring $D(a < r < b, o < x_3 < h)$, where h is its height, $r = \sqrt{(x_1{}^2 + x_2{}^2)}$, and a, b are the internal and external radii of the ring, respectively.

In ([4], p 82, p.415) for ring D a simplified case of Problem A is considered, in particular, when the boundary function $g(y) = g(y_1, y_2, y_3) = v$, for $y \in \{y \in S | r = b, \quad 0 < y_3 < h\}$ and $g(y) = 0$ on the remaining part of S. In the considered case the external circles of the bases of the ring are discontinuity curves. In the mentioned conditions the exact analytical solution to Problem A has the following form (in cylindrical coordinates)

$$u(r, x_3) = \frac{4v}{\pi} \sum_{m=0}^{\infty} \frac{I_0(c_m r)K_0(c_m a) - I_0(c_m a)K_0(c_m r)}{I_0(c_m b)K_0(c_m a) - I_0(c_m a)K_0(c_m b)}$$

$$\times \frac{\sin(c_m x_3)}{2m+1} \equiv \frac{4v}{\pi} \sum_{m=0}^{\infty} u_m(r, x_3), \tag{2.1}$$

where $a < r < b, \; 0 < x_3 < h, \; c_m = (2m+1)\pi/h, I_0$ and K_0 are first and second kind Bessel's functions of order zero with imaginary argument, respectively. For construction of (2.1) methods noted in Section 1 are applied.

In (2.1)(see, e.g., [8]):

$$I_0(t) \equiv J_0(it) = \sum_{k=0}^{\infty} \frac{\left(\frac{t}{2}\right)^{2k}}{(k!)^2}, \quad t \in R,$$

$$I_0(0) = 1, \quad I_0(t) \to \frac{e^t}{\sqrt{2\pi t}} \; for \; t \to \infty;$$

$$K_0(t) \equiv K_0(it) = -\Big(ln\frac{t}{2} + C\Big)I_0(t) + \sum_{k=0}^{\infty} \Phi(k)\frac{(\frac{t}{2})^{2k}}{(k!)^2}, \; t > 0,$$

where

$$\Phi(k) = \sum_{j=1}^{k} \frac{1}{j}, \;\; k \geq 1, \;\; \Phi(0) = 0,$$

$$K_0(t) \to \sqrt{\frac{\pi}{2t}} e^{-t} \;\; for\ t \to \infty, \; and\ C = 0.577215664901532$$

is the Euler-Mascheroni constant.

Besides in [8], it is known that $I_0(t)$ and $K_0(t)$ are linearly independent solutions of the following ordinary differential equation

$$y'' + \frac{1}{t} y\prime - y = 0, where\ y = y(t), \;\; t \in (0, \infty). \qquad (2.2).$$

I. Firstly we demonstrate that the general term $u_m(r, x_3)$ of (2.1) is harmonic in D. For this, as is obvious from (2.1), it is sufficient to show the harmonicity of functions

$$f_1 = I_0(c_m r) \sin(c_m x_3), \;\; f_2 = K_0(c_m r) \sin(c_m x_3).$$

On account of identity of these two processes we investigate only the function f_1.

Since, the boundary conditions are independent of the cylindrical coordinate φ,

$$\Delta f_1 = \frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r} + \frac{\partial^2 f_1}{\partial x_3^2}.$$

It is easy to see, that

$$\Delta f_1 = c_m^2 \sin(c_m x_3) \big[I_0''(c_m r) + \frac{1}{c_m r} I_0'(c_m r) - I_0(c_m r) \big],$$

then on the basis of (2.2) we have $\Delta f_1 = \Delta f_2 = 0$ or $\Delta u_m = 0$ in D (Q.E.D.).

II. Now, we investigate the question about character of convergence of the series (2.1). For a fixed value of r from interval (a, b) majorizing series of (2.1) when $0 < x_3 < h$ is a positive numerical series

$$\sum_{m=0}^{\infty} a_m(r), \qquad (2.3)$$

where

$$a_m(r) = \frac{I_0(c_m r) K_0(c_m a) - I_0(c_m a) K_0(c_m r)}{I_0(c_m b) K_0(c_m a) - I_0(c_m a) K_0(c_m b)}.$$

On the basis of the asymptotical formulas of functions $I_0(t)$ and $K_0(t)$, it is not difficult to show that $a_m(r) > 0$ $(m = 0, 1, 2, ...)$ for $a < r < b$. In order to determine convergence of series (2.3), in this case, it is more convenient to use D'Alembert's limit test. Namely, if

$$\lim_{m \to \infty} \frac{a_{m+1}(r)}{a_m(r)} = q \;\; (0 < q < 1),$$

then the series (2.3) is convergent.

It is easy to see that in our case

$$q = exp\Big(\frac{2(r - b)\pi}{h}\Big) \;\; (a < r < b)\ or\ 0 < q < 1,$$

respectively, on the basis of Weierstrass test the series (2.1) is uniformly convergent in D (Q.E.D.).

Concerning the convergence rate of series (2.1). It is very high for all points $x = (x_1, x_2, x_3) \in D$. In particular, the asymptotical behaviour of the general term of (2.1) is

$$u_m(r, x_3) \to \frac{1}{2m+1} \; exp(c_m(r-b)) \; for \; m \to \infty \; and \; a < r < b.$$

Besides, it is easy to see that $\sin(c_m(h/2+t)) = \sin(c_m(h/2-t))$ for $0 \le t \le h/2$, therefore, $u(r, h/2+t) = u(r, h/2-t)$ for $a \le r \le b$, and this fact is in exact accordance with the real physical picture.

For illustration we calculated the partial sum $S_p(r, x_3)$ of the series (2.1) for $m = \overline{0, p}$ at several interesting points. In numerical experiments $a = 1, b = 2, h = 2, v = 1$ are taken. Because of convergence rate of (2.1) when $(r, x_3) \in D$, the calculations have shown that for $p = 20, 50$ practically: $S_p(1.2, 1) = 0.221517$; $S_p(1.5, 1) = 0.519826$; $S_p(1.8, 1) = 0.846086$; $S_p(1.5, 1.5) = 0.424747$;$S_p(1.5, 0.5) = 0.425002$; $S_p(1.5, 1.8) = 0.217324$;$S_p(1.5, 0.2) = 0.217581$; $S_p(1.8, 1.5) = 0.750919$; $S_p(1, 8, 1.8) = 0.516492$;

Since boundary conditions are symmetric with respect to the plane $x_3 = 1$, therefore, for control the partial sum $S_p(r, x_3)$ is calculated also at the points which are symmetric to the same plane, the results are in a good accordance with the real physical picture of the field.

III. It is easy to see that for the solution $u(r, x_3)$, when $r = b$ and $0 < y_3 < h$,we have

$$u(b, y_3) = \frac{4v}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin(c_m y_3), \tag{2.4}$$

and it is zero on the remaining part of the surface. In order that (2.4) is equal to v, the equality

$$\sum_{m=0}^{\infty} \frac{1}{2m+1} \sin(c_m y_3) = \frac{\pi}{4}. \tag{2.5}$$

should be befulfilled when $0 < y_3 < h$.

It is not difficult to show that (2.5) is valid. Indeed, it is known (see e.g., [9]) that if the function $f(t)$ is the integrable in the interval [0,h], then its Fourier-series expansion only with respect to sines has the following form

$$f(t) = \sum_{m=1}^{\infty} b_m \sin\Big(\frac{m\pi t}{h}\Big), \tag{2.6}$$

where $b_m = \frac{2}{h} \int\limits_0^h f(t) \sin \frac{m\pi t}{h} dt \quad (m = 1, 2, 3, ...)$.

It should be noted that at points $t = 0$ and $t = h$ the sum of the series (2.6) is zero. Therefore, this can give us values $f(0)$ and $f(h)$, evidently, only in the case when these values are equal to zero.

In particular, if $f(t) = \pi/4$, then $b_m = 1/(2m+1)$ $(m = 0, 1, 2, ...)$. Thus, on the basis of (2.6) equality (2.5) is valid.

We demonstrate that series (2.4) is uniformly convergent in the interval $(0, h)$. For this, in the first place, we represent (2.4) in the following form

$$\frac{4v}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin \frac{(2m+1)\pi y_3}{h} = \sum_{m=0}^{\infty} a_m b_m(t) \tag{2.7}$$

where $a_m = 4v/(\pi(2m+1))$, $b_m(t) = sin((2m+1)t)$, and $t = \pi y_3/h$ $(0 < y_3 < h)$.

We show that the series (2.7) satisfies Dirichlet test for the uniform convergence: 1). $\{a_m\}$ is a positive numbers sequence, which monotonically tends to zero; 2). Let $B_n(t) = b_0(t) + b_1(t) + ... + b_n(t)$ $(n = 0, 1, ...)$ be partial sums of a series $\sum_{m=0}^{\infty} b_m(t)$, they must be bounded.

It is obvious that $\lim_{m\to\infty} a_m = 0$. Now we show that $|B_n(t)| \leq M$ for any $0 < t < \pi$ and n. We have

$$B_n(t) = \sum_{m=0}^{n} \sin[(2m+1)t] = \frac{1}{2\sin t} \sum_{m=0}^{n} 2\sin[(2m+1)t]\sin t$$

$$= \frac{1}{2\sin t} \sum_{m=0}^{n} [\cos((2m+1)t - t) - \cos((2m+1)t + t)] = \frac{\cos 0 - \cos[(2n+1)+1]t}{2\sin t}.$$

It is evident that $|B_n(t)| \leq 1/\sin t| \leq M$, since $t = \pi y_3/h \neq k\pi$ $(k = 0, 1, 2, ...)$. Thus, series (2.4) is uniformly convergent, when $0 < y_3 < h$ (Q.E.D.).

It is clear that if a point $y(y_1, y_2, y_3) \equiv (b, y_3)$ and tends to the discontinuity curve $l_k (k = 1, 2)$, then all the terms of series (2.4) tend to zero. Consequently, series (2.4) converges very slowly, therefore, the accuracy of the satisfaction of the boundary condition is very low.

In Table 1 the values of partial sum $S_p(b, x_3)$ of (2.4) are given at several points (b, y_3) for $p = 100, 500, 1000$ and the same parameters a, b, h, v.

Table 1: Results for partial sum of (2.4)

i	$(b, y_3$	S_{100}	S_{500}	S_{1000}
1	$(2, 1)$	1.00315	1.00064	1.00032
2	$(2, 1.5)$	0.99842	0.99968	0.99984
3	$(2, 1.8)$	0.99166	0.99833	0.99917
4	$(2, 1, 9)$	0.98068	0.99613	0.99807
5	$(2, 1.95)$	0.90301	0.90798	0.98989
6	$(2, 1.995$	0.87899	0.89175	0.97978
7	$(2, 1.999)$	0.20087	0.87393	0.92922
8	$(2, 1.9999)$	0.01986	0.019910	0.02002

From Table 1 it is clear that the accuracy of the satisfaction of the boundary condition is very low in the neighborhood of the discontinuity carves, as expected (see Section 1).

Performed calculations showed that the analytic solution (2.1) has the accuracy, which is enough for a wide group of practical problems. In addition, the results of calculations for inner control points are in a good accordance with the real physical picture of the field. Finally, we note from our viewpoint that the problem considered can be used in the role of a test with the help of the above-mentioned analytic solution.

Remark 2. If we consider such simple case, when boundary function $g(y) = const$ on the lower base of the noted cylindrical ring, and $g(y) = 0$ on the rest surface, then the analytic form of the solution of the Problem A is so difficult in the sense of numerical implementation, that it has only theoretical significance (see [4], p.82, 416-417pp).

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Authors' addresses:

M. Zakradze
N. Muskhelishvili Institute of Computational Mathematics of
Georgian Technical University
4, Grigol Peradze St., Tbilisi 0131
Georgia
E-mail: mamuliz@yahoo.com

M. Kublashvili
N. Muskhelishvili Institute of Computational Mathematics of Georgian Technical University
4, Grigol Peradze St., Tbilisi 0131
Georgia
E-mail: mkublashvili@mail.ru

Z. Tabagari
N. Muskhelishvili Institute of Computational Mathematics of Georgian Technical University
4, Grigol Peradze St., Tbilisi 0131
Georgia
E-mail: z.tabagari@hotmail.com