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ON THE SOLUTION OF THE BITSADZE-SAMARSKII PROBLEM FOR THE TWO-DIMESIONAL EQUATION OF STATICS OF THE THEORY OF ELASTIC MIXTURE BY VARIATIONAL METHOD

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Abstract. In the paper the Bitsadze-Samarskii nonlocal problem for the equation of statics of the linear theory of elastic mixture in a rectangle is solved by the variation method. The uniqueness theorem is proved and the necessary and sufficient condition, indicating when the vector-function minimizing the specially constructing functional is a solution of the considered problem is given.

Keywords and phrases: Elastic mixture, variation method, Bitsadze-Samarskii nonlocal problem, functional, minimizing vector-function.

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1. Introduction

The basic two-dimensional boundary value problems of statics of the linear theory of elastic mixture are studied in [1], [2], [6] and also by many other authors. Two-dimensional boundary value problems of statics is investigated by potential method and the theory of singular integral equations in [1]. Using potentials with complex densities the solutions of basic plane boundary value problems of statics are reduced to the solution of Fredholm's linear integral equations of second kind in [2]. By variation method the first boundary value problem of statics of the linear theory of elastic mixture, in the case of a finite simply connected plane domain is solved in [6].

In the paper, for the homogeneous equation of statics of the linear theory of elastics mixture in a rectangle the Bitsadze-Samarskii nonlocal problem is solved by variation method. To solve the problem we use the method described in [3], [4] and [5].

2. Some auxiliary formulas and operators

The homogeneous equation of statics of the linear theory of elastic mixture for the two-dimensional case can be written in the matrixs form as [1]

$$A(\partial x)U(x) = 0, \quad x = (x_1, x_2), \tag{2.1}$$

where

$$A(\partial x) = \begin{pmatrix} A^{(1)}(\partial x) & A^{(2)}(\partial x) \\ A^{(2)}(\partial x) & A^{(3)}(\partial x) \end{pmatrix}, \qquad A^{(p)}(\partial x) = [A^{(\rho)}_{Kj}(\partial x)]_{2\times 2}, \quad p = 1, 2, 3,$$

$$A^{2q-1}(\partial x) = a_q \delta_{kj}\Delta + b_q \frac{\partial^2}{\partial x_k \partial x_j}, \quad q = 1, 2, \quad k, j = 1, 2,$$

$$A^{2}(\partial x) = c\delta_{kj}\Delta + d\frac{\partial^2}{\partial x_k \partial x_j}; \quad k, j = 1, 2,$$

δ_{kj} is Kroneker's symbol and Δ is the Laplace operator $U=(u',u'')^T$, $u'=(u_1,u_2)^T$ and $u''=(u_3,u_4)^T$ are partial displacements, $x=(x_1,x_2)$

$$a_1=\mu_1-\lambda_5,\quad a_2=\mu_2-\lambda_5,\quad c=\mu_3+\lambda_5,\quad b_1=\mu_1+\lambda_1+\lambda_5-\alpha_2\rho_2/\rho,$$

$$b_2=\mu_2+\lambda_2+\lambda_5-\alpha_2\rho_1/\rho,\quad \alpha_2=\lambda_3-\lambda_4,\quad \rho=\rho_1+\rho_2,$$

$$d=\mu_3+\lambda_3-\lambda_5-\alpha_2\frac{\rho_1}{\rho}\equiv\mu_3+\lambda_4-\lambda_5+\alpha_2\frac{\rho_2}{\rho}.$$

Here $\mu_1,\mu_2,\mu_3,\lambda_p,\quad p=\overline{1,5}$ are elastic constants ρ_1 and ρ_2 are partial densities (Positive constants).

In the sequel it is assumed that [1]

$$\mu_1>0,\quad \lambda_5<0,\quad \mu_1\mu_2-\mu_3^2>0,\quad b_1-\lambda_5>0,$$
$$(b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_5)^2>0. \tag{2.2}$$

In the plane $0x_1x_2$, let us consider the rectangle $D=\{-l<x_1<l,\quad 0<x_2<1\}$, where $l>0$ is a given constant. By Γ we denote the boundary of the rectangle D.

A vector-function $U=(u',u'')^T=(u_1,u_2,u_3,u_4)^T$ is said to be regular in D if $U\in C^2(D)\bigcap C^1(D\bigcup\Gamma)$.

We note that, for a regular $U=(u_1,u_2,u_3,u_4)^T$ and $V=(v_1,v_2,v_3,v_4)^T$ vector-functions we have the Green formula [1]

$$\int_D[V(x)A(\partial x)U(x)+N(v,u)]dx=\int_{-l}^{l}V(x_1,1)NU(x_1,0)dx_1+$$
$$+\int_0^1 V(l,x_2)NU(l,x_2)dx_2-\int_{-l}^{l}V(x_1,1)NU(x_1,1)dx_1$$
$$-\int_0^1 V(-l,x_2)NU(-l,x_2)dx_2, \tag{2.3}$$

where

$$N=N(\partial x,n(x))=M_1\frac{\partial}{\partial n(x)}+M_2^0\frac{\partial}{\partial S(x)}+M_3\left(\frac{\partial}{\partial x},n(x)\right) \tag{2.4}$$

is the pseudostres operator;

$$M_1=\begin{bmatrix} a & 0 & c_0 & 0\\ 0 & a & 0 & c_0\\ c_0 & 0 & b & 0\\ 0 & c_0 & 0 & b\end{bmatrix},\qquad M_2^0=\begin{bmatrix} 0 & a-\frac{m_3}{\Delta_0} & 0 & c_0+\frac{m_2}{\Delta_0}\\ \frac{m_3}{\Delta_0}-a & 0 & -c_0-\frac{m_2}{\Delta_0} & 0\\ 0 & c_0+\frac{m_2}{\Delta_0} & 0 & b-\frac{m_1}{\Delta_0}\\ -c_0-\frac{m_2}{\Delta_0} & 0 & \frac{m_1}{\Delta_0}-b & 0\end{bmatrix},$$

$$M_3(\partial x, n(x)) = \begin{bmatrix} -b_1 n_2 \frac{\partial}{\partial x_2} & b_1 n_2 \frac{\partial}{\partial x_1} & -dn_2 \frac{\partial}{\partial x_2} & dn_2 \frac{\partial}{\partial x_1} \\ b_1 n_1 \frac{\partial}{\partial x_2} & -b_1 n_1 \frac{\partial}{\partial x_1} & dn_1 \frac{\partial}{\partial x_2} & -dn_1 \frac{\partial}{\partial x_1} \\ -dn_2 \frac{\partial}{\partial x_2} & dn_2 \frac{\partial}{\partial x_1} & -b_2 n_2 \frac{\partial}{\partial x_2} & b_2 n_2 \frac{\partial}{\partial x_1} \\ dn_1 \frac{\partial}{\partial x_2} & -dn_1 \frac{\partial}{\partial x_1} & b_2 n_1 \frac{\partial}{\partial x_2} & -_2 n_1 \frac{\partial}{\partial x_1} \end{bmatrix},$$

$$\frac{\partial}{\partial n(x)} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2}, \qquad \frac{\partial}{\partial s(x)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1},$$

$n = (n_1, n_2)^T$ is a unit vector.
Here

$$\Delta_0 = m_1 m_3 - m_2^2, \quad m_1 = l_1 + \frac{1}{2} l_4; \quad m_2 = l_2 + \frac{1}{2} l_5, \quad m_3 = l_3 + \frac{1}{2} l_6,$$

$$l_1 = \frac{a_2}{d_2}, \quad l_2 = -\frac{c}{d_2}, \quad l_3 = \frac{a_1}{d_2}, \quad d_2 = a_1 a_2 - c^2,$$

$$l_1 + l_4 = \frac{b}{d_1}, \quad l_2 + l_5 = -\frac{c_0}{d_1}, \quad l_3 + l_6 = \frac{a}{d_1},$$

$$a = a + b_1, \quad b = a_2 + b_2, \quad c_0 = c + d, \quad d_1 = ab - c_0^2.$$

$$N(u,v) = N(v,u) = \frac{1}{2}\left(2a - \frac{m_3}{\Delta_0}\right)\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right)$$

$$+\frac{1}{2}\left(2c_0 + \frac{m_2}{\Delta_0}\right)\left[\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_4}{\partial x_2}\right) + \left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_4}{\partial x_2}\right)\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right)\right]$$

$$+\frac{1}{2}\left(2b - \frac{m_1}{\Delta_0}\right)\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_4}{\partial x_2}\right)\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_4}{\partial x_2}\right)$$

$$+\frac{m_3}{2\Delta_0}\left[\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) + \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right)\right]$$

$$-\frac{m_2}{\Delta_0}\left[\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial v_3}{\partial x_1} - \frac{\partial v_4}{\partial x_2}\right) + \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)\left(\frac{\partial v_4}{\partial x_1} + \frac{\partial v_3}{\partial x_2}\right)\right]$$

$$+\frac{m_1}{2\Delta_0}\left[\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_4}{\partial x_2}\right)\left(\frac{\partial v_3}{\partial x_1} - \frac{\partial v_4}{\partial x_2}\right) + \left(\frac{\partial u_4}{\partial x_1} + \frac{\partial u_3}{\partial x_2}\right)\left(\frac{\partial v_4}{\partial x_1} + \frac{\partial v_3}{\partial x_2}\right)\right]$$

$$+\frac{1}{2}\left(2a_1 - \frac{m_3}{\Delta_0}\right)\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)$$

$$+\frac{1}{2}\left(2c + \frac{m_2}{\Delta_0}\right)\left[\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\left(\frac{\partial v_4}{\partial x_1} - \frac{\partial v_3}{\partial x_2}\right) + \left(\frac{\partial u_4}{\partial x_1} - \frac{\partial u_3}{\partial x_2}\right)\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\right]$$

$$+\frac{1}{2}\left(2a_2-\frac{m_1}{\Delta_0}\right)\left(\frac{\partial u_4}{\partial x_1}-\frac{\partial u_3}{\partial x_2}\right)\left(\frac{\partial v_4}{\partial x_1}-\frac{\partial v_3}{\partial x_2}\right) \tag{2.5}$$

From (2.3) when $V=U$ we get

$$\int_D [U(x)A(\partial x)U(x)]+N(u,u)dx=\int_{-l}^{l}U(x_1,0)NU(x_1,0)dx_1+\int_0^1 U(l,x_2)NU(l,x_2)dx_2$$
$$-\int_{-l}^{l}U(x_1,1)NU(x_1,1)dx_1-\int_0^1 U(-l,x_2)NU(-l,x_2)dx_2, \tag{2.6}$$

where (see (2.5))

$$N(u,u)=\frac{1}{2}\left(2a-\frac{m_3}{\Delta_0}\right)\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)^2$$
$$+\left(2c_0+\frac{m_2}{\Delta_0}\right)\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_4}{\partial x_2}\right)$$
$$+\frac{1}{2}\left(2b-\frac{m_1}{\Delta_0}\right)\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_4}{\partial x_2}\right)^2+\frac{m_3}{2\Delta_0}\left[\left(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\right)^2+\left(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\right)^2\right]$$
$$-\frac{m_2}{\Delta_0}\left[\left(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_4}{\partial x_2}\right)+\left(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\right)\left(\frac{\partial u_4}{\partial x_1}+\frac{\partial u_3}{\partial x_2}\right)\right]$$
$$+\frac{m_1}{2\Delta_0}\left[\left(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_4}{\partial x_2}\right)^2+\left(\frac{\partial u_4}{\partial x_1}+\frac{\partial u_3}{\partial x_2}\right)^2\right]$$
$$+\frac{1}{2}\left(2a_1-\frac{m_3}{\Delta_0}\right)\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)^2+\left(2c+\frac{m_2}{\Delta_0}\right)\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)\left(\frac{\partial u_4}{\partial x_1}-\frac{\partial u_3}{\partial x_2}\right)$$
$$+\frac{1}{2}\left(2a_2-\frac{m_1}{\Delta_0}\right)\left(\frac{\partial u_4}{\partial x_1}-\frac{\partial u_3}{\partial x_2}\right)^2. \tag{2.7}$$

Due to (2.2) it follows that $N(u,u)$ is the positively defined quadratic form, also note that [2] the following theorem is valid.

Theorem 2.1. *The equation $N(u,u)=0$ admits a solution $U=(\gamma_1,\gamma_2,\gamma_3,\gamma_4)^T$, where $\gamma_1,\gamma_2,\gamma_3$ and γ_4 are arbitrary real constants.*

Note that (see(2.6)) for a regular solution of equation (2.1) we have the Green formulas:

$$\int_{D_0} N(u,u)dx=\int_{-l}^{0}U(x_1,0)NU(x_1,0)dx_1+\int_0^1 U(0,x_2)NU(o,x_2)dx_2$$
$$-\int_{-l}^{0}U(x_1,1)NU(x_1,1)dx_1-\int_0^1 U(-l,x_2)NU(-l,x_2)dx_2, \tag{2.8}$$
$$D_0=\{-l<x_1<0;\quad 0<x_2<1\};$$

$$\int_D N(u,u)dx=\int_{-l}^{l}U(x_1,0)NU(x_1,0)dx_1+\int_0^1 U(l,x_2)NU(l,x_2)dx_2$$
$$-\int_{-l}^{l}U(x_1,1)NU(x_1,1)dx_1-\int_0^l U(-l,x_2)NU(-l,x_2)dx_2$$

3. Statement of the problem and uniqueness theorem

The Bitsadze-Samarskii nonlocal boundary value problem for equation of statics of the linear theory of elastic mixture is formulated as follows. Find a regular solution of equation (2.1) in D satisfying the boundary conditions:

$$U(x_1,0)=\varphi_1(x_1), U(x_1,1)=\varphi_2(x_1),\ \ x_1\in[-l,l], U(-l,x_2)=\varphi_3(x_2),\ \ x_2\in[0,1], \tag{3.1}$$

$$U(0,x_2)=U(l,x_2),\quad x_2\in[0,1], \tag{3.2}$$

where $\varphi_k=(\varphi_{k_1},\varphi_{k_2},\varphi_{k_3},\varphi_{k_4})^T,\quad k=1,2,3$ are given continuous vector-functions, which satisfy the conditions necessary for the continuity in D of the searching solution of the problem (2.1), (3.1), (3.2)

$$\varphi_1(-l)=\varphi_3(0),\quad \varphi_2(-l)=\varphi_3(1).$$

Using the Green formulas (2.8) and (2.9) it is easy to prove

Theorem 3.1. *Assume that (see (2.4))*

$$NU(0,x_2)=\left[M_1\frac{\partial U(x_1,x_2)}{\partial n(x)}+M_2^0\frac{\partial U(x_1,x_2)}{\partial s(x)}+M_3U(x_1,x_2)\right]_{x_1=0}=0,$$

$$0\le x_2\le 1.$$

Then the nonlocal (2.1), (3.1), (3.2) problem has at most one solution

4. Solution of the stated nonlocal problem

Let us consider the functional

$$M(U)=\int_D N(u,u)dx \tag{4.1}$$

where $N(u,u)$ is defined by (2.7)

On the basis of the above results (see(2.7) and Theorem 2.1 we have that (4.1) functional is a positively defined quadratic form. For the solution of the problem by the variational method we have used the way developed in [3], [4] and [5].

Let us now prove the following

Theorem 4.1. *The vector-function $U(x)$ which minimizes the functional (4.1) and satisfies the equality (3.4) is a solution of nonlocal (2.1), (3.1), (3.2) problem if and only if the conditions (3.1) and (3.2) are fulfilled.*

Proof. At first let us prove sufficiency of conditions (3.1) and (3.2). Let minimization of the vector-function $U(x)$ of the functional (4.1) satisfy equality (3.4) and (3.1) (3.2) conditions. Let us show that the vector-function $U(x)$ is the solution of problem (2.1) (3.1) (3.2)

To this end let us consider the vector-function $U(x)+\varepsilon h(x)$ where ε is an arbitrary real scalar constant, and $h=(h_1,h_2,h_3,h_4)^T\neq 0$ is an arbitrary vector-function in D and satisfies the condition

$$h^+(y)=0,\quad y\in\Gamma, \tag{4.2}$$

Elementary calculations yield (see (2.5), (2.6) and (4.1))

$$M(U+\varepsilon h)=M(U)+2\varepsilon M(U,h)+\varepsilon^2M(h)>0 \tag{4.3}$$

where

$$M(U,h) = \int_D N(u,h)dx, \tag{4.4}$$

From (2.3) if $V = h$ by virtue (4.2) and (4.4) we obtain

$$\int_D h(x)A(\partial x)U(x)dx = -M(U,h), \tag{4.5}$$

Let us note that since in (4.3) ε is an arbitrary real scalar constant and the $M(U)$ functional at $U(x)$ attains minimum, we have

$$M(U,h) = 0 \tag{4.6}$$

By virtue of the fact $h(x) \neq 0$ is an arbitrary regular vector-function in D therefore owing to (4.6) from (4.5) it follows that $U(x)$ is a solution of equation (2.1) in the domain D.

Finally, from the above arguments and owing to Theorem 3.1 we conclude that if (3.4) equality and (3.1), (3.2) conditions are fulfilled then the minimization vector-function $U(x)$ of the functional (4.1) is the solution of (2.1), (3.1), (3.2) problem.

Now let us show the necessity of condition (3.1) and (3.2). Since the minimization vector-function $U(x)$ of the functional (4.1) is the solution of problem (2.1) (3.1) (3.2) and also equality (3.4) is fulfilled, therefore owing to uniquness Theorem 3.1 we can conclude that conditions (3.1) and (3.2) are fulfilled.

Finally from Theorem 3.1 and Theorem 4.1 we conclude that vector-function $U(x)$ minimizing the functional (4.1) is the unique solution of problem (2.1), (3.1), (3.2).

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