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SPLITTING OF A FOUR-LAYER SEMIDISCRETE SCHEME OF SOLUTION OF AN EVOLUTION EQUATION WITH VARIABLE OPERATOR INTO TWO-LEVEL SCHEMES

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Abstract. In the present paper, using the perturbation algorithm, the purely implicit four-level semidiscrete scheme of an abstract evolution equation with variable operator is reduced to two-level schemes. Using the solutions of the latter two-level schemes an approximate solution to the original problem is constructed. Using the associated polynomials, the approximate solution error is proved in the Hilbert space.

Keywords and phrases: Evolutionary problem with variable operator, semidiscrete scheme, perturbation algorithm, approximate solution error estimate.

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1. Introduction

In the present paper, the purely implicit four-level semidiscrete scheme for an approximate solution of the Cauchy problem for an evolution equation with a variable operator is considered in the Hilbert space. Using the perturbation algorithm, the considered scheme is reduced to two-level schemes. The solutions of the latter schemes are used to construct an approximate solution of the initial problem. We note that the first two-layer scheme provides an approximate solution with an accuracy of first order, while the solution of each subsequent split scheme is the refinement of the preceding solution by one order.

Questions connected with the construction and investigation of approximate solution algorithms of evolution problems are considered for example in quite a lot of works, e.g. by S. K. Godunov and V. S. Ryabenki [1], G. I. Marchuk [2], R. Richtmayer and K. Morton [3], A. A. Samarski [4], N. N. Yanenko [5] and others.

The main difficulty which we encounter in the realization of multi-layer schemes (especially for multidimensional problems), consists in the necessity to use of a large operational access memory, which increases in proportion to the growth of dimensional of problem. One of the ways to cope with this difficulty is the splitting of multi-layer schemes. This is exactly the topic of the works [6], [7], [8], which are dedicated to the investigation of the splitting of purely implicit three and four-level semidiscrete schemes for an evolution equation with the constant operator. There, a purely implicit semidiscrete scheme for an evolution equation is reduced to two-level schemes and the explicit estimates are proved for an approximate solution, under quite general assumptions, of problems in the Banach [7] and the Hilbert [8] space. In the present paper, estimation of the approximate solution error, we applied the approach, proposed in [9], were the stability of linear many-step methods is investigated by the properties of the class of polynomials of many variables (which are called associated polynomials). They are a natural generalization of classical Chebyshev polynomials of second kind.

We would like to mention specially that the application of the perturbation algorithm to difference schemes for differential equations was considered in [10]. The perturbation algorithm is widely used for solving problems of mathematical physics (e.g., see [11]).

2. Splitting of a purely implicit four-layer scheme for an evolution problem into two-layer schemes

Let us consider the following evolutionary problem in the Hilbert space H:

$$\frac{du(t)}{dt} + A(t)u\,(t) = f(t), \quad t \in \,]0, T]\,, \tag{1}$$

$$u\,(0) = u_0, \tag{2}$$

where $A(t)$ is the self-adjoint positive definite operator in H whose domain of definition $D(A)$ does not depend on t; $f(t)$ is a continuously differentiable function having values in H; u_0 is a given vector from H; $u(t)$ is the unknown function.

On $[0, T]$ we introduce the grid $t_k = k\tau$, $k = 0, 1, \ldots, n$, with step $\tau = T/n$. Approximating the first derivative by the purely implicit four-level semidiscrete scheme, we can write equation (1) at the point $t = t_k$ as

$$\begin{aligned}\frac{u(t_k) - u(t_{k-1})}{\tau} + A_k u\,(t_k) + \frac{\tau}{2}\frac{\Delta^2 u(t_{k-2})}{\tau^2} + \frac{\tau^2}{3}\frac{\Delta^3 u(t_{k-3})}{\tau^3} \\ = f(t_k) - \tau^3 R_k(\tau, u),\end{aligned} \tag{3}$$

where $k = 3, \ldots, n$, $A_k = A(t_k)$, $\Delta u(t_{k-1}) = u(t_k) - u(t_{k-1})$, $R_k(\tau, u) \in H$.

Using the perturbation algorithm [10], from (3) we obtain the following system of equations

$$\frac{u_k^{(0)} - u_{k-1}^{(0)}}{\tau} + A_k u_k^{(0)} = f_k, \quad f_k = f(t_k), \quad u_0^{(0)} = u_0, \quad k = 1, \ldots, n, \tag{4}$$

$$\frac{u_k^{(1)} - u_{k-1}^{(1)}}{\tau} + A_k u_k^{(1)} = -\frac{1}{2}\frac{\Delta^2 u_{k-2}^{(0)}}{\tau^2}, \qquad k = 2, \ldots, n, \tag{5}$$

$$\frac{u_k^{(2)} - u_{k-1}^{(2)}}{\tau} + A_k u_k^{(2)} = -\frac{1}{2}\frac{\Delta^2 u_{k-2}^{(1)}}{\tau^2} - \frac{1}{3}\frac{\Delta^3 u_{k-3}^{(0)}}{\tau^3}, \qquad k = 3, \ldots, n. \tag{6}$$

We introduce the notation

$$v_k = u_k^{(0)} + \tau u_k^{(1)} + \tau^2 u_k^{(2)}, \quad k = 3, \ldots, n. \tag{7}$$

Let the vector v_k be an approximate value of the exact solution of problem (1), (2) for $t = t_k$, $u(t_k) \approx v_k$.

Note that in scheme (5) the starting vector $u_1^{(1)}$ is defined from the equality $v_1 = u_1^{(0)} + \tau u_1^{(1)}$, where $u_1^{(0)}$ is defined by scheme (4), and v_1 is an approximate value of $u(t_1)$ with an accuracy of $O(\tau^3)$. In a similar way, the starting vector $u_2^{(2)}$ is defined by the equality $v_2 = u_2^{(0)} + \tau u_2^{(1)} + \tau^2 u_2^{(2)}$, where $u_2^{(0)}$ and $u_2^{(1)}$ are found by schemes (4) and (5), respectively, and v_2 is an approximate value of $u(t_2)$ with an accuracy of $O(\tau^3)$.

We can show that the constructed solution v_k satisfies the equation

$$\frac{\frac{11}{6}v_k - 3v_{k-1} + \frac{3}{2}v_{k-2} - \frac{1}{3}v_{k-3}}{\tau} + A_k v_k = f_k + \widetilde{R}_k(\tau), \quad k = 5, \dots, n, \tag{8}$$

where for the residual $\widetilde{R}_k(\tau)$, the estimate

$$\left\| \widetilde{R}_k(\tau) \right\| \leq c\tau^3, \qquad c = const > 0, \qquad k = 5, \dots, n, \tag{9}$$

holds true provided that the initial vector is smooth enough. The study of scheme (4)–(7) is based on some facts concerning polynomials associated with a higher order difference equation.

3. A priori estimate for the error of an approximate solution

For the error $z_k = u(t_k) - v_k$ we have

$$\frac{\frac{11}{6}z_k - 3z_{k-1} + \frac{3}{2}z_{k-2} - \frac{1}{3}z_{k-3}}{\tau} + A_k z_k = r_k(\tau), \quad k = 5, \dots, n, \tag{10}$$

where $r_k(\tau) = -\left(\tau^3 R_k(\tau, u) + \widetilde{R}_k(\tau)\right)$.

Taking (9) into account, we conclude that if the solution of problem (1), (2) is a sufficiently smooth function, then $\|r_k(\tau)\| = O(\tau^3)$.

The following theorem is true.

Theorem 1. *Let $A(t)$ be a self-adjoint positive definite operator in H with the domain of definition $D(A)$ not depending on t, and for any t', t'' and s from $[0,T]$ the following be fulfilled:*

$$\|(A(t') - A(t''))A^{-1}(s)\| \leq c_0|t' - t''|, \quad c_0 = const > 0.$$

Then we have

$$\|z_{k+2}\| \leq c\left(\|z_2\| + \|z_3\| + \|z_4\| + \tau\sum_{i=3}^{k}\|r_{i+2}(\tau)\|\right), \tag{11}$$

where $c = const > 0$, $k = 3, \dots, n-2$.

Let consider the main stages of the proof. Using (10), we get

$$z_{k+1} = \frac{18}{11}L_{k+1}z_k - \frac{9}{11}L_{k+1}z_{k-1} + \frac{2}{11}L_{k+1}z_{k-2} + \frac{6}{11}\tau L_{k+1}r_{k+1}(\tau), \tag{12}$$

where

$$L_k = \left(I + \frac{6}{11}\tau A_k\right)^{-1}.$$

If we introduce the notation

$$L_k^1=\frac{18}{11}L_k,\quad L_k^2=-\frac{9}{11}L_k,\quad L_k^3=\frac{2}{11}L,\quad g_{k+1}=\frac{6}{11}\tau L_{k+1}r_{k+1}(\tau),$$

then (12) takes the form

$$z_{k+1}=L_{k+1}^1z_k+L_{k+1}^2z_{k-1}+L_{k+1}^3z_{k-2}+g_{k+1}.$$

Hence, by induction, we obtain

$$\begin{aligned}z_{k+2}=&(\overset{3}{U}_{k-1}L_3^1+\overset{4}{U}_{k-2}L_4^2+\overset{5}{U}_{k-3}L_5^3)z_2+(\overset{3}{U}_{k-1}L_3^2+\overset{4}{U}_{k-2}L_4^3)z_1\\&+\overset{3}{U}_{k-1}L_3^3z_0+\frac{6}{11}\tau\sum_{i=1}^{k}\overset{i+2}{U}_{k-i}L_{i+2}r_{i+2}(\tau),\end{aligned}\tag{13}$$

where the operators $\overset{i}{U}_k$ are defined by the recurrent relation

$$\overset{i}{U}_k=L_{k+i}^1\overset{i}{U}_{k-i}+L_{k+i}^2\overset{i}{U}_{k-2}+L_{k+i}^3\overset{i}{U}_{k-3},$$

$$\overset{i}{U}_0=I,\quad \overset{i}{U}_{-1}=\overset{i}{U}_{-2}=0.$$

Let us consider the following difference homogeneous equation

$$\frac{11}{6}w_k-3w_{k-1}+\frac{3}{2}w_{k-2}-\frac{1}{3}w_{k-3}+\tau Aw_k=0,\quad k=3,\dots,n,\tag{14}$$

where A is a self-adjoint positive definite operator.

From (14) we have

$$w_k=\beta_1Lw_{k-1}+\beta_2Lw_{k-2}+\beta_3Lw_{k-3},\tag{15}$$

where $\beta_1=\frac{18}{11}$, $\beta_2=-\frac{9}{11}$, $\beta_3=\frac{2}{11}$,

$$L=(I+\tau\beta_0A)^{-1},\quad \beta_0=\frac{6}{11}.$$

Hence we obtain (see [9], Chapter I, §3)

$$\begin{aligned}w_{k+2}=&(\beta_1LU_{k-1}+\beta_2LU_{k-2}+\beta_3LU_{k-3})w_2\\&+(\beta_2LU_{k-1}+\beta_3LU_{k-2})w_1+\beta_3LU_{k-1}w_0,\end{aligned}\tag{16}$$

the operator polynomials are defined by the recurrent relation

$$\begin{aligned}U_k(\beta_1L,\beta_2L,\beta_3L)=&\beta_1LU_{k-1}(\beta_1L,\beta_2L,\beta_3L)\\&+\beta_2LU_{k-2}(\beta_1L,\beta_2L,\beta_3L)+\beta_3LU_{k-3}(\beta_1L,\beta_2L,\beta_3L),\end{aligned}\tag{17}$$

$$U_0=I,\quad U_{-1}=U_{-2}=0.$$

In equation (14) we replace the operator A by the operator $A(t_j)$ (j is fixed) and write the resulting equation in the form

$$\frac{11}{6}w_k - 3w_{k-1} + \frac{3}{2}w_{k-2} - \frac{1}{3}w_{k-3}$$

$$+\tau A(t_{k+j})w_k = \tau(A(t_{k+j}) - A(t_j))w_k, \quad k = 3, 4, \ldots \ .$$

Hence

$$w_k = \beta_1 L_{k+j} w_{k-1} + \beta_2 L_{k+j} w_{k-2} + \beta_3 L_{k+j} w_{k-3}$$
$$+\tau\beta_0 L_{k+j}(A(t_{k+j}) - A(t_j))w_k.$$

By virtue of formula (13) we obtain

$$w_{k+2} = (\beta_1 \overset{j+3}{U}_{k-1} L_{j+3} + \beta_2 \overset{j+4}{U}_{k-2} L_{j+4} + \beta_3 \overset{j+5}{U}_{k-3} L_{j+5})w_2$$

$$+(\beta_2 \overset{j+3}{U}_{k-1} L_{j+3} + \beta_3 \overset{j+4}{U}_{k-2} L_{j+4} +)w_1 + \beta_3 \overset{j+3}{U}_{k-1} L_{j+3} w_0$$

$$+\tau\beta_0 \sum_{i=1}^{k} \overset{i+m}{U}_{k-i} L_{i+m}(A(t_{i+m}) - A(t_j))w_{i+2}, \quad m = j+2. \tag{18}$$

Equating the right-hand parts of formulas (16) and (18) and assuming that

$$w_0 = w_1 = 0,$$

we obtain

$$U_k w_2 = \overset{m}{U}_k w_2 + \tau\beta_0 \sum_{i=1}^{k} \overset{i+m}{U}_{k-i} L_{i+m}(A(t_{i+m}) - A(t_j))U_i w_2.$$

Since w_2 is arbitrary, we get

$$\overset{m}{U}_k = U_k - \tau\beta_0 \sum_{i=1}^{k} \overset{i+m}{U}_{k-i} L_{i+m}(A(t_{i+m}) - A(t_j))U_i. \tag{19}$$

Here the operator polynomials $U_k = U_k(\beta_1 L, \beta_2 L, \beta_3 L)$ are defined by the recurrent relation (17), where

$$L = (I + \tau\beta_0 A(t_j))^{-1}.$$

Since $Sp(L) \subset [0; 1]$ ($Sp(L)$ is the spectrum of the operator L), the $S = L^{\frac{1}{3}}$ exists.

The following formula is easily proved by induction

$$U_k(sx_1, s^2x_2, s^3x_3) = s^k U_k(x_1, x_2, x_3), \quad s > 0.$$

By virtue of this formula we have

$$U_k\left(\beta_1 L, \beta_2 L, \beta_3 L\right) = S^k U_k\left(\beta_1 S^2, \beta_2 S, \beta_3 I\right). \tag{20}$$

As is well known, when the argument is a self-adjoint bounded operator, the norm of the operator polynomial is equal to the C -norm of the corresponding scalar polynomial on the spectrum of this operator (see e.g. [12], Ch. IX, §5).

By virtue of this fact we have

$$\begin{aligned} \|U_k\left(\beta_1 S^2, \beta_2 S, \beta_3 I\right)\| &= \left\|U_k\left(\frac{18}{11}S^2, -\frac{9}{11}S, \frac{2}{11}I\right)\right\| \\ &\le \max_{x\in[0,1]}\left|U_k\left(\frac{18}{11}x^2, -\frac{9}{11}x, \frac{2}{11}\right)\right|. \end{aligned} \tag{21}$$

It is obvious that the polynomials

$$P_k(x) = U_k\left(\frac{18}{11}x^2, -\frac{9}{11}x, \frac{2}{11}\right)$$

satisfy, by definition, the following recurrent relation:

$$\begin{gathered} P_k(x) = \frac{18}{11}x^2 P_{k-1} - \frac{9}{11}xP_{k-2} + \frac{2}{11}P_{k-3}, \\ P_0 = I, \quad P_{-1} = P_{-2} = 0, \quad x \in [0,1]. \end{gathered} \tag{22}$$

The characteristic equation of the difference equation (22) has the form

$$Q_2(\lambda) = \lambda^3 - \frac{18}{11}x^2\lambda^2 + \frac{9}{11}x\lambda - \frac{2}{11} = 0, \quad x \in [0,1]. \tag{23}$$

Let us show that for any $x \in [0,1]$, the real root of equation (23) is in the unit circle, while the other two roots are complex-conjugate and belong to one and the same circle lying in the unit circle. Then the polynomials

$$P_k(x) = U_k\left(\frac{18}{11}x^2, -\frac{9}{11}x, \frac{2}{11}\right)$$

are uniformly bounded. (see [9], Ch. I, §3).

First we show that the discriminant of equation (23) is negative, i.e.

$$D = -108\left(\frac{q^2}{4} + \frac{p^3}{27}\right) < 0,$$

where

$$q = \frac{2a^3}{27} - \frac{ab}{3} + c, \quad p = b - \frac{a^2}{3}.$$

Hence, as is known, it follows that one root is real, while the other two roots are complex-conjugate.

It is obvious that in our case we have

$$p(x) = \frac{9}{11}x(1 - x^3) - \frac{9}{11^2}x^4 \ge -\frac{9}{11^2},$$

$$q(x) = -\frac{2^4 \cdot 3^3}{11^3}x^6 + \frac{2 \cdot 3^3}{11^2}x^3 - \frac{2}{11} \leq -\frac{5}{11 \cdot 2^4},$$

where $x \in [0,1]$.

This implies that

$$p^3(x) \geq -\frac{3^6}{11^6}, \quad q^2(x) \geq \frac{5^2}{11^2 \cdot 2^8}.$$

By virtue of these estimates we have

$$D = -108\left(\frac{q^2}{4} + \frac{p^3}{27}\right) \leq -108 \cdot \frac{1}{11^2}\left(\frac{5^2}{2^{10}} - \frac{3^3}{11^4}\right)$$

$$\leq -108 \cdot \frac{1}{11^2}\left(\frac{5^2}{2^{10}} - \frac{3^3}{3 \cdot 2^{10}}\right) < 0.$$

Thus we see that one root of equation (23) is real, while the other two roots are complex-conjugate.

Let us show that there exists $\alpha > 1$, such that for any x from $[0,1)$ the real root of (23) lies in the interval $[\frac{2}{11}\alpha, 1]$.

Obviously, there exists $\alpha \in (1, \frac{11}{4})$, such that for any $x \in [0,1]$ we have

$$Q_2\Big(\frac{2}{11}\alpha\Big) = -9\alpha^2(15 \cdot 11^2 - 64\alpha^3) < 0.$$

Since

$$Q_2(1) = -\frac{9}{11}(2x^2 - x - 1) > 0, \quad x \in [0,1),$$

the real root λ_1 of equation (23) belongs to the interval $[\frac{2}{11}\alpha, 1]$.

Since the other roots λ_2 and λ_3 are complex-conjugate ($\lambda_3 = \overline{\lambda}_2$), by the Viéte theorem we have

$$|\lambda_1 \cdot \lambda_2 \cdot \overline{\lambda}_2| = \frac{2}{11}.$$

Hence

$$|\lambda_2|^2 = \frac{2}{11\lambda_1} < \frac{1}{\alpha} < 1.$$

Thus the real root of equation (23) is in the unit circle, while the other two roots are complex-conjugate and belong to one and the same circle lying in the unit circle. Hence it follows that the polynomials $U_k(\frac{18}{11}x^2, -\frac{9}{11}x, \frac{2}{11})$ are uniformly bounded, i.e.

$$\max_{x\in[0,1]}\left|U_k\left(\frac{18}{11}x^2, -\frac{9}{11}x, \frac{2}{11}\right)\right| \leq c_1, \quad c_1 = const > 0, \quad k = 0,1,\dots\ . \tag{24}$$

Hence, by virtue of (21), it follows that

$$\|U_k\| \leq c_1, \quad c_1 = const > 0, \quad k = 0,1,\dots\ . \tag{25}$$

Further, since

$$\tau A(t_j)L^{\frac{k}{3}} = \frac{1}{\beta_0}(I - L)L^{\frac{k}{3}-1}, \quad k = 3,4,\dots\ ,$$

we have

$$\|\tau A(t_j)L^{\frac{k}{3}}\| \leq \frac{1}{\beta_0}\max_{x\in[0,1]}(1-x)x^{\frac{k}{3}-1} \leq \frac{3}{\beta_0 k}. \tag{26}$$

Using the conditions of Theorem 1 and inequalities (24) and (26) we obtain

$$\|(A(t_{k+m})-A(t_j))U_k(\beta_1 L,\beta_2 L,\beta_3 L)\|$$

$$\leq \|(A(t_{k+m})-A(t_j))A^{-1}(t_j)\|\|A(t_j)L^{\frac{k}{q}}\|\|U_k(\beta_1 S^2,\beta_2 S,\beta_3 I)\|$$
$$\leq c_0\frac{t_{k+m}-t_j}{\tau}\cdot\frac{3}{\beta_0 k}\cdot c_1 = c_0c_1\frac{3}{\beta_0}\frac{k+2}{k} \leq c_0c_1\frac{9}{\beta_0} = c. \tag{27}$$

If in (19) we pass over to the norms, we get

$$\|\overset{m}{U}_k\| \leq \|U_k\| + \tau\beta_0\sum_{i=1}^{3}\|\overset{i+m}{U}_{k-i}\|\|L_{i+m}(A(t_{i+m})-A(t_j))U_i\|$$

$$+\tau\beta_0\sum_{i=4}^{k}\|\overset{i+m}{U}_{k-i}\|\|L_{i+m}\|\|(A(t_{i+m})-A(t_j))U_i\|. \tag{28}$$

Note that the operator $(A(t)-A(t_j))L_i$ is bounded. Indeed, due to the condition of the theorem we have

$$\|(A(t)-A(t_j))L_i\| \leq \|(A(t)-A(t_j))A^{-1}(t_i)\|\|A(t_i)L_i\| \leq c_0\beta_0^{-1}\tau^{-1}|t-t_j|. \tag{29}$$

Since by the condition of the theorem, the definition domain of the operator $A(t)$ does not depend on t, we have $L_iA(t)\subset(A(t)L_i)^*$. The same is true if $A(t)$ is replaced by $A(t)-A(t_j)$. In view of this fact and estimates (25) and (29) we obtain

$$\|L_{i+m}(A(t_{i+m})-A(t_j))U_iv\| = \|((A(t_{i+m})-A(t_j))L_{i+m})^*U_iv\|$$

$$\leq c_1\|((A(t_{i+m})-A(t_j))L_{i+m})^*\|\|v\| = c_1\|((A(t_{i+m})-A(t_j))L_{i+m})\|\|v\|$$

$$\leq c_0c_1\beta_0^{-1}|i+m-j|\|v\| = c_0c_1\beta_0^{-1}|i+2|\|v\|$$

$$\leq 5c_0c_1\beta_0^{-1}\|v\|, \quad i=1,2,3, \quad v\in H. \tag{30}$$

From (28), by virtue of (25), (27) and (30), we have

$$\|\overset{m}{U}_k\| \leq c + c\tau\sum_{i=1}^{k}\|\overset{i+m}{U}_{k-i}\|, \quad c = const > 0.$$

If we replace k by $(n-m)$, then

$$\|\overset{m}{U}_{n-m}\| \leq c + c\cdot\tau\sum_{i=1}^{n-m}\|\overset{i+m}{U}_{n-m-i}\|,$$

or, which is the same,

$$\|\overset{m}{U}_{n-m}\| \leq c + c \cdot \tau \sum_{s=m+1}^{n} \|\overset{s}{U}_{n-s}\|.$$

If we introduce the notation $\|\overset{s}{U}_{n-s}\| = x_s$, then we get

$$x_m \leq c + c\tau(x_{m+1} + \cdots + x_n), \quad m = 2, 3, \cdots, n-1.$$

This by induction implies the estimate

$$x_{n-i} \leq c(1 + c\tau)^{i-1}(1 + \tau x_n). \tag{31}$$

Since

$$x_n = \|\overset{n}{U}_0\| = 1,$$

from (31) we obtain

$$x_{n-i} \leq c(1 + c\tau)^i, \quad i = 0, 1, \cdots, n-m.$$

Therefore we have

$$\|\overset{n-i}{U}_i\| \leq c(1 + c\tau)^i \leq ce^{ct_i}.$$

Replacing i by $(n - i - m)$, we obtain

$$\|\overset{i+m}{U}_{n-m-i}\| \leq ce^{ct_{n-m-i}}, \quad i = 0, 1, \cdots, n-m,$$

or, which is the same,

$$\|\overset{i+m}{U}_{k-i}\| \leq ce^{ct_{k-i}}, \quad i = 0, 1, \cdots, k.$$

Again using the notation $m = (j + 2)$, we obtain

$$\|\overset{i+j+2}{U}_{k-i}\| \leq ce^{ct_{k-i}}.$$

Taking this estimate into account, from (13) we get the a priori estimate (11).

This completes the proof of the theorem.

Further, we note that if $\|u(t_k) - v_k\| = O(\tau^3),\ k = 1, 2$, then for sufficiently smooth initial data it easy to prove the estimates

$$\|u(t_k) - v_k\| = O(\tau^3), \quad k = 3, 4. \tag{32}$$

Finally, taking $\|r_k(\tau)\| = O(\tau^3)$ into account, from (11) and (32) there follows the following assertion.

Theorem 2. *Let the operator $A(t)$ satisfy the conditions of Theorem 1 and let the solution of problem* (1), (2) *be a sufficiently smooth function. Then for $\|u(t_k) - v_k\| = O(\tau^3),\ k = 1, 2$, the estimate*

$$\|u(t_k) - v_k\| = O(\tau^3), \qquad k = 3, \ldots, n.$$

holds true.

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