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ON HIGHER ORDER NONLINEAR DIFFERENCE EQUATIONS WITH PROPERTY **B**

Koplatadze R.

Abstract. We study oscillatory properties of solutions of the functional difference equation of the form

$$\Delta^{(n)}u(k) + F(u)(k) = 0,$$

where $n \geq 2$, $F : S(\mathbb{N};\mathbb{R}) \to \mathbb{S}(\mathbb{N};\mathbb{R})$ (By $S(\mathbb{N};\mathbb{R})$ denote the set of discrete functions whose set of values is $\mathbb{R}$).

Sufficient conditions for the above equation to have the co-cold Property B are established. Analogous results for oscillation of solutions of linear ordinary and nonlinear functional differential equations see in [1-3, 5-10].

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1. Introduction

Let $\tau \in S(\mathbb{N};\mathbb{R})$, $\lim\limits_{k\to+\infty} \tau(k) = +\infty$. Denote by $V(\tau)$ the set of mappings $F : S(\mathbb{N};\mathbb{R}) \to \mathbb{S}(\mathbb{N};\mathbb{R})$ satisfying the condition $F(x)(k) = F(y)(k)$ holds for any $k \in \mathbb{N}$ and $x; y \in S(\mathbb{N};\mathbb{R})$ provided that $x(s) = y(s)$ for $\tau(k) \leq s$, $s \in \mathbb{N}$.

This work is dedicated to the study of oscillatory properties of solutions of the functional difference equation

$$\Delta^{(n)}u(k) + F(u)(k) = 0, \tag{1.1}$$

where $n \geq 2$, $F \in V(\tau)$, $\Delta^{(1)}u(k) = u(k+1) - u(k)$, $\Delta^i = \Delta^{(1)} \circ \Delta^{(i-1)}$ $(i = 2, \dots, n)$.

For any $k_0 = \mathbb{N}$ we denote by $H_{k_0,\tau}$ the set of all discrete functions $u \in S(\mathbb{N};\mathbb{R})$ satisfying $u(k) \neq 0$ for $k_* \leq k \in \mathbb{N}$, where $k_* = \min\big\{k_0, \tau_*(k_0)\big\}$, $\tau_*(k) = \inf\big\{\tau(s) : k \leq s, \ \ s \in \mathbb{N}\big\}$.

Throughout the work whenever the notation $V(\tau)$ and $H_{k_0,\tau}$ occurs it will be understood, unless specified otherwise, that function τ satisfies the conditions stated above.

It will always be assumed that the condition

$$F(u)(k)\, u(k) \leq 0 \quad \text{for} \quad u \in H_{k_0,\tau}, \quad k_0 \in N \tag{1.2}$$

is fulfilled.

The following notation will be used throughout the work $N_{k_0}^+ = \{k_0, k_0+1, \dots\}$ $\big(N_{k_0}^- = \{1, 2, \dots, k_0\}\big)$.

Definition 1.1. Let $k_0 \in \mathbb{N}$. We will call a function $u : \mathbb{N}_{k_0}^+ \to \mathbb{R}$ a proper solution of the equation (1.1), if it satisfies (1.1) on $\mathbb{N}_{k_0}^+$ and

$$\sup\big\{|u(i)| : i \in \mathbb{N}_k^+\big\} > 0 \quad \text{for any} \quad k \in \mathbb{N}_{k_0}^+.$$

Definition 1.2. We say that a proper solution $u : \mathbb{N}_{k_0}^+ \to \mathbb{R}$ of equation (1.1) is oscillatory, if for any $k \in \mathbb{N}_{k_0}$ there exist $k_1; k_2 \in \mathbb{N}_k^+$ such that $u(k_1)\, u(k_2) \leq 0$. Otherwise the equation is called nonoscillatory.

Definition 1.3. We say that equation (1.1) has Property B if any of its proper solutions either is oscillatory or satisfies

$$\left|\Delta^{(i)}u(k)\right| \downarrow 0 \quad \text{as} \quad k \uparrow +\infty, \quad k \in \mathbb{N} \quad (i = 0, \dots, n-1), \tag{1.3}$$

when n is even or

$$\left|\Delta^{(i)}u(k)\right| \uparrow +\infty \quad \text{as} \quad k \uparrow +\infty, \quad k \in \mathbb{N} \quad (i = 0, \dots, n-1). \tag{1.4}$$

Sufficient conditions of higher order Emden-Fowler type difference equation to have Property A can be found in [4,17-19]. The problem of establishing sufficient conditions for the oscillation of all solutions to the second order linear and nonlinear difference equations see in [11–16].

2. On some classes of nonoscillatory discrete functions

Lemma 2.1. *Let $n \geq 2$, $k_0 \in \mathbb{N}$, $u : \mathbb{N}^+_{k_0} \to \mathbb{R}$ and $u(k) > 0$, $\Delta^{(n)}u(k) \geq 0$, $\Delta^{(n)}u(k) \not\equiv 0$ for any $s \in \mathbb{N}^+_{k_0}$ and $k \in \mathbb{N}^+_s$. Then there exist $k_1 \in \mathbb{N}^+_{k_0}$ and $\ell \in \{0, \dots, n\}$, such that $\ell + n$ even and*

$$\begin{aligned} \Delta^{(i)}u(k) > 0 \quad &\textit{for} \quad k \in \mathbb{N}^+_{k_1} \quad (i = 0, \dots, \ell), \\ (-1)^{i+\ell}\Delta^{(i)}u(k) > 0 \quad &\textit{for} \quad k \in \mathbb{N}^+_{k_1} \quad (i = \ell, \dots, n-2), \\ \Delta^{(n)}u(k) \geq 0 \quad &\textit{for} \quad k \in \mathbb{N}^+_{k_1}. \end{aligned} \tag{2.1}$$

Proof. The lemma follows immediately from the fact that, if $u(k) > 0$ and $\Delta^{(2)}u(k) \leq 0$ for $k \in \mathbb{N}^+_{k_0}$, than there exist $k_1 \in \mathbb{N}^+_{k_0}$ such that $\Delta^{(1)}u(k) > 0$ for $k \in \mathbb{N}_{k_1}$.

Remark 2.1. It is obvious that if $u_1; u_2 : \mathbb{N} \to \mathbb{R}$ and $\Delta^{(i)}u_1(k_0) = \Delta^{(i)}u_2(k_0)$ $(i = 0, \dots, n-1)$ and $\Delta^{(n)}u_1(k) = \Delta^{(n)}u_2(k)$ for $k \in \mathbb{N}$. Then $u_1(k) = u_2(k)$ for $k \in \mathbb{N}$.

Lemma 2.2. *([19]) Let $u : \mathbb{N} \to \mathbb{R}$, $m; s \in \mathbb{N}$. Then*

$$\begin{aligned} \Delta^{(i)}u(k) = &\sum_{j=1}^{m-1} \frac{\Delta^{(j)}u(s)}{(j-i)!} \prod_{r=1}^{j-i}(k - s - r + 1) \\ &+ \frac{1}{(m-i-1)!} \sum_{j=s}^{k} \prod_{r=1}^{m-i-1} (k - j - r + 1)\Delta^{(m)}u(j-1), \\ &\qquad (i = 0, \dots, m-1) \quad \textit{for} \quad k \in \mathbb{N}^+_s, \end{aligned} \tag{2.2}$$

where

$$\Delta^{(m)}u(s-1) = 0, \quad \prod_{r=1}^{0}(k - s - r + 1) = 1$$

and

$$\begin{aligned} \Delta^{(i)}u(k) = &\sum_{j=1}^{m-1} \frac{\Delta^{(j)}u(s)}{(j-i)!} \prod_{r=1}^{j-i}(k - s - r + 1) \\ &- \frac{1}{(m-i-1)!} \sum_{j=k}^{s} \prod_{r=1}^{m-i-1} (k - j - r + 1)\Delta^{(m)}u(j-1), \\ &\qquad (i = 0, \dots, m-1) \quad \textit{for} \quad k \in \mathbb{N}^-_s, \end{aligned} \tag{2.3}$$

where

$$\Delta^{(m)}u(s)=0, \quad \prod_{r=1}^{0}(k-s-r+1)=1.$$

Lemma 2.3. *([19]) Let $u:\mathbb{N}\to\mathbb{R}$, $m;s\in\mathbb{N}$. Then the equality holds*

$$\sum_{i=s}^{k} i^{m-j-1}\Delta^{(m)}u(i)=\sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(k+1)\Delta^{m-i-1}(k+i+1-m)$$
$$-\sum_{i=j}^{m-i-1}\Delta^{(i)}u(s+1)\Delta^{(m-i-1)}(s+i+1-m)^{m-j-1}, \quad \text{for} \quad k\in\mathbb{N}_s^+, \tag{2.4}$$

where

$$\Delta^{(m)}u(s)=0$$

and

$$-\sum_{i=1}^{s}(i+1)^{m-j-1}\Delta^{(m)}u(j+1)=\sum_{i=1}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(k+1)\Delta^{m-i-1}(k+i+1-m)$$
$$-\sum_{i=j}^{m-1}(-1)^{m+i-1}\Delta^{(i)}u(s+1)\Delta^{(m-i-1)}(s+i+1-m)^{m-j-1} \quad \text{for} \quad k\in\mathbb{N}_s^-, \tag{2.5}$$

where

$$\Delta^{(m)}u(s+1)=0.$$

Lemma 2.4. *Let $u:\mathbb{N}\to\mathbb{R}$, $k_0;n\in\mathbb{N}$, n is even and*

$$(-1)^i\Delta^{(i)}u(k)>0 \quad (i=0,\dots,n-1), \quad \Delta^{(n)}u(k)\geq 0 \quad \text{for} \quad k\in\mathbb{N}_{k_0}^+. \tag{2.6}$$

Then

$$\sum_{k=1}^{+\infty}k^{n-1}\Delta^{(n)}u(k)<+\infty \tag{2.7}$$

and

$$\left|\Delta^{(i)}u(k)\right|\geq\frac{1}{(n-i-1)!}\sum_{j=k}^{+\infty}\prod_{r=1}^{m-i-1}(j-k+r-1)\Delta^{(n)}u(j) \tag{2.8}$$
$$\text{for} \quad k\in\mathbb{N}_{k_0}^+ \quad (i=0,\dots,n-1).$$

Proof. Let $k_0\leq k<s$. It can by assumed without loss generality that $\Delta^{(n)}u(s)=0$. Let $m=n$. Then from (2.3) we have

$$(-1)^i\Delta^{(i)}u(k)=\sum_{j=1}^{n-1}\frac{(-1)^j\Delta^{(j)}u(s)}{(j-i)!}\prod_{r=1}^{j-i}(s+r-k-1)$$
$$+\frac{1}{(n-i-1)!}(-1)^n\sum_{j=k}^{s}\prod_{r=1}^{n-i-1}(j+r-k-1)\Delta^{(n)}u(k). \tag{2.9}$$

Therefore, since n is even, by (2.6), (2.7) holds.

According to (2.6) from (2.9) with $s \to +\infty$ we can readily (2.8).

Lemma 2.5. *Let $u : \mathbb{N} \to \mathbb{R}$ and for some $k \in \mathbb{N}$ and $\ell \in \{1, \dots, n-2\}$, where $\ell + n$ is even,* (2.1) *be fulfilled. Then*

$$\sum_{k=1}^{+\infty} k^{n-\ell-1} \Delta^{(n)} u(k) < +\infty, \tag{2.10}$$

there exist $k_2 \in \mathbb{N}_k^+$, such that

$$\left|\Delta^{(i)} u(k)\right| \geq \frac{1}{(n-i-1)!} \sum_{j=k}^{+\infty} \prod_{r=1}^{n-i-1} (j+r-k-1) \Delta^{(n)} u(j) \tag{2.11}$$

$$\text{for} \quad k \in \mathbb{N}_{k_2} \quad (i = \ell, \dots, n-1),$$

$$\Delta^{(i)} u(k) \geq \Delta^{(i)} u(k_2) + \frac{1}{(\ell-i-1)!(n-\ell-1)!} \sum_{j=k_2}^{k-1} \prod_{r=1}^{\ell-i-1} (k+r-s-1)$$

$$\times \sum_{j=s}^{+\infty} \prod_{r=1}^{n-\ell-1} (j+r-s-1) \Delta^{(n)} u(j) \tag{2.12}$$

$$\text{for} \quad k \in \mathbb{N}_{k_2+1}^+ \quad (i = 0, \dots, \ell-1).$$

If in addition

$$\sum_{k=1}^{+\infty} k^{n-\ell} \Delta^{(n)} u(k) = +\infty, \tag{2.13}$$

then

$$\frac{u(k)}{\prod\limits_{i=0}^{\ell-1} (k-i)} \downarrow, \qquad \frac{u(k)}{\prod\limits_{i=1}^{\ell-1} (k-i)} \uparrow, \tag{2.14}$$

for large k

$$u(k) \geq \frac{1+o(1)}{\ell!}\, k^{\ell-1} \Delta^{(\ell-1)} u(k) \tag{2.15}$$

and

$$\Delta^{(\ell-1)} u(k) \geq \frac{k}{(n-\ell)!} \sum_{i=k}^{+\infty} i^{n-\ell-1} \Delta^{(n)} u(k) + \frac{1}{(n-\ell)!} \sum_{i=k_2}^{k} i^{n-\ell} \Delta^{(n)} u(i) \tag{2.16}$$

$$\text{for} \quad k \in \mathbb{N}_{k_2}^+.$$

Proof. Let $s; k \in \mathbb{N}$ and $s < k$. Assume that $\Delta^{(n)} u(s) = 0$. By (2.1), from equality (2.4), with $j = \ell$ and $m = n$ we have

$$\sum_{i=s} (-1)^{n+\ell} i^{n-\ell-1} \Delta^{(n)} u(i) = -\sum_{i=\ell}^{n-1} (-1)^{\ell+i} \Delta^{(i)} u(k+1) \Delta^{(n-i-1)} (k+i+1-n)^{n-\ell-1}$$

$$+ \sum_{i=\ell}^{n-1} (-1)^{\ell+i} \Delta^{(i)} u(s+1) \Delta^{(n-i-1)} (s+i+1-n)^{n-\ell-1}. \tag{2.17}$$

Since $\ell+n$ is even and $(-1)^{\ell+i}\Delta^{(i)}u(s+1)\ge 0$ $(i=\ell,\dots,n-1)$, from (2.17) we obtain

$$\sum_{i=s}^{k} i^{n-\ell-1}\Delta^{(n)}u(i)\le\sum_{i=\ell}^{n-1}\big|\Delta^{(i)}u(s+1)\big|\Delta^{(n-i-1)}(s+i+1-n)^{n-\ell-1}.$$

From the last inequality, when $k\to+\infty$ $(k\in\mathbb{N}_s^+)$ we obtain (2.10). From equality (2.5) also follows the inequality

$$\sum_{i=\ell}^{k}\big|\Delta^{(i)}u(k+1)\big|\Delta^{(n-i-1)}(k+i+1-n)^{n-\ell-1}\ge\sum_{i=k}^{+\infty} i^{n-\ell-1}\Delta^{(n)}u(i+1) \quad \text{for} \quad k\in\mathbb{N}_{k_2}^+. \tag{2.18}$$

On account of (2.1) and (2.10), from (2.2) with $s=k_2$ and $m=\ell$, we get

$$\Delta^{(i)}u(k)\ge\Delta^{(i)}u^{(i)}u(k_2)+\frac{1}{(\ell-i-1)!}\sum_{j=k_2}^{k}\prod_{r=1}^{\ell-i-1}(k-i+r-1)\Delta^{(\ell)}u(j-1) \tag{2.19}$$
$$(i=0,\dots,\ell-1)\quad \text{for}\quad k\in\mathbb{N}_{k_2}^+.$$

Hence, by (2.11) and (2.19) we obtain (2.12).

Using (2.1), from (2.4) with $j=\ell-1$, $m=n$ and $s=k_2$ we have

$$\begin{aligned}\Delta^{(\ell-1)}u(k)&=\frac{1}{(n-\ell)!}\sum_{j=k_2}^{k} i^{n-\ell}\Delta^{(n)}u(i)\\&+\frac{1}{(n-\ell)!}\sum_{i=\ell}^{n-1}\big|\Delta^{(i)}u(k+1)\big|\Delta^{(n-i-1)}(k+i+1-n)^{n-\ell}\\&+\frac{1}{(n-\ell)!}\sum_{i=\ell-1}^{n-1}(-1)^{n+i+1}\Delta^{(i)}u(k_2+1)\Delta^{(n-i-1)}(k_2+i+1-n)^{n-\ell}.\end{aligned}$$

Therefore, according to (2.13) there exist $k^*>k_2$ such that

$$\begin{aligned}\Delta^{(\ell-1)}u(k+1)&\ge\frac{1}{(n-\ell)!}\sum_{i=k*}^{k} i^{n-\ell}\Delta^{(n)}u(i)\\&+\frac{1}{(n-\ell)!}\sum_{i=\ell}^{n-1}\big|\Delta^{(i)}u(k+1)\big|\Delta^{(n-i-1)}(k+i+1-n)^{n-\ell}\quad\text{for}\quad k\in\mathbb{N}_{k^*}^+.\end{aligned}$$

From the last inequality, by (2.13) we have

$$\Delta^{(\ell-1)}u(k+1)-(k+\ell+1-n)\Delta^{(\ell)}u(k+1)\to+\infty \tag{2.20}$$

and by (2.18) inequality (2.16) holds.

Let $k_0\in\mathbb{N}$ and for any $k\in\mathbb{N}_{k_0}^+$ and $i\in\{1,\dots,\ell\}$ put

$$\rho_i(k)=i\Delta^{(\ell-i)}u(k)-(k+1-i)\Delta^{(\ell+i+1)}u(k), \tag{2.21}$$
$$\gamma_i(k)=(k-i)\Delta^{(\ell-i+1)}u(k)-(i-1)\Delta^{(\ell-i)}u(k). \tag{2.22}$$

Applying (2.20) and Lopital's rule, we have

$$\lim_{k\to+\infty}\frac{\Delta^{(\ell-i)}u(k)}{\prod\limits_{j=1}^{i=1}(k-j)}=+\infty \quad (i=1,\dots,\ell). \tag{2.23}$$

(Here it is meant that $\prod_{j=1}^{0}(k-j)=1$).

Since

$$\Delta^{(1)}\left(\frac{\Delta^{(\ell-i)}u(k)}{\prod\limits_{j=1}^{i=\ell}(k-j-1)}\right)=\frac{\gamma_i(k)}{\prod\limits_{j=1}^{i=\ell}(k-j-1)},$$

by (2.23) there exist $k_\ell>\cdots>k_1>k_0$ such that $\gamma_i(k_i)>0$ $(i=1,\dots,\ell)$. Therefore by (2.20), $\rho_1(k)\to+\infty$ as $k\to+\infty$, $\Delta^{(1)}\rho_{i+1}(k)=\rho_i(k)$, $\Delta^{(1)}\gamma_{i+1}(k)=\gamma_i(k)$ and $\gamma_1(k)=(k-1)\Delta^{(\ell)}u(k)>0$ for $k\in\mathbb{N}^+_{k_0}$ $(i=1,\dots,\ell-1)$, we find that $\rho_i(k)\to+\infty$ as $k\to+\infty$ and $\gamma_i(k)>0$ for$k\in\mathbb{N}^+_{k_i}$ $(i=1,\dots,\ell)$. These facts along with (2.20)–(2.23) prove (2.14).

On the other hand, since $\rho_i(k)\to+\infty$, by (2.21) for large k

$$i\Delta^{\ell-i)}u(k)>(k+1-i)\Delta^{(\ell-i+1)}u(k) \quad (i=1,\dots,\ell),$$

which implies (2.15).

3. Necessary condition for the existence of conditions of type 2.1

The results of this section play on important role in establishing sufficient conditions for equation (1.1) to have Property B.

Let $k_0\in\mathbb{N}$ and $\ell\in\{1,\dots,n-2\}$. By U_{ℓ,k_0}we denote the set of all solutions of equation (1.1) satisfying the condition (2.1).

Theorem 3.1. *Let for some $k_0\in\mathbb{N}$, condition* (2.2) *and*

$$\big|F(u)(k)\big|\ge p(k)\big|u(\sigma(k))\big|^{\lambda} \quad \textit{for} \quad k\in\mathbb{N}^+_{k_0}, \quad u\in H_{k_0}\tau, \tag{3.1}$$

be fulfilled, $\ell\in\{1,\dots,n-2\}$ with $\ell+n$ is even and

$$\sum_{k=1}^{+\infty}k^{n-\ell}\big(\sigma(k)\big)^{\lambda(\ell-1)}p(k)=+\infty, \tag{3.2}$$

where

$$0<\lambda<1, \quad p(k)\ge 0, \quad \sigma(k)\ge k+1 \quad \textit{for} \quad k\in\mathbb{N}. \tag{3.3}$$

If, moreover, $U_{\ell,k_0}\neq\varnothing$, then for any $\delta\in[0,\lambda]$ and $i\in\mathbb{N}$, we have

$$\sum_{k=1}^{+\infty}k^{n-\ell}(\sigma(k))^{\lambda(\ell-1)}\big(\rho_{i,\ell}(\sigma(k))^{\delta}p(k)<+\infty \quad (i=1,2,\dots), \tag{3.4}$$

where

$$\rho_{1,\ell}(k)=\Big(\frac{1-\lambda}{\ell!(n-\ell)!}\sum_{i=1}^{k-1}\sum_{j=i}^{+\infty}j^{n-\ell-1}(\sigma(j))^{\lambda(\ell-1)}p(j)\Big)^{\frac{1}{1-\lambda}}, \tag{3.5}$$

$$\rho_{s,\ell}(k)=\Big(\frac{1-\lambda}{\ell!(n-\ell)!}\sum_{j=1}^{k-1}\sum_{j=i}^{+\infty}j^{n-\ell-1}(\sigma(j))^{\lambda(\ell-1)}(\rho_{s-1,\ell}(\sigma(j))\lambda\Big)^{\lambda} \quad (s=2,3,\dots). \tag{3.6}$$

Proof. Let $k_0 \in \mathbb{N}$ and $U_{\ell,k_0} \neq \varnothing$. By definition of the set U_{ℓ,k_0}, equation (1.1) has a proper solution $u \in U_{\ell,k_0}$ satisfying the condition (2.1). By (2.1), (3.1) and (3.2) it is clear that the condition (2.13) holds. Thus by Lemma 2.5, (2.13)–(2.15) are fulfilled and by (1.1), (2.15) and (2.16), we have

$$\Delta^{(\ell-1)}u(k) \geq \frac{k}{\ell!(n-\ell)!}\sum_{i=k}^{+\infty} i^{n-\ell-1}(\sigma(i))^{\lambda(\ell-1)}\big(\Delta^{(\ell-1)}u(\sigma(i)\big)^{\lambda}p(i)$$
$$+\frac{1}{\ell!(n-\ell)!}\sum_{i=k_*}^{k} i^{n-\ell}(\sigma(i))^{\lambda(\ell-1)}\big(\Delta^{(\ell-1)}u(\sigma(i)\big)^{\lambda}p(i) \text{ for } k \in \mathbb{N}^+_{k_*}, \tag{3.7}$$

where k_* is a sufficiently large natural number. By the identity

$$\sum_{i=k_*}^{k} u(i)\Delta^{(1)}v(i) = u(k)v(k+1) - u(k_*-1)v(k_*) - \sum_{i=k_*}^{k} v(i)\Delta^{(1)}u(i-1)$$

we have

$$\sum_{i=k_*}^{k} i^{n-\ell}(\sigma(i))^{\lambda(\ell-1)}\big(\Delta^{(\ell-1)}u(\sigma(i))\big)^{\lambda}p(i)$$
$$-\sum_{i=k_*}^{k}\Delta^{(1)}\sum_{s=i}^{+\infty} s^{n-\ell-1}(\sigma(s))^{\lambda(\ell-1)}\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s)$$
$$= -k\sum_{i=k}^{+\infty}(\sigma(s))^{\lambda(\ell-1)}\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s)$$
$$+(k_*-1))\sum_{i=k_*}^{+\infty} s^{n-\ell-1}(\sigma(s))^{\lambda(\ell-1)}\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s)$$
$$+\sum_{i=k_*}^{k}\Big(\sum_{s=i}^{+\infty} s^{n-\ell-1}(\sigma(s))^{\lambda(\ell-1)}\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s)\Big).$$

Therefore, from (3.7) we get

$$\Delta^{(\ell-1)}u(k) \geq \frac{1}{\ell!(n-\ell)!}\sum_{i=k_*}^{k}\Big(\sum_{s=i}^{+\infty} s^{n-\ell-1}(\sigma(s))^{\lambda(\ell-1)}\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s)\Big) \quad k \in \mathbb{N}^+_{k_*}. \tag{3.8}$$

Denote

$$x(k) = \frac{1}{\ell!(n-\ell)!}\sum_{i=k_*}^{k}\Big(\sum_{s=i}^{+\infty} s^{n-\ell-1}(\sigma(s))^{\lambda(\ell-1)}\big(\Delta^{(\ell-1)}u(\sigma(s))\big)^{\lambda}p(s)\Big) \quad k \in \mathbb{N}^+_{k_*}.$$

Since $\Delta^{(\ell-1)}u(k)$ is a nondecreasing and $\sigma(k) \geq k+1$, by (3.7) we have

$$\Delta^{(1)}x(k) \geq \frac{\big(\Delta^{(\ell-1)}u(k+1)\big)^{\lambda}}{\ell!(n-\ell)!}\sum_{s=k}^{+\infty} s^{n-\ell-1}(\sigma(s))^{\lambda(\ell-1)}p(s)$$

$$\geq \frac{x^{\lambda}(k+1)}{\ell!(n-\ell)!}\sum_{s=k}^{+\infty} s^{n-\ell-1}(\sigma(s))^{\lambda(\ell-1)}p(s) \quad \text{for} \quad k\in\mathbb{N}^{+}_{k_*}.$$

Therefore,

$$\sum_{j=k_*}^{k-1}\frac{\Delta^{(1)}x(j)}{x^{\lambda}(j+1)} \geq \frac{1}{\ell!(n-\ell)!}\sum_{i=k_*}^{k-1}\Big(\sum_{i=j}^{+\infty} i^{n-\ell-1}(\sigma(s))^{\lambda(\ell-1)}p(s)\Big) \quad k\in\mathbb{N}^{+}_{k_*}. \tag{3.9}$$

Since

$$\sum_{j=k_*}^{k-1}\frac{\Delta^{(1)}x(j)}{x^{\lambda}(j+1)} = \sum_{j=k_*}^{k-1} x^{-\lambda}(j+1)\int_{x(j)}^{x(j+1)} dt$$

and $x^{-\lambda}(j+1)\leq t^{-\lambda}$, when $x_j\leq t\leq x(j+1)$, we have

$$\sum_{j=k_*}^{k-1}\frac{\Delta^{(1)}x(j)}{x^{\lambda}(j+1)} \leq \sum_{j=k_*}^{k-1}\int_{x(j)}^{x(j+1)} t^{-\lambda}dt = \frac{1}{1-\lambda}\big(x^{1-\lambda}(k)-x^{1-\lambda}(k_*)\big) \leq \frac{1}{1-\lambda}x^{1-\lambda}(k).$$

That's why, from (3.9) we get

$$x(k) \geq \Big(\frac{1-\lambda}{\ell!(n-\ell)!}\sum_{j=k_*}^{k-1}\Big(\sum_{i=j}^{+\infty} i^{n-\ell-1}(\sigma(i))^{\lambda(\ell-1)}p(i)\Big)\Big)^{\frac{1}{1-\lambda}}. \tag{3.10}$$

I.e.

$$\Delta^{(\ell-1)}u(k) \geq \rho_{1,\ell,k_*}(k) \quad \text{for} \quad k\in\mathbb{N}^{+}_{k_*}, \tag{3.11}$$

where

$$\rho_{1,\ell,k_*}(k) = \Big(\frac{1-\lambda}{\ell!(n-\ell)!}\sum_{j=k_*}^{k-1}\Big(\sum_{i=j}^{+\infty} i^{n-\ell-1}(\sigma(i))^{\lambda(\ell-1)}p(i)\Big)\Big)^{\frac{1}{1-\lambda}}.$$

Thus, by (3.8) and (3.11) we get

$$\Delta^{(\ell-1)}u(k) \geq \rho_{s,\ell,k_*}(k) \quad \text{for} \quad k\in\mathbb{N}^{+}_{k_s} \quad (s=2,3,\dots), \tag{3.12}$$

where

$$\rho_{s,\ell,k_*}(k) = \frac{1}{\ell!(n-\ell)!}\sum_{j=k_*}^{k-1}\Big(\sum_{i=j}^{+\infty} i^{n-\ell-1}(\sigma(i))^{\lambda(\ell-1)}p(i)\big(\rho_{s-1,\ell,k_*}(\sigma(i))\big)^{\lambda}.$$

On the other hand, by (2.1), (3.3), (3.5) and (3.6) from (3.7), for any $\delta\in[0,\lambda]$, we have

$$\Delta^{(\ell-1)}u(k+1) \geq \frac{1}{\ell!(n-\ell)!}\sum_{i=k_*}^{k}\Big(\sum_{j=i}^{+\infty} j^{n-\ell-1}(\sigma(j))^{\lambda(\ell-1)}$$
$$p(j)\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta}\big(\Delta^{(\ell-1)}u(\sigma(j))\big)\Big)^{\lambda-\delta} \quad (s=1,2,\dots)$$

and

$$\Delta^{(\ell-1)}u(k+1) \geq \frac{k-k_*}{\ell!(n-\ell)!}\sum_{i=k}^{+\infty}(\sigma(j))^{\lambda(\ell-1)}p(j)$$
$$\times\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta}\big(\Delta^{(\ell-1)}u(\sigma(j))\big)^{\lambda-\delta} \quad (s=1,2,\dots). \tag{3.13}$$

If $\delta = \lambda$, then from (3.13) we get

$$\sum_{j=k}^{+\infty} j^{n-\ell-1}(\sigma(j))^{\lambda(\ell-1)}p(j)\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\lambda} \le \frac{\ell!(n-\ell)!(k+1)}{k-k_*} \cdot \frac{\Delta^{(\ell-1)}u(k+1)}{k+1}.$$

By first for condition of (2.14), we get

$$\sum_{j=k}^{+\infty} j^{n-\ell-1}(\sigma(j))^{\lambda(\ell-1)}p(j)\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\lambda} < +\infty \quad (s = 1, 2, \dots). \tag{3.14}$$

Let $\delta \in [0, \lambda)$. Then from (3.13)

$$\frac{\Delta^{(\ell-1)}u(k+1)}{\sum\limits_{j=k}^{+\infty} j^{n-\ell-1}(\sigma(j))^{\lambda(\ell-1)}p(j)\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta}\big(\Delta^{(\ell-1)}u(\sigma(j))\big)} \ge \frac{k-k_*}{\ell!(n-\ell)!} \quad \text{for} \quad k \in \mathbb{N}^+_{k^*}.$$

Therefore

$$\frac{\big(\Delta^{(\ell-1)}u(k+1)\big)^{\lambda-\delta}k^{n-\ell-1}p(k)\big(\sigma^{\lambda(\ell-1)}\big)^{\delta}}{\sum\limits_{j=k}^{+\infty} j^{n-\ell-1}(\sigma(j))^{\lambda(\ell-1)}p(j)\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\lambda-\delta}\big)^{\lambda-\delta}}$$
$$\ge \Big(\frac{k-k_*}{\ell!(n-\ell)!}\Big)^{\lambda-\delta}k^{n-\ell-1}p(k)(\sigma(k))^{\lambda(\ell-1)}\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta}. \tag{3.15}$$

Denote

$$a_k = \sum_{j=k}^{+\infty} j^{n-\ell-1}(\sigma(j))^{\lambda(\ell-1)}p(j)\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta}\big(\Delta^{(\ell-1)}u(j+1)\big)^{\lambda-\delta}.$$

Since $\Delta^{(\ell-1)}u(k)$ is a nondecreasing function, from (3.15), we get

$$\frac{a_k - a_{k+1}}{a_k^{\lambda-\delta}} \ge \Big(\frac{k-k_*}{\ell!(n-\ell)!}\Big)^{\lambda-\delta}k^{n-\ell-1}p(k)(\sigma(k))^{\lambda(\ell-1)}\big(\rho_{s,\ell,k_*}(\sigma(j))\big)^{\delta}. \tag{3.16}$$

Hence from the last inequality we get

$$\sum_{i=k_0}^{k} \frac{a_i - a_{i+1}}{a_i^{\lambda-\delta}} \ge \Big(\frac{1}{\ell!(n-\ell)!}\Big)^{\lambda-\delta}\sum_{i=k_0}^{k}(i-k_*)^{\lambda-\delta}i^{n-\ell-1}p(i)(\sigma(i))^{\lambda(\ell-1)}\big(\rho_{s,\ell,k_*}(\sigma(i))\big)^{\delta}.$$

Since

$$\sum_{i=k_0}^{k} \frac{a_i - a_{i+1}}{a_i^{\lambda}} = \sum_{i=k_0}^{k} a_i^{\delta-\lambda}\int_{a_{i+1}}^{a_i} dt \le \sum_{i=k_*}^{k}\int_{a_{i+1}}^{a_i} t^{\delta-\lambda}dt \le \int_0^{a_{k_*}} t^{\delta-\lambda}dt = \frac{a_{k_*}^{1+\delta-\lambda}}{1-\delta-\lambda}\,.$$

Thus, from (3.16) we get

$$\sum_{i=k_0}^{k}(i-k_*)^{\lambda-\delta}i^{n-\ell-1}p(i)(\sigma(i))^{\lambda(\ell-1)}\big(\rho_{s,\ell,k_*}(\sigma(s))\big)^{\delta} \le \frac{a_{k_0}^{1+\delta-\lambda}(\ell!(n-\ell)!)^{\lambda-\delta}}{1+\delta-\lambda}\,. \tag{3.17}$$

Without loss of generality, by (3.14) we can assume $a_{k_*} \le 1$. Thus from (3.17) we have

$$\sum_{i=k_0}^{k} (i-k_*)^{\lambda-\delta} i^{n-\ell-1} p(i)(\sigma(i))^{\lambda(\ell-1)} \big(\rho_{s,\ell,k_*}(\sigma(i))\big)^{\delta} \le \frac{(\ell!(n-\ell)!)^{\lambda-\delta}}{1+\delta-\lambda}. \tag{3.18}$$

According to (3.14) and (3.16), for any $\delta \in [0,\lambda]$ and $s \in \mathbb{N}$, we have

$$\sum_{i=k_*}^{+\infty} i^{n-\ell-1+\lambda-\delta} p(i)(\sigma(i))^{\lambda(\ell-1)} \big(\rho_{s,\ell,k_*}(\sigma(i))\big)^{\delta} < +\infty. \tag{3.19}$$

Since

$$\frac{\rho_{s,\ell}(k)}{\rho_{s,\ell,k_*}(k)} \to 1 \quad \text{for} \quad k \to +\infty,$$

by (3.19) it is obvious that (3.4) holds, which proves the validity of the theorem.

4. Sufficient conditions of nonexistence of solutions of type (2.1)

Theorem 4.1 *Let conditions* (1.2), (3.1)–(3.3 *be fulfilled,* $\ell \in \{1,\dots,n-2\}$, *with* $\ell+n$ *even and for some* $\delta \in [0,\lambda]$ *and* $s \in \mathbb{N}$

$$\sum_{k=1}^{+\infty} k^{n-\ell-1+\lambda-\delta}(\sigma(k))^{\lambda(\ell-1)} \big(\rho_{s,\ell}(\sigma(k))\big)^{\delta} p(k) = +\infty, \tag{4.1}$$

where $\rho_{s,\ell}$ *is defined by* (3.5) *and* (3.6). *Then* $U_{\ell,k_0} = \varnothing$ *for any* $k_0 \in \mathbb{N}$.

Proof. Assume the contrary. Let there exist $k_0 \in \mathbb{N}$ such that $U_{\ell,k_0} \ne \varnothing$. Thus equation (1.1) has a proper solution $u : \mathbb{N}^+_{k_0} \to (0,+\infty)$ satisfying the condition (2.1).

Since conditions of Theorem 3.1 are fulfilled (3.2) holds for any $\delta \in [0,\lambda]$ and $s \in \mathbb{N}$, which contradicts condition (4.1). The obtained contradiction proves the validity of the theorem.

From this theorem, with $\delta = 0$, immediately follow

Theorem 4.1′. *Let conditions* (1.2), (3.1)–(3.3) *be fulfilled* $\ell \in \{1,\dots,n-2\}$, *with* $\ell+n$ *even and*

$$\sum_{k=1}^{+\infty} k^{n+\lambda-\ell-1}(\sigma(k))^{\lambda(\ell-1)} p(k) = +\infty. \tag{4.2}$$

Then for any $k_0 \in \mathbb{N}$, $U_{\ell,k_0} = \varnothing$.

Theorem 4.2. *Let conditions* (1.2), (3.1)–(3.3) *be fulfilled,* $\ell \in \{1,\dots,n-2\}$, *with* $\ell+n$ *even and for some* $\alpha \in (1,+\infty)$ *and* $\gamma \in (\lambda,1)$

$$\liminf_{k\to+\infty} k^{\gamma} \sum_{j=k}^{+\infty} j^{n-\ell-1}(\sigma(j))^{\lambda(\ell-1)} p(j) > 0, \tag{4.3}$$

$$\liminf_{k\to+\infty} \frac{\sigma(k)}{k^{\alpha}} > 0. \tag{4.4}$$

If moreover, at last one of conditions

$$\alpha\lambda \ge 1, \tag{4.5}$$

or if $\alpha\lambda < 1$, *for some* $\varepsilon > 0$

$$\sum_{k=1}^{+\infty} k^{n-\ell-1+\frac{\alpha\lambda(1-\gamma)}{1-\alpha\lambda}}(\sigma(k))^{\lambda(\ell-1)} p(k) = +\infty \tag{4.6}$$

holds, then $U_{\ell,k_0}=\varnothing$, *for any* $k_0\in\mathbb{N}$.

Proof. It is sufficient to show that condition (4.1) satisfies for some $s\in\mathbb{N}$ and $\delta=\lambda$. Indeed, according to (4.3) there exist $\alpha>1$, $\gamma\in(\lambda,1)$, $c>0$ and $k_0\in\mathbb{N}$ such that

$$k^{\gamma}\sum_{j=k}^{+\infty} j^{n-\ell-1}(\sigma(\lambda))^{\lambda(\ell-1)}p(j)\geq c \quad \text{for} \quad k\in\mathbb{N}_{k_0}^{+} \tag{4.7}$$

and

$$\sigma(k)\geq ck^{\alpha} \quad \text{for} \quad k\in\mathbb{N}_{k_0}^{+}. \tag{4.8}$$

By (4.3), (4.4) and (3.5) it is obvious that $\lim_{k\to+\infty}\rho_{1,\ell}(k)=+\infty$. Therefore without loss of generality we can assume that $\rho_{1,\ell}(k)\geq 1$ for $k\in\mathbb{N}_{k_0}^{+}$. Thus, by (4.8), (3.4) and (3.5) we get

$$\begin{aligned}\rho_{2,\ell}(k)&\geq \frac{c}{\ell!(n-\ell)!}\sum_{i=k_0}^{k-1} i^{-\gamma}=\frac{c}{\ell!(n-\ell)!}\sum_{i=k_0}^{k-1} i^{-\gamma}\int_i^{i+1}dt\\&\geq \frac{c}{\ell!(n-\ell)!}\sum_{i=k_0}^{k-1}\int_i^{i+1} t^{-\gamma}dt=\frac{c}{\ell!(n-\ell)!}\int_{k_0}^{k} t^{-\gamma}dt\\&=\frac{c}{\ell!(n-\ell)!(1-\gamma)}(k^{1-\gamma}-k_0^{1-\gamma}).\end{aligned}$$

We can choose $k_1\in\mathbb{N}_{k_0}^{+}$, such that

$$\rho_{2,\ell}(k)\geq\frac{c}{2\ell!(n-\ell)!(1-\gamma)}k^{1-\gamma} \quad \text{for} \quad k\in\mathbb{N}_{k_1}^{+}.$$

Thus, by (4.8), from (3.6), we have

$$\rho_{3,\ell}(k)\geq\Big(\frac{c}{2\ell!(n-\ell)!(1-\gamma)}\Big)^{1+\lambda}k^{(1-\gamma)(1+\alpha\lambda)} \quad \text{for} \quad k\in\mathbb{N}_{k_2}^{+}$$

where $k_2\in\mathbb{N}_{k_1}^{+}$,is a sufficiently large natural number. Therefore for any $s\in\mathbb{N}$, there exists $k_s\in\mathbb{N}$ such that

$$\rho_{s,\ell}(k)\geq\Big(\frac{c}{2\ell!(n-\ell)!(1-\gamma)}\Big)^{1+\lambda+\cdots+\lambda^{s-2}}k^{(1-\gamma)(1+\alpha\lambda+\cdots+(\alpha\lambda)^{s-2})} \quad \text{for} \quad k\in\mathbb{N}_{k_s}^{+}. \tag{4.9}$$

Assume that (4.5) is fulfilled. Choose $s_0\in\mathbb{N}$ such that $(1-\gamma)(s_0-1)\geq\frac{1}{\lambda}$. Then according to (4.9), $\rho_{s_0,\ell}^{(k)}\geq c_0k$ for $k\in\mathbb{N}_{k_s}^{+}$, where $c_0>0$. Therefore, by (4.9), it is obvious that (4.1) holds, for $\delta=\lambda$ and $s=s_0$. In the case, when (4.5) holds, the validity of the theorem has been already proved.

Assume now that $0<\alpha\lambda<1$ and (4.6) holds. Let $\varepsilon>0$ and by (4.9), choose $s_o\in\mathbb{N}$ such that

$$\rho_{\ell}(k)\geq c_1k \quad \text{for} \quad k\in\mathbb{N}_{k_{s_0}}^{+},$$

where $c_1>0$. Therefore, by (4.6), holds (4.1) for $s=s_0$. The proof of the theorem is proved.

5. Difference equations with Property B

Theorem 5.1 *Let conditions* (1.2), (3.1), (3.3) *be fulfilled and for any* $\ell\in\{1,\dots,n-2\}$ *with* $\ell+n$ *even, let as well* (4.1) *hold. Moreover, if*

$$\sum_{k=1}^{+\infty}\big(\sigma(k)\big)^{\lambda(n-1)}p(k)=+\infty \tag{5.1}$$

and with n is even

$$\sum_{k=1}^{+\infty} k^{n-1} p(k) = +\infty, \tag{5.2}$$

then equation (1.1) *has Property* **B**.

Proof. Let equation (1.1) have a proper nonoscillatory solution $u : \mathbb{N}_{k_0} \to (0, +\infty)$. By (1.1), (1.2) and Lemma 2.1, there exist $\ell \in \{0, \dots, n\}$, such that $\ell + n$ is even and the condition (2.1) holds. Since conditions of Theorem 4.1 are fulfilled, for any $\ell \in \{1, \dots, n-2\}$, with $\ell + n$ even, we have $\ell \notin \{1, \dots, n-2\}$. Therefore, $\ell = n$ or $\ell = 0$ and n is even.

Assume that $\ell = n$. To complete the proof, it suffices to show that (1.4) is valid. From (2.1), when $\ell = n$, we have $u(\sigma(k)) \geq c(\sigma(k))^{n-1}$ for $k \in \mathbb{N}^+_{k_1}$, where $c > 0$ and $k_1 \in \mathbb{N}^+_{k_0}$ is a sufficiently large natural number. Therefore, by (1.2), (5.1) and (2.1), when $\ell = n$ from (1.1) we get

$$\Delta^{(n-1)} u(k) \geq \Delta^{(n-1)} u(k_1) + c^{\lambda} \sum_{j=k_1}^{k-1} p(j)(\sigma(j))^{\lambda(n-1)} \to +\infty \quad \text{for} \quad k \to +\infty.$$

Now assume that, n is even and $\ell = 0$. Then we will show that, condition (1.3) hold. If that is not the case, there exist $c > 0$ such that $u(k) \geq c$ for sufficiently large k. According to (2.1), with $\ell = 0$, by (3.1), from (1.1) we have

$$\sum_{j=k_0}^{k} j^{n-1} \Delta^{(n)} u(j) + c \sum_{j=k_0}^{k} j^{n-1} p(j) \leq 0, \tag{5.3}$$

where $k_0 \in \mathbb{N}$ is a sufficiently large natural number.

On the other hand, in view of the identity

$$\sum_{j=k_0}^{k} j^{n-1} \Delta^{(n)} u(j) = k^{n-1} \Delta^{(n-1)} u(k+1) - (k_0 - 1)^{n-1} \Delta^{(n-1)} u(k_0) - \sum_{j=k_0}^{k} \Delta^{(n-1)} u(j) \Delta(j-1)^{n-1}$$

it is easy to show that

$$\sum_{j=k_0}^{k} j^{n-1} \Delta^{(n)} u(j) = \sum_{i=0}^{n-1} (-1)^i \Delta^i (k-i)^{n-1} \Delta^{(n-i-1)} u(k+1) - \sum_{i=0}^{n-1} (-1)^i (k_0 - i - 1)^{n-i-1} \Delta^{(n-i-1)} u(k_0).$$

Since $(-1)^i \Delta^{(i)} u(k) \geq 0$, from (5.3), we have

$$c \sum_{j=k_0}^{k} j^{n-1} p(j) \leq \sum_{i=0}^{n-1} (k_0 - i - 1)^{n-i-1} \left| \Delta^{(n-i-1)} (k_0) \right|,$$

which contradicts the condition (5.2). Therefore equation (1.1) has Property **B**.

From this theorem, with $\delta = 0$, immediately follow

Theorem 5.1$'$. *Let condition* (1.2), (3.1), (3.2), (5.1.), (5.2) *and for any* $\ell \in \{1, \dots, n-2\}$ *with* $\ell + n$ *even, the condition* (4.2) *holds. Then equation* (1.2) *has Property* **B**.

Theorem 5.2. *Let conditions* (1.2), (3.1), (3.2) *as well as* (5.3) *be fulfilled for even* n *and*

$$\liminf_{k\to+\infty} \frac{\sigma^\lambda(k)}{k} > 0. \tag{5.4}$$

Then the condition

$$\sum_{k=1}^{+\infty} k^{n+\lambda-2} p(k) = +\infty, \tag{5.5}$$

for odd n *and the condition*

$$\sum_{k=1}^{+\infty} k^{n+\lambda-3} (\sigma(k))^\lambda p(k) = +\infty \tag{5.6}$$

for even n *is sufficient, for equation* (1.1) *have Property* **B**.

Proof. It is obvious that, according to (5.4), (5.5) for any $\ell \in \{1, \dots, n-2\}$, where $\ell + n$ even, condition (4.2) holds. Therefore, all conditions of Theorem 5.1$'$ hold, which proves the validity of the theorem.

Theorem 5.3. *Let conditions* (1.2), (3.1), (5.1) *be fulfilled and let*

$$\limsup_{k\to+\infty} \frac{\sigma^\lambda(k)}{k} < +\infty. \tag{5.7}$$

Then for equation (1.1), *to have Property* **B** *it is sufficient that*

$$\sum_{k=1}^{+\infty} k^{1+\lambda} (\sigma(k))^{\lambda(n-3)} p(k) = +\infty. \tag{5.8}$$

Proof. It is obvious that, according to (5.7), (5.8) and since $0 < \lambda < 1$ and $\sigma(k) \geq k+1$ conditions (5.2) and for any $\ell \in \{1, \dots, n-2\}$, with $\ell + n$ even, conditions (4.2) holds. Therefore condition of the Theorem 5.1$'$ holds, which proved the validity of the theorem.

Theorem 5.4. *Let conditions* (1.2), (3.1), (5.1) *and for any* $\ell \in \{1, \dots, n-\ell\}$, *with* $\ell + n$ *even* (4.2)–(4.4) *be fulfilled. Moreover, of conditions* (4.5) *or* (4.6) *hold, Then for equation* (1.1), *to have Property* **B**.

Proof. Let equation (1.1) have a proper nonoscillatory solution $u : \mathbb{N}^+_{k_0} \to (0, +\infty)$ (the case $u(k) < 0$ is similar). Then by (1.1), (1.2) and Lemma 2.1, there exist $\ell \in \{1, \dots, n-2\}$, such that $\ell + n$ is even and condition (2.1) holds. Since all conditions of the Theorem 4.2 are fulfilled for any $\ell \in \{1, \dots, n-2\}$ with $\ell + n$ even, we have $\ell \notin \{1, \dots, n-2\}$. Therefore, $\ell = n$ or n is even and $\ell = 0$. It is obvious that, since $\gamma \in (0, 1)$ by first condition of (4.2), satisfied the condition (5.1), (5.2). Therefore, analogously Theorem 5.1, by (4.2) and (5.1), (5.2) we can prove that condition(1.3) and (1.4) hold. That is, equation (1.1) has Property **B**.

Theorem 5.5. *Let conditions* (1.2), (3.1), (4.4), (4.5) *or if* $0 < \alpha\lambda < 1$, *then for some* $\varepsilon > 0$ *and* $\gamma \in (\lambda, 1)$

$$\sum_{k=1}^{\infty} k^{n-2+\frac{\alpha\lambda(1-\gamma)}{1-\alpha\gamma}} p(k) = +\infty. \tag{5.9}$$

and

$$\liminf_{k\to+\infty} k^{\gamma}\sum_{j=k}^{+\infty} j^{n-2}p(j) > 0. \tag{5.10}$$

Then equation (1.1) *has Property* **B**.

Proof. It is obvious that by (4.4), (4.5), (5.9) and (5.10) all conditions of the Theorem 5.4 hold, which proves the validity of the theorem.

Analogously we can prove

Theorem 5.6. *Let conditions* (1.2), (3.1), (4.4), (4.5), (5.7) *or if* $0 < \alpha\lambda < 1$, *then for some* $\varepsilon > 0$ *and* $\gamma \in (1, \lambda)$

$$\liminf_{\to+\infty} k^{\gamma}\sum_{j=k}^{+\infty} j(\sigma(j))^{\lambda(n-3)}p(j) > 0,$$

$$\sum_{k=1}^{+\infty} k^{1+\frac{\alpha\lambda((1-\gamma}{1-\alpha\gamma}}(\sigma(k))^{\lambda(n-3)}p(k) = +\infty$$

holds. Then equation (1.1), *has Property* **B**.

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Author's address:

I. Javakhishvili Tbilisi State University, Department of Mathematics
and I. Vekua Institute of Applied Mathematics
2, University St., Tbilisi 0143
Georgia
E-mail: r_koplatadze@yahoo.com
roman.koplatadze@tsu.ge