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ON ONE PROBLEM OF THE PLANE THEORY OF VISCOELASTICITY FOR A DOUBLY-CONNECTED DOMAIN BOUNDED BY POLYGONS

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Abstract. In this research the quasi-static boundary value problem of the coupled theory of elasticity for porous materials is examined. The problem of equilibrium of a spherical layer is reviewed and the explicit solution of the Dirichlet boundary value problem is given as absolutely and uniformly convergent series. The paper considers the problem of the plane theory of viscoelasticity for a doubly-connected domain bounded by convex polygons. It is assumed that absolutely smooth rigid punches are applied to the outer boundary while the inner polygon has a smooth washer whose dimensions are slightly different from the dimensions of the rectangle so that the boundary points receive constant normal displacements without friction. The problem consists of determining the corresponding complex potentials characterizing the equilibrium of the plate by the Kelvin-Voigt model.

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Statement of the problem

Let the viscoelastic plate on the plane z of a complex variable occupy a finite doubly-connected domain S, bounded by the convex polygons (A) and (B) (so we have the polygonal domain (A) with a hole (B)). we will denote by A_j $(j=\overline{1,n})$ and B_k $(k=\overline{1,m})$ the vertices (and their affixes), and $L_0^{(j)}$ and $L_1^{(k)}$ the sides of the polygons A and B. The values of the internal angles of the domain S at the vertices A_j and B_k will be denoted by $\pi\alpha_j^0$ and $\pi\beta_k^0$, while the angles between the x axis and outward normals to the contours $L_0=\bigcup\limits_{j=1}^{n} L_0^{(j)}$ and $L_1=\bigcup\limits_{k=1}^{m} L_1^{(k)}$ will be denoted by $\alpha(\sigma)$ and $\beta(\sigma)$, respectively. So $\alpha(\sigma)=\alpha_0^{(j)}=const,\ \beta(\sigma)=\beta_1^{(k)}=const,\ \sigma\in L_0\cup L_1$. It is assumed that absolutely smooth rigid punches are applied to the outer boundary are under the action of normal stresses with the principal vectors $N_j^{(0)}$ $(j=\overline{1,n})$ while the polygon (B) has a smooth washer whose dimensions are slightly different from the dimensions of the rectangle so that on the boundary $L_1^{(k)}$ $(k=\overline{1,m})$ are given normal displacements $v_n=v_1^{(k)}=const$ and without friction.

The problem consists of determining the corresponding complex potentials characterizing the distribution of stresses and displacements of the plate S by the Kelvin-Voigt model.

Solution of the problem

The problem is solved by the methods of conformal mappings and the theory of boundary value problems of analytic functions. Relying on the well-known KolosovMuskhelishvili's formulas, the problem formulated with respect to unknown complex potentials is reduced to the two RiemannHilbert problems for a circular ring.

In the view that a given domain is doubly connected, it is advisable to use functions $\Phi(z,t)$ and $\Psi(z,t)$ which are also one-valued in the case of a multiply connected domain, however, when composed boundary conditions containing boundary values of displacements, it is necessary to differentiate these values.

1. The function $z=\omega(\zeta)$ conformally maps the circular ring $D=\{1<|\zeta|<R\}$ onto the domain S. Its derivative is the solution of the Riemann-Hilbert problem for the circular ring D (see [1])

$$\mathrm{Re}[i\sigma e^{-i\gamma(\sigma)}\omega'(\sigma)]=0, \quad \sigma\in l_0\cup l_1, \tag{1}$$

$$l_0=\{|\zeta|=R\},\ l_1=\{|\zeta|=1\},\ \gamma(\sigma)=\alpha(\sigma),\ \sigma\in l_0;\ \gamma(\sigma)=\beta(\sigma),\ \sigma\in l_1,$$

and if

$$\prod_{k=1}^{n}\left(\frac{a_k}{R}\right)^{\alpha^0_{k-1}}\prod_{i=1}^{m}(b_i)^{\beta^0_{i-1}}=1,$$

is given by the formula

$$\begin{aligned}\omega'(\zeta)=K^0\prod_{k=1}^{n}\left(\tfrac{a_k}{R}\right)^{\frac{\alpha^0_k-1}{2}}\left(1-\tfrac{\zeta}{a_k}\right)^{\alpha^0_k-1}\prod_{i=1}^{m}\left(1-\tfrac{b_i}{\zeta}\right)^{\beta^0_i-1}\\ \times\prod_{j=1}^{\infty}\prod_{k=1}^{n}\left(1-\tfrac{\zeta}{R^{2j}a_k}\right)^{\alpha^0_k-1}\left(1-\tfrac{a_k}{R^{2j}\zeta}\right)^{\alpha^0_k-1}\\ \times\prod_{i=1}^{m}\left(1-\tfrac{\zeta}{R^{2j}b_i}\right)^{\beta^0_i-1}\left(1-\tfrac{b_i}{R^{2j}\zeta}\right)^{\beta^0_i-1}\end{aligned} \tag{2}$$

where K^0 is an arbitrary real constant.

2. The first and second basic boundary value problems of the viscoelasticity plane S for the Kelvin-Voigt linear model have the following forms

$$\varphi(\sigma,t)+\sigma\overline{\varphi'(\sigma,t)}+\overline{\psi(\sigma,t)}=i\int_0^s(X_n+iY_n)ds_0+c_1+ic_2, \tag{3}$$

$$\begin{aligned}&\textstyle\int_0^t[x^*e^{k(\tau-t)}+2e^{m(\tau-t)}]\varphi(\sigma,\tau)d\tau-\int_0^t e^{m(\tau-t)}[\varphi(\sigma,\tau)+\tau\overline{\varphi'(\sigma,\tau)}+\overline{\psi(\sigma,\tau)}]d\tau\\ &=2\mu^*(u+iv),\quad \sigma\in L=L_0\cup L_1,\end{aligned} \tag{4}$$

here and then the coordinate t is the parameter of the time.

From (3) and (4) we have

$$\Gamma\varphi(\sigma,t)=M\Big[i\int_0^s(X_n+iY_n)ds_0+c_1+ic_2\Big]+2\mu^*(u+iv), \tag{5}$$

where Γ and M are operators of the time t

$$\begin{aligned}\Gamma\varphi(\sigma,t)&=\int_0^t[(x^*e^{k(\tau-t)}+2e^{m(\tau-t)}]\varphi(\sigma,\tau)d\tau,\\ M[c(\sigma,t)]&=\textstyle\int_0^t e^{m(\tau-t)}C(\sigma,\tau)d\tau,\end{aligned} \tag{6}$$

Considering the equality

$$X_n+iY_n=(N+iT)e^{i\gamma(\sigma)};\ \ (\gamma(\sigma)=\alpha(\sigma),\ \ \sigma\in L_0;\ \gamma(\sigma)=\beta(\sigma),\ \ \sigma\in L_1),$$

$$u+iv=(v_n+iv_s)e^{i\gamma(\sigma)},\ \ T(\sigma)=0,\ \ \sigma\in L=L_0\cup L_1,\ \ v_n(\sigma)=v_n^k=const,$$

$$\sigma \in L_0^{(k)}; \ \ v_n(\sigma) = v_n^i = const, \ \ \sigma \in L_1^{(i)} \ \ (k = \overline{1,n}; \ \ i = \overline{1,m}),$$

from boundary conditions (3) and (4) we have

$$\begin{aligned} &Re[e^{-i\alpha(\sigma)}\Gamma\varphi(\sigma,t)] = f_0(\sigma,t), \ \ \sigma \in L_0, \\ &Re[e^{-i\beta(\sigma)}\Gamma\varphi(\sigma,t)] = f_1(\sigma,t), \ \ \sigma \in L_1, \end{aligned} \tag{7}$$

$$f_j(\sigma,t) = 2\mu^* v_n^{(j)} + M[C^{(j)}(\sigma)], \quad j = 0,1,$$

$$C^{(j)}(\sigma) = \text{Re}[i\int_0^s N^{(j)}(\sigma_0)e^{i[\gamma(\sigma_0)-\gamma(\sigma)]}ds_0 + e^{-i\gamma(\sigma)}[c_1^{(j)} + ic_2^{(j)}]$$

$$= \sum_{k=1}^{r(j)} N_k^{(j)} \sin[\gamma_{r^j}(\sigma) - \gamma_{r(j)}(\sigma_0)] + c_1^{(j)}\cos\gamma(\sigma) - c_2^j \sin\gamma(\sigma), \ \ \sigma \in L_j,$$

$j = 0,1$, $r^{(0)} = \overline{1,n}$, $r^{(1)} = \overline{1,m}$, $N_k^{(j)} = \int_{L_j^{(k)}} N(\sigma_0)ds_0$, $c_1^{(j)}$ and $c_2^{(j)}$ are arbitrary real constants.

Let's assume that at the initial moment of time the points of the contour L are given of the normal displacement $v_k(\sigma)$ (this state will be considered starting) this state will be assumed to be the initial one and is subsequently held in this position (so $v_k(\sigma) = 0$, when $t \geq 0$), and the corresponding stresses are determined through the complex potentials $\Phi(z) = \varphi'(z,t)$ and $\Psi(z) = \psi'(z,t)$. With that said, it is easy to notice that functions $f_j(z,t)$ $(j = 0,1)$ are written in the forms

$$f_j(\sigma,t) = C^{(j)}(\sigma)F(t), \quad F(t) = \frac{1}{m}[1 - e^{-mt}], \ \ j = 0,1.$$

Introducing the notation

$$\varphi_1(z,t) = \frac{1}{F(t)}\Gamma\varphi(z,t), \tag{8}$$

from (7) we obtain the boundary condition

$$\text{Re}[e^{-i\gamma(\sigma)}\varphi_1(\sigma,t)] = C^{(j)}(\sigma), \ \ \sigma \in L_j, \ \ j = 0.1, \tag{9}$$

however we will fined the solution of problem (9) in the class $h(a_1, a_2, ..., a_3)$ (see [5]).

Differentiating (9) with respect to the arc abscissa s and taking into account the fact the functions $\gamma(\sigma)$ and $C^{(j)}(\sigma)$ are piecewise constants, we obtain

$$\text{Re}i\Gamma\varphi_1'(\sigma,t) = 0, \quad \sigma \in L. \tag{10}$$

Let the function $z = \omega(\zeta)$ map conformally the domain S onto the circular ring D (see formula (2)). Introducing the notation $\varphi[\omega(\zeta),t] = \varphi_0(\zeta,t)$ and from the equality $\varphi_0'(\zeta,t) = \varphi'[\omega(\zeta),t]\cdot\omega'(\zeta)$ for the function

$$\Phi_0(\zeta,t) = \varphi_0'(\zeta,t) = \varphi'[\omega(\zeta),t]\omega'(\zeta) \tag{11}$$

we obtain for the circular ring D the following Riemann-Hilbert boundary value problem

$$\text{Re}\Big[\frac{i}{F(t)}\Gamma\Phi_0(\xi,t)\Big] = 0, \quad \xi \in l = l_0 \cup l_1. \tag{12}$$

The solution of problem (12) is given by the formula

$$\frac{1}{F(t)}\Gamma\Phi_0(\zeta,t)=K_0,$$

where K_0 is a real constant.

Based on what has been said and from (6) for determining $\Phi_0(\zeta,t)$ we obtain the equation

$$\varkappa^*\int_0^t e^{k(\tau-t)}\Phi_0(\zeta,\tau)d\tau+2\int_0^t e^{m(\tau-t)}\Phi_0(\zeta,t)d\tau=K_0F(t),$$

or

$$\varkappa^*\int_0^t e^{k\tau}\Phi_0(\zeta,\tau)d\tau+2e^{(k-m)t}\int_0^t e^{m\tau}\Phi_0(\zeta,\tau)d\tau=K_0e^{kt}F(t). \tag{13}$$

Differentiating (13) with respect to the variable t and the obtained equality pluses (13) which is multiplied by $m-k$ we have

$$\varkappa^*(m-k)\int_0^t e^{k\tau}\Phi_0(\zeta,\tau)d\tau+(\varkappa^*+2)e^{kt}\Phi_0(\zeta,t)=K_0e^{kt}. \tag{14}$$

From this, after differentiating with t, we easily obtain the differential equation

$$\dot{\Phi}_0(\zeta,t)+a\Phi_0(\zeta,t)=b, \tag{15}$$

where $\dot{\Phi}_0(\zeta,t)$ denotes derivatives in time t

$$a=\frac{\varkappa^*m+2}{\varkappa^*+2};\quad b=\frac{kK_0}{\varkappa^*+2}. \tag{16}$$

From (14) we have

$$\Phi_0(\zeta,0)=\frac{K_0}{\varkappa^*+2}. \tag{17}$$

The solution of the differential equation (15) for condition (16) has the form

$$\Phi_0(\zeta,t)=\frac{K_0}{\varkappa^*+2}[(k+a)e^{-at}-k], \tag{18}$$

where a is defined by formula (16).

After finding the function $\Phi_0(\zeta,t)$, for the definition of the function $\Psi_0(\zeta,t)=\psi_0'(\zeta,t)=\psi_0'(\omega(\zeta),t)$ we use the first basic condition which after the conformally mapping is rewritten in the form

$$\begin{aligned}&\Phi_0(\eta,t)+\overline{\Phi_0(\eta,t)}-\tfrac{\eta^2}{\omega'(\eta)}\left[\overline{\omega(\eta)}\Phi_0'(\eta,t)+\omega'(\eta)\Psi_0(\eta,t)\right]\\&=N(\eta,t)-iT(\eta,t),\quad \eta\in l.\end{aligned} \tag{19}$$

On the basis of (18) and the equation $T(\eta,t)=0$, from (19) we have

$$\mathcal{I}_m\Big[\frac{\eta^2}{\overline{\omega'(\eta)}}\omega'(\eta)\Psi_0(\eta,t)\Big]=0,\quad \eta\in l,$$

or

$$\frac{\eta\omega'(\eta)}{\overline{\eta}\overline{\omega'(\eta)}}\,\Psi_0(\eta,t)-\frac{\overline{\eta}\overline{\omega'(\eta)}}{\eta\omega'(\eta)}\,\overline{\Psi_0(\eta,t)}=0,\ \ \eta\in l, \tag{20}$$

From (20) for the function $\Omega(\zeta,t)=\zeta^2\omega'^2(\zeta)\Psi_0(\zeta,t)$ we obtain the Dirichlet problem for a circular ring

$$\mathcal{I}_m\Omega(\eta,t)=0,\ \ \eta\in l. \tag{21}$$

The solution of problem (21) has the form $\Omega(\zeta,t)=K_1$ (K_1 is the real constant) and for the function $\Psi_0(\zeta,t)$ we obtain the formula

$$\Psi_0(\zeta,t)=\frac{K_1}{\zeta^2\omega'(\zeta)}.$$

On the basis of (12) we have

$$\varphi'(\zeta,t)=\frac{\Phi_0(\zeta,t)}{\omega'(\zeta)}. \tag{22}$$

Based on the results obtained in (see [6]) near the angles points we have

$$z-B_k=(\zeta-b_k)^{\beta_k^0}\Omega_k(\zeta),\ \ (\Omega_k(b_k)\neq 0,\ \ k=\overline{1,m}), \tag{23}$$

and thus from (18), (2) and (23) near a point B (B is one of the point B_k $(k=\overline{1,m})$) we have the estimate

$$|\varphi'(z,t)|<M|z-B|^{\frac{1}{\beta_0}-1},\ \ M=const.$$

So near the points B_k $(k=\overline{1,m})$, $\varphi'(z,t)$ has integral singularity. Similarly it can be proved that near the points A_k $(k=\overline{1,n})$ the fuction $\varphi'(z,t)$ is bounded.

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