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SOME COMMENTS ON HIERARCHICAL MODELS

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Abstract. The main aim of the present comments is by quotations, brought from competent publications, to emphasize the importance of hierarchical models, their purpose and requirements for them.

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Let us start by quoting from the Encyclopedia of Computational Mechanics [1], pp. 2-10:

"1.4 Families of Problems

We will address two types of problems on our thin domain Ω^d: (i) find the displacement u solution to the equilibrium equation $div\sigma(u) = f$ for a given load f, (ii) Find the (smallest) vibration eigen-modes (Λ, u) of the structure. For simplicity of exposition, we assume in general that the structure is clamped (this condition is also called 'condition of place') along its lateral boundary Γ^d and will comment on other choices for lateral boundary conditions. On the remaining part of the boundary $\partial\Omega^d \backslash \Gamma^d$ ('top' and 'bottom') traction free condition is assumed."

"1.5 Computational obstacles

With the twofold aim of improving the precision of the models and their approximability by finite elements, the idea of hierarchical models becomes natural: Roughly, it consists of an Ansatz of polynomial behavior in the thickness variable, with bounds on the degrees of the three components of the 3-D displacement. The introduction of such models in variational form is due to Vogelius and Babuška (1981) and Szabo and Sahrmann (1988). Earlier beginnings in that direction can be found in Vekua (1955, 1965). The hierarchy (increasing the transverse degrees) of models obtained in that way can be discretized by the p-version of the elements."

3.1 The concepts of the hierarchical models

"The idea of hierarchical models is a natural and efficient extension to that of limiting models and dimension reduction. In the finite element framework, it has been firstly formulated in Szabo and Sahmann (1988) for isotropic domains, mathematically investigated in Babuška and Li (1991, 1992), and generalized to laminated composites in Babuška, Szabo and Actis (1992) and Actis, Szabo and Schwab (1999)."

"Any model that belongs to the hierarchical family has to satisfy three requirements; see Szabo and Babuška (1991) Chap. 14.5:

1. Approximability. At any fixed thickness $\varepsilon > 0$:

$$\lim_{q\to\infty} \|u^\varepsilon - u^{\varepsilon,q}\|_{E(\Omega^d)} = 0$$

2. Asymptotic consistency. For any fixed degree triple q:

$$\lim_{\varepsilon\to 0}\frac{\|u^{\varepsilon}-u^{\varepsilon,q}\|_{E(\Omega^d)}}{\|u^{\varepsilon}\|_{E(\Omega^d)}}=0$$

3. Optimality of the convergence rate. There exists a sequence of positive exponents $\gamma(q)$ with the growth property $\gamma(q)<\gamma(q')$ if $q\prec q'$, such that 'in the absence of boundary layers and edge singularities'

$$\|u^{\varepsilon}-u^{\varepsilon,q}\|_{E(\Omega^d)}\le C\varepsilon^{\gamma(q)}\,\|u^{\varepsilon}\|_{E(\Omega^d)}$$

$\|\cdots\|_{E(\Omega^d)}$ means the strain energy norm".

As we see, here the question is the convergence of the approximate solutions(models) to the corresponding 3D solutions (models), asymptotic consistency of the approximate models, and optimality of the convergence rate. There is no question of satisfying boundary conditions on the face surfaces (which could have caused violation of the hierarchicality models) of the plate, just as in the Kirchhoff-Love model it is not required to satisfy the boundary conditions on face surfaces of the plate, but here the effect of the load is significant (it is taken into account in the right-hand side of the governing equation) along with satisfaction (which is approximate as well in the above sense) of boundary conditions at the boundary of the plate (i.e., on the lateral boundary of the 3D body).

The accuracy of the approximation was the interest of I. Vekua in [2] (see §11), that is why he raised the question in such a form:

Теперь, очевидно, встает вопрос, **в какой мере** приближения вида

$$U_N(x^1,x^2,x^3)=\sum_{k=1}^{N}\overset{(k)}{U}(x^1,x^2)P_k\left(\frac{x^3-\bar h}{h}\right),\tag{1}$$

и

$$P^i(U_N)=\sum_{k=1}^{N}\overset{(k)}{P^i}(x^1,x^2)P_k\left(\frac{x^3-\bar h}{h}\right),\tag{2}$$

удовлетворяют краевым условям на лицевых поверхностях S^+ и S^-, где мы считаем заданными напряжения $\overset{(+)}{P}$ и $\overset{(-)}{P}$ соответственно, i.e., "**in which measure**" approximations (1) and (2) satisfy boundary conditions on the face surfaces S^+ and S^-, where we assume prescribed tractions $\overset{(+)}{P}$ and $\overset{(-)}{P}$, respectively.

In other words, I. Vekua was interested in the accuracy of approximations (1) and (2) constructed by him in §11, "**in which measure**" they met the boundary condition of the face surfaces, to which he himself responded, but was not satisfied and presented a paper of D. Gordeziani [3], which was dedicated to the accuracy of the approximate solutions constructed by Vekua, for publication in the Journal "Reports of the Academy of Sciences of the Soviet Union". This paper is cited in [1] as well, where it is also cited the paper of M. Avalishvili and D. Gordeziani [4], which is dedicated to the same problem and a paper of Schwab [5] in which, Vekua's hierarchical models for the plates of constant thickness are investigated in accordance with the requirements, mentioned at the beginning of the present comments. Here should be also mentioned the following articles of D. Gordeziani, G. Avalishvili, and M. Avalishvili [6,7].

Remark. It should be noted that by means of traditional methods (e.g., based on Korn's inequality) used for investigation of hierarchical models it is not possible to reveal completely peculiarities of setting the boundary conditions on cusped edges (ends). The reason is that in this case governing equations are singular differential equations and peculiarities of setting the boundary conditions. Besides, the principal part of equations depend on values of coefficients of less order derivatives on the boundary, belonging to the line of degeneration of equations (see [8]) or on behaviour of that in a neighbourhood (see [9]) of the line of degeneration of equations.

Now, in the spirit of the present paper we shortly discuss a question of satisfaction of boundary conditions on the so called face surfaces within the framework of 2D models. Let us start with the simplest model of plane deformation in a finite cylinder. In this case in order to maintain the plane deformation we are forced to apply $+\sigma_{33}$ and $-\sigma_{33}$, which are calculated after solving the 2D BVP under BCs on the lateral surface of the cylinder, at top and bottom bases, which play a role of the face surfaces, of the cylinder, respectively. Therefore, they cannot be prescribed arbitrarily.

Now, a quotation from the preface of the book "Theories of Plates and Shells, Critical Review and new Applications [10], p. VII:

"Plate and shell theories are inherently approximative in character since the in-fact three-dimensional state of stress and deformation is described by quantities, which live on a two-dimensional surface. Therefore the establishment of consistent theories attracted broad attention."

About consistent plate theories, we follow R. Kienzler, D.K. Rose [11], pp. 147, 148, and R. Kienzler [12], and R. Kienzler [13] pp. 94, 95."

"Whereas Kirchoff's plate theory is rather well established since more than 150 years, various version of plate theories exist, which take shear deformations and change-of-thickness effects into account cf, e.g., Wang, Reddy 2000. Most prominently, Mindlin's and Reissner's theories are involved. The number of engineering (sometimes self-contradictory) a priori assumptions necessary to establish these theories can be reduced considerably by the application of the consistent approximation technique [14]. The idea is to approximate the complete set of governing equations uniformly, i.e., to the same degree of accuracy. This method has been extended [12,13] by the proper demand that not only during the derivation of the equations, terms of a certain order of magnitude are to be retained (and terms of higher order to be neglected) but also during the reduction of the equation system using an elimination process. All theories in common is an integration process with respect to the plate thickness giving rize to a plate parameter $c^2 = h^2/(12a^2)$ (h is the characteristic length in thickness direction assumed to be constant throughout the paper, and a is the characteristic in-plane dimension), This plate parameter is usually small $c^2 << 1$. If only terms up to the zeroth order $O[(c^2)^0]$ are considered, only rigid body motions of the plate are admitted. The consistent first-order approximation $O[(c^2)^1]$ delivers exactly Kirchhoffs theory limiting effective shear forces for a proper formulation of the boundary conditions and the second-order approximation $O[(c^2)^2]$ results in a consistent using deformable plate theory. It has been shown that various higher-order plate theories coincide within the second-order approximation [12,13]".

"A Consistent Second-order Plate Theory: in a first step, the displacements are expanded in thickness direction into a power series involving, among others, the transvers displacements in the plate-middle-surface w and the change of the slope of the straight line normal to the undeformed midplane ψ_α $(\alpha = 1, 2,)$. Comparing coefficients of equal order in x_3, a series expansion of the strain tensor tends into the kinematic relations. With these, the strain-energy density W and the potential of external forces V can be calculated".

"With the differential equations

$$\frac{K}{a^2}\Delta\Delta w = a\Big[{}^0P + \frac{c^2}{10(1-\nu)}(5\nu\Delta^2 P - 6(4+\nu)\Delta\,{}^0P)\Big], \tag{3}$$

and

$$c^2\left(\psi - \frac{3}{2}c^2\Delta\psi\right) = 0, \tag{4}$$

the boundary conditions

$$\begin{aligned} M^*_{nn} &= M_{nn} \;\text{ or }\; \psi^*_n = -(1+c^2k\Delta)w - \frac{6}{5}c^2\psi^*, \\ M^*_{nt} &= M_{nt} \;\text{ or }\; \psi^*_t = -(1+c^2k\Delta)w^* + \frac{6}{5}c^2\psi', \\ Q^*_n &= Q_n \;\text{ or }\; w^* = (1+3c^2\varepsilon\Delta)w. \end{aligned} \tag{5}$$

and the equations for the relevant stress resultants

$$\begin{aligned} M_{\alpha\beta} = -\frac{K}{a}\Big[&(1+kc^2\Delta)(1-\nu)w,_{\alpha\beta} + \nu\delta_{\alpha\beta}w,_{\gamma\gamma} \\ &+\frac{3}{5}(1-\nu)(\varepsilon_{\alpha\gamma}\psi,_{\beta\gamma} + \varepsilon_{\beta\gamma}\psi,_{\alpha\gamma})\Big] + \delta_{\alpha\beta}c^2a^2\varepsilon\tilde{P}, \end{aligned}$$

$$Q_\alpha = -\frac{K}{a^2}\Big[(1+kc^2\Delta)\Delta w,_\alpha + \frac{2}{5}(1-\nu)\varepsilon_{\alpha\beta}\psi,_\beta\Big] + c^2a\varepsilon\tilde{P},_\alpha\,,$$

we have established a consistent second-order plate theory without introducing any a priori assumptions. Hereby we used the displacement representation

$$u_\alpha = a(\psi_\alpha\zeta + \Psi_\alpha\zeta^3 + \mathcal{H}_\alpha\zeta^5),$$

$$u_3 = a(w - \omega\zeta^2 + \Omega\zeta^4),$$

involving nine parameters w, ω, Ω, ψ_α, Ψ_α and $\mathcal{H}_\alpha$, or, alternatively w, ω, Ω, φ, ψ, Φ, Ψ, $\mathcal{P}$, and $\mathcal{H}$. It turns out that the quantities Ω, $\mathcal{P}$ and $\mathcal{H}$ or Ω and $\mathcal{H}_\alpha$ are not specified by the calculation. They can be chosen arbitrarily, without changing the governing equations, i.e. a cubic representation in ζ with the six parameters w, ω, φ, ψ, Φ, and Ψ (or w, ω, ψ_α, Ψ_α) would be sufficient. In addition we have

$$c^2\Phi = c^2\frac{1}{6}\frac{2-\nu}{1-\nu}\Delta\Delta w + c^4\hat{\Phi},$$

$$c^2\Psi = -c^2\frac{1}{9}\psi + c^4\hat{\Psi},$$

and it follows also that specific restriction is put on the terms multiplied by c^4, i.e., $\hat{\Phi}$ and $\hat{\Psi}$. Instead of neglecting the terms Ω, $\mathcal{P}$, $\mathcal{H}$, $\hat{\Phi}$ and $\hat{\Psi}$ one can use them to satisfy a posteriori the boundary conditions for the tractions on the plate faces $x_3 = \pm h/2$.

The uniform-approximation technique leads to self-consistent plate theories of any desired degree of accuracy without the necessity of introducing any a priori assumptions or neglections. The second-order theory, discussed in the paper, is governed by two uncoupled differential equations (3), (4), one of fourth order in the transverce displacement w and one of second order in ψ, the quantity describing the influence of shear deformation. Along the boundary, three boundary conditions (5) have to be satisfied to guarantee a unique solution".

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