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ON A PROBLEM OF INTEGER VALUED OPTIMIZATION

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Abstract. We obtain an expression for the maximal value of the product of finite sequence of positive integers when their sum is fixed. We show also that this value in general is strictly less than the estimation obtained by means of the classical mean arithmetic-mean geometric inequality.

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1. Introduction

In historical notes to Chapter 1 of [1] it is written: "The concepts of arithmetic mean and geometric mean of two positive quantities go back to antiquity, in particular the pythagoreans made of them one of their favorite subjects. It is also probable that the inequality

$$\sqrt{ab} \leq \frac{1}{2}(a+b)$$

between these means was well-known to them; in any case it is proved by Euclid in connection with the problem of maximizing the product of two numbers whose sum is given."

The mentioned problem for not necessarily two numbers can be formulated as follows:

Let $n>1$ be a natural number, $L>0$ and $x_i>0, i=1,\dots,n$ real numbers; maximize $\prod_{i=1}^n x_i$ when $\sum_{i=1}^n x_i = L$.

A solution of this *extremum problem* is contained in the following version of the arithmetic mean- geometric mean inequality, for short, the *AM-GM inequality*:

Let $n>1$ be a natural number, $L>0$ and let $x_i>0, i=1,\dots,n$ be real numbers with $\sum_{i=1}^n x_i = L$. Then

$$\prod_{i=1}^n x_i \leq \left(\frac{L}{n}\right)^n, \tag{1}$$

and the equality we have if and only if $x_i = \frac{L}{n}$, $i=1,\dots,n$. There are many different proofs of this inequality; e.g., in [7] is presented O. L. Cauchy's proof (it is mentioned also in [8], however it is not mentioned in [5]); 74 proofs are included in [2]. One more proof will appear also in [4].

Seemingly the second author of this paper was the first who considered the above-mentioned maximization problem for natural numbers; he observed that for them the bound $\left(\frac{L}{n}\right)^n$ was not the best possible and found the correct bound. His result first appeared in (unpublished) preprint [3] and then (in a slightly more general setting) in [6].

In the present paper we formulate an assertion which covers the results of [3] and of [6] and shows also, that in general *the sharper bound* $\left[\left(\frac{L}{n}\right)^n\right]$ *may not be the best possible as well.*

2. Main result and remarks

Our assertion looks as follows:

Theorem 1. Let natural numbers L, n and an integer $k \geq 0$ be such $d := L - n(1+k) \geq 0$. Let, moreover, $[q]$ be the integer part of $q := \frac{L}{n}$ and $r := L - n[q]$. Then the following statements are valid.

(a)

$$\mathbb{Z}_+(L,n;k) := \left\{(x_1,\dots,x_n) \in \mathbb{Z}_+^n \,:\, \sum_{i=1}^n x_i = L,\, x_i \geq k, i = 1,\dots,n\right\}$$

is a finite non-empty set and

$$b(L,n;k) := \max\left\{\prod_{i=1}^n x_i : (x_1,\dots,x_n) \in \mathbb{Z}_+(L,n;k)\right\} \geq (1+k)^{n-1}(1+k+d)\,.$$

(b) We have the equality:

$$b(L,n;k) = (1+[q])^r [q]^{n-r}\,. \tag{2}$$

(c) If $r = 0$, then

$$b(L,n;k) = q^n \tag{3}$$

and if $r > 0$, then

$$b(L,n;k) \leq [q^n]\,. \tag{4}$$

Proof. (a) is easy to verify. (b) is proved in [6] (see also [4]). (c): (3) follows from (2); an easy derivation of (4) from AM-GM-inequality (1) is omitted.

Remark 1. The inequality (4) from Theorem 1 can be rewritten as follows:

$$(1+[q])^r[q]^{n-r} \leq \left[\left([q] + \frac{r}{n}\right)^n\right] \tag{5}$$

and the equality in (5) takes place only in the following cases:
Case 1. $n = 2$.
Case 2. $n = 3$ and $L \in \{4,5,7,8\}$.
Case 3. $n > 3$ and $L = n+1$.

The proof of this statements is contained in the next remark.

Remark 2. Let $n > 1$ and $1 \leq r < n$ be natural numbers. Then we have:

$$(1+x)^r x^{n-r} \leq \left[\left(x + \frac{r}{n}\right)^n\right],\ x = 1,2,\dots \tag{6}$$

and the following statements give a complete description set

$$V_{n,r} := \left\{x \in \mathbb{N} : (1+x)^r x^{n-r} = \left[\left(x + \frac{r}{n}\right)^n\right]\right\}\,.$$

(a) $V_{2,1} = \mathbb{N}$, $V_{3,1} = V_{3,2} = \{1,2\}$ and $V_{n,1} = \{1\}$, $n = 4,5,\dots$

(b) $n > 3, r > 1 \implies V_{n,r} = \emptyset$.

Proof of Remark 2. Clearly, (6) is equivalent to (5) and we also have:

$$V_{n,r} = \{x \in \mathbb{N} : \left(x + \frac{r}{n}\right)^n - (1+x)^r x^{n-r} < 1\}\,; \tag{7}$$

$$\left(x+\frac{r}{n}\right)^n-(1+x)^r x^{n-r}=\sum_{k=0}^{n}\left(\frac{r^k}{n^k}C_{n,k}-C_{r,k}\right)x^{n-k}\,, \tag{8}$$

where $C_{n,k}:=\frac{n!}{k!(n-k)!}$ and $C_{r,k}=0$ if $k>r$;

$$\frac{r^k}{n^k}C_{n,k}-C_{r,k}=0\,,\ k=0,1 \tag{9}$$

$$\frac{r^k}{n^k}C_{n,k}-C_{r,k}>0\,,\ k=2,\dots,n\,. \tag{10}$$

(The equalities (7), (8) and (9) are easy to verify; (10) is equivalent to the inequality

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right)>\left(1-\frac{1}{r}\right)\left(1-\frac{2}{r}\right)\dots\left(1-\frac{k-1}{r}\right),$$

which is of course true when $r<n$.)

Proof of Remark 2 (*a*).
The equalities $V_{2,1}=\mathbb{N}$ and $V_{3,1}=V_{3,2}=\{1,2\}$ are easy to verify by using (7).

We have $1\in V_{n,1}$ from (7) and from the known inequality $\left(1+\frac{1}{n}\right)^n-2<1$. The remaining relation

$$x\ge 2,\ n>3\implies x\notin V_{n,1}$$

will follow from the next more general statement

$$x\ge 2,\ n>3\implies x\notin V_{n,r}\,. \tag{11}$$

To prove (11), note that from (8),(9) and (10) we can write

$$\left(x+\frac{r}{n}\right)^n-(1+x)^r x^{n-r}>\left(\frac{r^2}{n^2}C_{n,2}-C_{r,2}\right)x^{n-2}=\frac{x^{n-2}r}{2}\left(1-\frac{r}{n}\right).$$

From this relation we get (11). Indeed, if $n\ge 4$ and $x\ge 2$, then

$$\frac{x^{n-2}r}{2}\left(1-\frac{r}{n}\right)\ge 2^{n-3}\frac{r(n-r)}{n}$$

and since $r(n-r)$ achieves its minimum value at $r=1$ or at $r=n-1$ we have the estimation

$$\frac{x^{n-2}r}{2}\left(1-\frac{r}{n}\right)\ge 2^{n-3}\frac{r(n-r)}{n}\ge 2^{n-3}(1-\frac{1}{n})>1$$

and from (7) we conclude that $x\notin V_{n,r}$.

Proof of Remark 2 (*b*). From (11) we have that if $x\ge 2$, then $x\notin V_{n,r}$.

Now let us prove, that if $n\ge 4$ and $r>1$, then $1\notin V_{n,r}$ as well. For this purpose according to (7) it is sufficient to show that the following (*slightly unexpected*) inequality holds

$$n\ge 4,\ 1<r<n\implies\left(1+\frac{r}{n}\right)^n-2^r>1\,. \tag{12}$$

Therefore, it remains now to prove the next inequality:

$$n \geq 4,\ 1 < r < n \implies \left(\left(1 + \frac{r}{n}\right)^{\frac{n}{r}}\right)^r - 2^r > 1\,. \tag{13}$$

It is clear that for given r the left-hand side of (13) increases as n increases. Let us consider separately two cases $r = 2$ and $r \geq 3$.

If $r = 2$ it is enough to check that (13) is true when $n = 4$. Plugging in (13) $r = 2$ and $n = 4$ we are getting the true inequality $\frac{17}{16} > 1$ and hence it's over.

For $r \geq 3$ it is enough to show that (13) holds when $n = r + 1$. In this case the left-hand side of (13) will be

$$\left(2 - \frac{1}{r+1}\right)^{r+1} - 2^r = \frac{C_{r+1,2}2^{r-1}}{(r+1)^2} - \frac{C_{r+1,3}2^{r-2}}{(r+1)^3} + \sum_{k=4}^{r+1} C_{r+1,k}2^{r+1-k}\frac{(-1)^k}{(r+1)^k}\,.$$

Since the sum $\sum_{k=4}^{r+1} C_{r+1,k}2^{r+1-k}\frac{(-1)^k}{(r+1)^k}$ is nonnegative, it is enough to show that $\frac{C_{r+1,2}2^{r-1}}{(r+1)^2} - \frac{C_{r+1,3}2^{r-2}}{(r+1)^3} > 1$. We have

$$\frac{C_{r+1,2}2^{r-1}}{(r+1)^2} - \frac{C_{r+1,3}2^{r-2}}{(r+1)^3} = 2^{r-2}\left(1 - \frac{1}{r+1}\right)\left(\frac{5}{6} + \frac{1}{3(r+1)}\right) > 2^{r-2}\left(1 - \frac{1}{r+1}\right)\frac{5}{6}$$

$$\geq 2 \times \frac{3}{4} \times \frac{5}{6} = \frac{15}{12} > 1$$

and so, (13) holds when $n = r + 1$. □

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