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EXPLICIT SOLUTION OF THE DIRICHLET TYPE BOUNDARY VALUE PROBLEM OF ELASTICITY FOR THE POROUS SPHERICAL LAYER

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Abstract. In this research the quasi-static boundary value problem of the coupled theory of elasticity for porous materials is examined. The problem of equilibrium of a spherical layer is reviewed and the explicit solution of the Dirichlet boundary value problem is given as a absolutely and uniformly convergent series.

Keywords and phrases: Porous spherical ring, explicit solutions, quasi-static theory.

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1. Introduction

In most of naturally or manufactured solids is not completely filled. In nearly every body there are empty interspaces, which are called pores through which the liquid or gas may flow. Many materials such as rocks, sand, soil etc., which occur on and below the surface of the earth, are known as porous materials. In some bodies there are immediately visible, in others the pores are recognized only with a magnifier. For example, the human skin has a larger number of pores, bone tissue could be assumed to be transversely isotropic and most closely describes mechanical anisotropy of bone and cancellous bone is considered as a porous material.

The foundations of the theory of elastic materials with voids were first proposed by Cowin and Nunziato [1,2]. They investigated the linear and nonlinear theories of elastic materials with voids. In these theories the independent variables are displacement vector field and the change of volume fraction of pores. Such materials include, in particular, rocks and soils, granulated and some other manufactured porous materials.

Elastic materials which contain a multi-porous structure has a multitude of applications in real life. The history of development of porous body mechanics, the main results and the sphere of their application are set forth in detail in the monographs [3-6] (see references therein). The generalization of the theory of elasticity and thermoelasticity for materials with double void pores belongs to Ieşan and Quintanilla [7]. In [8] Svanadze consider the coupled linear model of porous elastic solids by combining the following three variables: the displacement vector field the volume fraction of pores; and the pressure of the fluid. The basic internal and external BVPs (boundary value problems) of steady vibrations are investigated, Green's formulas are obtained, the uniqueness and the existence theorems are proved by means of the potential method and the theory of singular integral equations (see references therein). In [9] the coupled linear quasi-static theory of elasticity for porous materials is considered. The fundamental solution is constructed, and its basic properties are established. Green's formulas are obtained, and the uniqueness theorems of the internal and external boundary value problems are proved, the existence theorems for classical solutions of the BVPs are proved by means of the potential method and the theory of singular integral equations.

For applications, it is especially important to construct the solutions of boundary value problems in an explicit form because such solutions enable us to effectively perform quantitative analysis of the investigated problem. Many researchers have studied questions related to this topic, for example, in [10-30], where the explicit solutions are constructed for some boundary value problems of porous elasticity for the concrete domains, by applying different methods, such as analytical, numerical, ect.

In this paper the BVP of coupled linear quasi-static theory of elasticity is considered for an isotropic porous elastic materials.The paper is outlined as follows: in Section 2, the basic equations are presented in terms of the displacement vector field, the changes of volume fraction of pores, and fluid pressure in the pore network. The Dirichlet BVP is formulated for a spherical layer. In Section 3, some basic theorems, which are useful in the sequel, are given. In Section 4, the solution of the Dirichlet type BVP for a spherical layer is obtained in the form of absolutely and uniformly convergent series.

2. Basic equations. Formulation of the problem

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point in the Euclidean three-dimensional space E_3. Let us assume that D is a spherical layer, $R_1 < |\mathbf{x}| < R_2$, centered at point $O(0,0,0)$ in the space E_3, S_1 is a spherical surface of radius R_1, S_2 is a spherical surface of radius R_2 and $S = S_1 \cup S_2$. Let us assume that the domain D is filled with an isotropic porous materials.

Following Svanadze [8] and Mikelashvili [9] the basic system of equations of motion in the coupled linear quasi-static theory of elasticity for porous elastic materials expressed in terms of the displacement vector $\mathbf{u}$, changes of volume fraction of pores φ and the change of fluid pressure in pore network p has the following form [8, 9]

$$\begin{cases} \mu\Delta\mathbf{u} + (\lambda+\mu)\mathrm{graddiv}\mathbf{u} + b\mathrm{grad}\varphi - \beta\mathrm{grad}p = 0, \\ \\ (\alpha\Delta - \alpha_1)\varphi - b\ \mathrm{div}\mathbf{u} + mp = 0, \\ \\ (k\Delta + i\omega\ a)p + i\omega\beta\ \mathrm{div}\mathbf{u} + i\omega\ m\varphi = 0, \end{cases} \tag{1}$$

where $\mathbf{u} := (u_1, u_2, u_3)^\top$, λ and μ are the Lamé constants, β is the effective stress parameter, $k = \frac{k'}{\mu'}$, μ' is the fluid viscosity, k' is the macroscopic intrinsic permeability associated with the pore network, α, b, m, α_1, are constitutive coefficients, the value a measures the compressibility of pores, $\omega > 0$ is the oscillation frequency, i is the imaginary unit, Δ is the Laplacian. Throughout this paper the superscript $^\top$ stands for the transpose operation.

Definition. A vector-function $\mathbf{U} = (\mathbf{u}, \varphi, p)$ defined in the domain D is called regular if

$$\mathbf{U} \in C^2(D) \cap C^1(\overline{D}).$$

For the equations (1) we consider the following basic BVP.

Problem 1. Find in the domain D a regular solution $\mathbf{U} = (\mathbf{u}, \varphi, p)$ of equations (1) with the boundary conditions on S :

$$\begin{aligned} &\mathbf{u}^-(\mathbf{z}) = \boldsymbol{F}^-(\mathbf{z}), \quad \varphi^-(\mathbf{z}) = f_4^-(\mathbf{z}), \quad p^- = f_5^-(\mathbf{z}), \quad \rho = R_1, \\ &\mathbf{u}^+(\mathbf{z}) = \boldsymbol{F}^+(\mathbf{z}), \quad \varphi^+(\mathbf{z}) = f_4^+(\mathbf{z}), \quad p^+ = f_5^+(\mathbf{z}), \quad \rho = R_2. \end{aligned}$$

where the vector-function $\mathbf{F}(\mathbf{z}) = (f_1, f_2, f_3)$ and the functions $f_4(\mathbf{z})$, $f_5(\mathbf{z})$ are given functions on S, at $\mathbf{z}$. Moreover we assume that the boundary values are absolute integrable functions. The symbol $\mathbf{U}^+(\mathbf{U}^-)$ denotes the limits of $\mathbf{U}(\mathbf{x}) = (\mathbf{u}, \varphi, p)$ on $\mathbf{z} \in S$ from D

$$\mathbf{U}^+(\mathbf{z}) = \lim_{D \ni \mathbf{x} \to \mathbf{z} \in S_2} \mathbf{U}(\mathbf{x}), \quad \mathbf{U}^-(\mathbf{z}) = \lim_{D \ni \mathbf{x} \to \mathbf{z} \in S_1} \mathbf{U}(\mathbf{x}).$$

We assume that

$$\mu > 0, \quad \lambda + 2\mu > 0, \quad k > 0, \quad a > 0, \quad \alpha > 0, \quad (3\lambda + 2\mu)\alpha_1 > 3b^2.$$

The following assertion holds (for details see [9]).

Theorem 1. *The Dirichlet type BVP has at most one regular solution in the finite domain D.*

3. Preliminaries

In this section, some basic theorems from [13], which are useful in our subsequent, are given and we cite it without proof.

Theorem 2. *If $\boldsymbol{U} := (\boldsymbol{u}, \varphi, p)$ is a regular solution of the homogeneous system (1) then $\boldsymbol{u}$, $\operatorname{div}\boldsymbol{u}$, φ and p satisfy the following equations*

$$\Delta\Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)\mathbf{u} = 0, \ \Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)\mathbf{\Phi} = 0, \tag{2}$$

where $\mathbf{\Phi} = (\operatorname{div}\mathbf{u}, \varphi, p)$, λ_j^2, $\quad j = 1, 2$ are roots of algebraic equation

$$\alpha\mu_0 k\xi^2 - A_1\xi + i\omega A_2 = 0, \quad \mu_0 = \lambda + 2\mu, \ A_1 = \mu_0(a\alpha i\omega - \alpha_1 k) + kb^2 + i\omega\alpha\beta^2,$$

$$A_2 = \mu_0(-\alpha_1 a - m^2) + ab^2 - \alpha_1\beta^2 + 2bm\beta.$$

We may assume without loss of generality that $Im\lambda_j^2 > 0$ and λ_j^2 are distinct and different from zero [9].

Theorem 3. *The regular solution $\boldsymbol{U} = (\boldsymbol{u}, \varphi, p)$ of the system (1) admits a representation*

$$\begin{cases} \mathbf{u} = \mathbf{\Psi} - \operatorname{grad}\left[(k_0 - 1)h_0 + \sum\limits_{j=1}^{2} \dfrac{h_j}{\lambda_j^2}\right], \\[2ex] \varphi = B_0 h + \sum\limits_{j=1}^{2} B_j h_j, \\[2ex] p = C_0 h + \sum\limits_{j=1}^{2} C_j h_j, \end{cases} \tag{3}$$

where

$$\begin{cases} \operatorname{div}\mathbf{u} = h + \sum\limits_{j=1}^{2} h_j, \quad \operatorname{div}\mathbf{\Psi} = k_0 h, \quad \Delta h_0 = h, \quad \Delta h = 0, \quad (\Delta + \lambda_j^2)h_j = 0, \\[2ex] B_0 = \dfrac{ab + m\beta}{\delta_0}, \quad C_0 = \dfrac{\beta\alpha_1 - mb}{\delta_0}, \quad k_0 = \dfrac{A_2}{\mu\delta_0}, \\[2ex] B_j = \dfrac{i\omega\delta_0 B_0 - bk\lambda_j^2}{\delta_j}, \quad C_j = i\omega\dfrac{\delta_0 C_0 + \alpha\beta\lambda_j^2}{\delta_j}, \quad j = 1, 2, \\[2ex] \delta_0 = -a\alpha_1 - m^2, \quad \delta_j = -(\alpha_1 + \alpha\lambda_j^2)(i\omega a - k\lambda_j^2) - i\omega m^2, \\[2ex] \mu_0 + bB_0 - \beta C_0 = \dfrac{A_2}{\delta_0}, \quad bB_J - \beta C_j = -\mu_0. \end{cases} \tag{4}$$

Herein it is assumed that, the functions $\boldsymbol{\Psi}$ and h_0 are interrelated by the following relations:

$$\Delta\boldsymbol{\Psi} = 0, \quad \mathrm{div}\boldsymbol{\Psi} = k_0 h, \quad \Delta h_0 = h, \quad \Delta h = 0.$$

From relations (3) we conclude that the representation of a solution of $\mathbf{u}$ contains a harmonic, bi-harmonic, and a meta-harmonic functions, while the representations of φ and p contain only a harmonic and a meta-harmonic functions.

4. Explicit solution of Problem 1

Let us introduce the spherical coordinates and the following notations:

$$\begin{cases} x_1 = \rho\sin\xi\cos\eta, \quad x_2 = \rho\sin\xi\sin\eta, \quad x_3 = \rho\cos\xi, \quad x \in D, \\ y_1 = R\sin\xi_0\cos\eta_0, \quad y_2 = R\sin\xi_0\sin\eta_0, \quad y_3 = R\cos\xi_0, \quad y \in S, \\ |\mathbf{x}| = \rho = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad 0 \le \xi \le \pi, \quad 0 \le \eta \le 2\pi \quad 0 \le \rho \le R. \end{cases} \tag{5}$$

If $\mathbf{g}(\mathbf{x}) = \mathbf{g}(g_1, g_2, g_3)$ and $\mathbf{q}(\mathbf{x}) = \mathbf{q}(q_1, q_2, q_3)$, then by symbols $(\mathbf{g}.\mathbf{q})$ and $[\mathbf{g}.\mathbf{q}]$ we mean the scalar product and vector product of the two vectors, respectively. The operator $\frac{\partial}{\partial S_k(x)}$ is defined as follows:

$$[\mathbf{x}\cdot\nabla]_k = \frac{\partial}{\partial S_k(x)}, \qquad k = 1, 2, 3, \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right).$$

Making use of the identity, $(\mathbf{x}\cdot\mathrm{grad}) = \rho\frac{\partial}{\partial\rho}$, from (3) we obtain

$$\begin{cases} (\mathbf{x}\cdot\mathbf{u}) = (\mathbf{x}\cdot\boldsymbol{\Psi}) - \rho\dfrac{\partial}{\partial\rho}\left[(k_0 - 1)h_0 + \displaystyle\sum_{j=1}^{2}\frac{h_j}{\lambda_j^2}\right], \\ \varphi = B_0 h + \sum\limits_{j=1}^{2} B_j h_j, \quad p = C_0 h + \sum\limits_{j=1}^{2} C_j h_j. \end{cases} \tag{6}$$

By direct calculation it can be shown, that the function $(\mathbf{x}\cdot\boldsymbol{\Psi})$ is a solution of the equation

$$\Delta(\mathbf{x}\cdot\boldsymbol{\Psi}) = 2\mathrm{div}\ \boldsymbol{\Psi} = 2k_0 h.$$

From here we yield, that

$$(\mathbf{x}\cdot\boldsymbol{\Psi}) = \Omega + 2k_0 h_0, \tag{7}$$

where Ω is an arbitrary harmonic function $\Delta\Omega = 0$ and the function h_0 is a bi-harmonic function and chosen so that $\Delta h_0 = h$.

Substituting in (6), expression (7), we get

$$\begin{cases} (\mathbf{x}\cdot\mathbf{u}) = \Omega + 2k_0 h_0 - \rho\dfrac{\partial}{\partial\rho}\left[(k_0 - 1)h_0 + \displaystyle\sum_{j=1}^{2}\frac{h_j}{\lambda_j^2}\right], \\ \varphi = B_0 h + \sum\limits_{j=1}^{2} B_j h_j, \quad p = C_0 h + \sum\limits_{j=1}^{2} C_j h_j, \quad \sum\limits_{k=1}^{3}\dfrac{\partial u_k}{\partial S_k(\mathbf{z})} = \displaystyle\sum_{k=1}^{3}\frac{\partial\Psi_k}{\partial S_k(\mathbf{z})}.. \end{cases} \tag{8}$$

In the sequel we us the following notation

$$(\mathbf{z}\cdot\mathbf{F})^{\pm}=q_1^{\pm}(\mathbf{z}),\quad (\mathrm{div}\mathbf{F})^{\pm}=q_2^{\pm}(\mathbf{z}),\quad \sum_{k=1}^{3}\frac{\partial u_k}{\partial S_k(\mathbf{z})}=q_3^{\pm}. \tag{9}$$

Let us assume that functions q_k, $k=1,2$, and f_k, $k=4,5$ are representable in the form of the series

$$q_k(\mathbf{y})=\sum_{n=0}^{\infty}q_{kn}(\xi_0,\eta_0), \tag{10}$$

where q_{kn}, $k=1,2,3$ are the spherical harmonics of order n

$$q_{kn}=\frac{2n+1}{4\pi}\iint_S P_n(\cos\gamma)q_k(\mathbf{y})dS_y,$$

P_n is a Legender polynomial of the n-th order, and γ is an angle formed by radius-vectors Ox and Oy,

$$\cos\gamma=\frac{1}{|\mathbf{x}||\mathbf{y}|}\sum_{k=1}^{3}x_ky_k.$$

From (8), passing to the limit as $\rho\to R_1$, $\rho\to R_2$ and taking into account the boundary conditions for determining the unknown values h, h_j and Ω, we obtain the following system of algebraic equations:

for $\rho=R_1$

$$\begin{cases}\Omega^{-}+2k_0h_0-R_1\dfrac{\partial}{\partial\rho}\left[(k_0-1)h_0+\displaystyle\sum_{j=1}^{2}\frac{h_j}{\lambda_j^2}\right]=q_1^{-}, \quad \displaystyle\sum_{k=1}^{3}\frac{\partial u_k}{\partial S_k(\mathbf{z})}=q_3^{-},\\[2ex] B_0h^{-}+\sum_{j=1}^{2}B_jh_j^{-}=f_4^{-},\quad C_0h^{-}+\sum_{j=1}^{2}C_jh_j^{-}=f_5^{-},\quad h^{-}+\sum_{j=1}^{2}h_j^{-}=q_2^{-},\end{cases} \tag{11}$$

for $\rho=R_2$

$$\begin{cases}\Omega^{+}+2k_0h_0-R_2\dfrac{\partial}{\partial\rho}\left[(k_0-1)h_0+\displaystyle\sum_{j=1}^{2}\frac{h_j}{\lambda_j^2}\right]=q_1^{+}, \quad \displaystyle\sum_{k=1}^{3}\frac{\partial u_k}{\partial S_k(\mathbf{z})}=q_3^{+},\\[2ex] B_0h^{+}+\sum_{j=1}^{2}B_jh_j^{+}=f_4^{+},\quad C_0h^{+}+\sum_{j=1}^{2}C_jh_j^{+}=f_5^{+},\quad h^{+}+\sum_{j=1}^{2}h_j^{+}=q_2^{+}.\end{cases} \tag{12}$$

Taking into account the identities

$$\mu_0+bB_0-\beta C_0=\frac{A_2}{\delta_0},\quad bB_J-\beta C_j=-\mu_0$$

at first we find the functions $h^{\pm}$ from systems (11) and (12)

$$\begin{cases}h^{-}=\dfrac{\delta_0}{A_2}[bf_4^{-}-\beta f_5^{-}+\mu_0q_2^{-}]=G^{-},\\[2ex] h^{+}=\dfrac{\delta_0}{A_2}[bf_4^{+}-\beta f_5^{+}+\mu_0q_2^{+}]=G^{+}.\end{cases} \tag{13}$$

Now we consider the following equivalent system to equations $(11)_2,\ (11)_3,\ (12)_2$ and $(12)_3$

$$\begin{cases} \sum\limits_{j=1}^{2} B_j h_j^- = f_4^- - B_0 G^- = g_1^-, \quad \sum\limits_{j=1}^{2} h_j^- = q_2^- - G^- = g_2^-, \\ \\ \sum\limits_{j=1}^{2} B_j h_j^+ = f_4^+ - B_0 G^+ = g_1^+, \quad \sum\limits_{j=1}^{2} h_j^+ = q_2^+ - h^+ = g_2^+, \end{cases} \tag{14}$$

$B_j,\ (j=0,1,2)$ are given by (4).

On the basis of Theorem 1, we conclude that the determinant of system (14) is different from zero and the system (14) is always solvable. It follows from Eqs. (14), that

$$h_j^{\pm} = \frac{(-1)^j}{d}\left[B_0 G^{\pm} - f_4^{\pm} + \frac{B_1 B_2}{B_j}(q_2^{\pm} - G^{\pm})\right] = G_j^{\pm}, \quad j=1,2. \tag{15}$$

where

$$d = B_1 - B_2 = -i\omega\frac{(\lambda_1^2 - \lambda_2^2)\beta\delta_0 K_0}{\mu_0\delta_1\delta_2} = \frac{(\lambda_1^2-\lambda_2^2)\alpha k\mu_0\beta}{K_0},$$

$$K_0 = kbC_0 + \alpha\beta i\omega B_0, \quad \delta_1\delta_2 = -\frac{i\omega\delta_0^2 K_0^2}{\alpha k\mu_0^2}.$$

Thus, the functions h and h_j are known and from $(11)_1$ and $(12)_1$ it follows that

$$\begin{cases} \Omega^- = -2k_0 h_0^- + R_1\dfrac{\partial}{\partial\rho}\left[(k_0-1)h_0 + \sum\limits_{j=1}^{2}\dfrac{h_j}{\lambda_j^2}\right] + q_1^- = G_3^-, \quad \rho = R_1, \\ \\ \Omega^+ = -2k_0 h_0^+ + R_2\dfrac{\partial}{\partial\rho}\left[(k_0-1)h_0 + \sum\limits_{j=1}^{2}\dfrac{h_j}{\lambda_j^2}\right] + q_1^+ = G_3^+, \quad \rho = R_2, \end{cases} \tag{16}$$

Let the functions $h,\ h_j,\ j=1,2,\ \sum\limits_{k=1}^{3}\frac{\partial u_k}{\partial S_k(\mathbf{z})}$ and Ω be sought in the form [31]

$$\begin{cases} h(\mathbf{x}) = \sum\limits_{n=0}^{\infty}\left[\left(\dfrac{R_1}{\rho}\right)^{n+1} Z_n + \left(\dfrac{\rho}{R_2}\right)^n Z_{0n}\right], \\ \\ \Omega(\mathbf{x}) = \sum\limits_{n=0}^{\infty}\left[\left(\dfrac{R_1}{\rho}\right)^{n+1} Y_n + \left(\dfrac{\rho}{R_2}\right)^n Y_{0n}\right], \\ \\ \sum\limits_{k=1}^{3}\dfrac{\partial u_k}{\partial S_k(\mathbf{z})} = \sum\limits_{n=0}^{\infty}\left[\left(\dfrac{R_1}{\rho}\right)^{n+1} Y_{3n} + \left(\dfrac{\rho}{R_2}\right)^n Y_{4n}\right], \\ \\ h_j(\mathbf{x}) = \sum\limits_{n=0}^{\infty}\left[\phi_n(\lambda_j\rho)Y_{jn} + \Psi_n(\lambda_j\rho)Z_{jn}\right], \qquad R_1 < \rho < R_2 \end{cases} \tag{17}$$

where $Y_n,\ Y_{0n},\ Z_n,\ Z_{0n},\ Y_{jn}$ and $Z_{jn},\ j=1,2$ are the unknown spherical harmonics of order n,

$$\phi_n(\lambda_k\rho) = \frac{\sqrt{R_2}J_{n+\frac{1}{2}}(\lambda_k\rho)}{\sqrt{\rho}J_{n+\frac{1}{2}}(\lambda_k R_2)} \qquad k=1,2,$$

$J_{n+\frac{1}{2}}(\lambda_k\rho)$ is the Bessel function,

$$\Psi_m(\lambda_k\rho)=\frac{\sqrt{R_1}H^{(1)}_{m+\frac{1}{2}}(\lambda_k\rho)}{\sqrt{\rho}H^{(1)}_{m+\frac{1}{2}}(\lambda_k R_1)},$$

is the Hankel function.

On the basis of equation $\Delta h_0=h$, the function h_0 can be represented in the following form

$$h_0(\mathbf{x})=\frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{\rho^2}{3+2n}\left(\frac{\rho}{R_2}\right)^n Z_{0n}(\vartheta,\eta)+\frac{\rho^2}{1-2n}\left(\frac{R_1}{\rho}\right)^{n+1}Z_n(\vartheta,\eta)\right]. \tag{18}$$

On the other hand, as seen from equations (13), (15), (16) and (17), for determining unknown functions, we get

$$\begin{cases} Z_n+\left(\dfrac{R_1}{R_2}\right)^n Z_{0n}=G^-_n, \quad \left(\dfrac{R_1}{R_2}\right)^{n+1}Z_n+Z_{0n}=G^+_n,\\[2ex] Y_n+\left(\dfrac{R_1}{R_2}\right)^n Y_{0n}=G^-_{3n}, \quad \left(\dfrac{R_1}{R_2}\right)^{n+1}Y_n+Y_{0n}=G^+_{3n},\\[2ex] Y_{3n}+\left(\dfrac{R_1}{R_2}\right)^n Y_{4n}=q^-_{3n}, \quad \left(\dfrac{R_1}{R_2}\right)^{n+1}Y_{3n}+Y_{4n}=q^+_{3n},\\[2ex] \Phi_n(\lambda_j R_1)Y_{jn}+Z_{jn}=G^-_{jn}, \quad Y_{jn}+\Psi_n(\lambda_j R_2)Z_{jn}=G^+_{jn}, \end{cases} \tag{19}$$

where $G^{\pm}_n$, G^-_{3n}, G^-_{jn}, G^+_{3n}, and G^+_{jn}, are the known spherical harmonics of order n.

By applying Theorem 1 we conclude that the determinant of system (19) for $n\geq 0$ is different from zero. Therefore, system (19) is uniquely solvable.

$$\begin{cases} Z_n=\dfrac{R_2^{n+1}}{R_2^{2n+1}-R_1^{2n+1}}[R_2^nG^-_n-R_1^nG^+_n],\\ Z_{0n}=\dfrac{R_2^n}{R_2^{2n+1}-R_1^{2n+1}}[R_2^{n+1}G^+_n-R_1^{n+1}G^-_n],\\ Y_{3n}=\dfrac{R_2^{n+1}}{R_2^{2n+1}-R_1^{2n+1}}[R_2^nq^-_n-R_1^nq^+_n],\\ Y_{4n}=\dfrac{R_2^n}{R_2^{2n+1}-R_1^{2n+1}}[R_2^{n+1}q^+_n-R_1^{n+1}q^-_n],\\ Y_n=\dfrac{R_2^{n+1}}{R_2^{2n+1}-R_1^{2n+1}}[R_2^nG^-_{3n}-R_1^nG^+_{3n}],\\ Y_{0n}=\dfrac{R_2^n}{R_2^{2n+1}-R_1^{2n+1}}[R_2^{n+1}G^+_{3n}-R_1^{n+1}G^-_{3n}],\\ Y_{jn}=\dfrac{1}{\Phi_n(\lambda_jR_1)\Psi_n(\lambda_jR_2)-1}[\Psi_n(\lambda_jR_2)G^-_{jn}-G^+_{jn}],\\ Z_{jn}=\dfrac{1}{\Phi_n(\lambda_jR_1)\Psi_n(\lambda_jR_2)-1}[\Phi_n(\lambda_jR_1)G^+_{jn}-G^-_{jn}], \quad j=1,2. \end{cases} \tag{20}$$

Substituting relations (20) into (17) and then obtained functions in (3), we get the solution of Problem 1, where functions h, h_j, Ω and h_0 are defined from (17) and (18).

For absolutely and uniformly convergence of series, together with their first derivatives, it is sufficient to assume that

$$f_j \in C^5(S), \quad j = 1, 2, .., 5.$$

Under these conditions the resulting series are absolutely and uniformly convergent.

Thus, the considered problem is completely solved.

Remark. By using the above-mentioned method, it is possible to construct explicitly the solutions of basic BVPs for systems of equations in modern linear theories of poroelasticity and thermoelasticity for materials with micro-structures for simple cases of 3D domains (sphere, space with a spherical cavity and etc.) in the form of absolutely and uniformly convergent series.

5. Conclusions

This paper with the coupled linear quasi-static equations of the theory of elasticity for porous elastic materials, in which the basic equations are expressed in terms of the displacement vector $\mathbf{u}$, the changes of the volume fraction φ of pores and the fluid pressure p in pore network. The following results are obtained:

1. Efficient solutions of the Dirichlet type BVP is obtained for a porous spherical layer.
2. The obtained solution is represented as absolutely and uniformly convergent series.

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