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A DELAY OPTIMIZATION PROBLEM FOR THE LINEAR CONTROL SYSTEM WITH THE MIXED INITIAL CONDITION

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Abstract. The necessary conditions of optimality of delays parameters, of the initial vector, of the initial and control functions are proved for the optimization problem with constant delays in the phase coordinates and controls. The necessary conditions are concretized for the optimization problem with the integral functional and with the fixed right end.

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1. Problem statement and optimality conditions

Let $\mathbb{R}^n_x$ be the n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$, where T denotes transposition. Let $p \in \mathbb{R}^k_p$ and $q \in \mathbb{R}^m_q$, with $k + m = n$ and $x = (p, q)^T$. Furthermore, let

$$\tau_2 > \tau_1 > 0,\ \sigma_2 > \sigma_1,\ \theta_2 < \theta_1$$

be given numbers. Let $I = [t_0, t_1]$, with $t_0 + \tau_2 < t_1; I_1 = [\hat{\tau}, t_0]$ and $I_2 = [t_0 - \theta_2, t_1]$, where $\hat{\tau} = t_0 - \max\{\tau_2, \sigma_2\}$. Denote by C^1_φ the space of continuous differentiable functions

$$\varphi : I_1 \to \mathbb{R}^k_p$$

and by C^1_g the space of continuous differentiable functions

$$g : I_1 \to \mathbb{R}^m_q.$$

Next, denote by AC_u the space of absolutely continuous control functions

$$u : I_2 \to \mathbb{R}^r_u.$$

Let us introduce the sets

$$\Phi = \{\varphi \in C^1_\varphi : \varphi(t) \in K, t \in I_1\},\ G = \{g \in C^1_g : g(t) \in M, \in I_1\},$$

$$\Omega = \{u \in AC_u : u(t) \in U,\ |\dot{u}(t)| \le const,\ t \in I_2\},$$

where $K \subset \mathbb{R}^k_p, M \subset \mathbb{R}^m_q$ and $U \subset \mathbb{R}^r_u$ are convex and compact sets.

To any element

$$w = (\tau, \sigma, \theta, p_0, \varphi, g, u) \in W = (\tau_1, \tau_2) \times (\sigma_1, \sigma_2) \times (\theta_1, \theta_2) \times P$$

$$\times \Phi \times G \times \Omega$$

we assign the linear control delay differential equation

$$\dot{x}(t) = (\dot{p}(t), \dot{q}(t))^T = A(t)x(t) + B(t)p(t - \tau) + C(t)q(t - \sigma)$$
$$+D(t)u(t) + E(t)u(t - \theta), t \in \tag{1}$$

with the mixed initial condition

$$\begin{cases} x(t) = (p(t), q(t))^T = (\varphi(t), g(t))^T, t \in [\hat{\tau}, t_0), \\ x(t_0) = (p_0, g(t_0))^T. \end{cases} \tag{2}$$

Here $P \subset R_p^k$ is a convex and compact set; $A(t),\ B(t),\ C(t),\ D(t)$ and $E(t)$ are the measurable and bounded matrix functions with dimensions $n \times n,\ n \times k,\ n \times m,\ n \times r$ and $n \times r$, respectively.

Condition (2) is said to be the mixed initial condition, because it consists of two parts: the first part is

$$p(t) = \varphi(t), t \in [\hat{\tau}, t_0), p(t_0) = p_0,$$

the discontinuous part, since in general $p(t_0) \neq p_0$ (discontinuity at the initial moment may be related to the instant change in a dynamic process, for example, changes of investment, environment); the second part is

$$q(t) = g(t), t \in I_1,$$

the continuous part, since always $q(t_0) = g(t_0)$.

Definition 1. Let $w = (\tau, \sigma, \theta, p_0, \varphi, g, u) \in W$. A function $x(t) = x(t; w), t \in [\hat{\tau}, t_1]$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element w, if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

From the linearity of equation (1) it follows that for every element $w \in W$ there exists a corresponding solution.

Let the scalar-valued functions $z^i(\tau, \sigma, \theta, p, x),\ i = \overline{0, l}$, be continuously differentiable on

$$[\tau_1, \tau_2] \times [\theta_1, \theta_2] \times [\sigma_1, \sigma_2] \times R_p^k \times R_x^n.$$

Definition 2. An element $w = (\tau, \sigma, \theta, p_0, \varphi, g, u) \in W$ is said to be admissible if the corresponding solution $x(t) = x(t; w)$ satisfies the boundary conditions

$$z^i(\tau, \sigma, \theta, p_0, x(t_1)) = 0, i = \overline{1, l}. \tag{3}$$

Denote by W_0 the set of admissible elements.

Definition 3. An element $w_0 = (\tau_0, \sigma_0, \theta_0, p_{00}, \varphi_0, g_0, u_0) \in W_0$ is said to be optimal if for an arbitrary element $w \in W_0$ the inequality

$$z^0(\tau_0, \sigma_0, \theta_0, p_{00}, x_0(t_1)) \leq z^0(\tau, \sigma, \theta, p_0, x(t_1)) \tag{4}$$

holds, where $x_0(t) = x(t; w_0),\ x(t) = x(t; w)$.

(1)-(4) is called the delay optimization problem.

Theorem 1. *Let w_0 be an optimal element and let $x_0(t) = (p_0(t), q_0(t))^T$ be the corresponding solution. Moreover, the function $B(t)$ is continuous at the point $t_{00} + \tau_0$. There exist a vector $\pi = (\pi_0, ..., \pi_l) \neq 0$, with $\pi_0 \leq 0$, and a solution $\psi(t) = (\psi_1(t), ..., \psi_n(t))$ of the equation*

$$\dot{\psi}(t) = -\psi(t)A(t) - \psi(t+\tau_0)\Big(B(t+\tau_0), \Theta_{n\times m}\Big) - \psi(t+\sigma_0)\Big(\Theta_{n\times k}, C(t+\sigma_0)\Big) \tag{5}$$

with the initial condition

$$\psi(t_1) = \pi Z_{0x}, \quad \psi(t) = 0, t > t_1 \tag{6}$$

where $\Theta_{n\times m}$ *is the* $n \times m$ *zero matrix and*

$$Z = (z^0, ..., z^l)^T, Z_{0x} = \frac{\partial Z(\tau_0, \sigma_0, \theta_0, p_{00}, x_0(t_1))}{\partial x},$$

such that the following conditions hold:

1) *the condition for the delay* τ_0

$$\pi Z_{0\tau} = \psi(t_0 + \tau_0)B(t_0 + \tau_0)[p_{00} - \varphi_0(t_{00})] + \int_{t_0}^{t_1} \psi(t)B(t)\dot{p}_0(t - \tau_0)dt;$$

2) *the condition for the delay* σ_0

$$\pi Z_{0\sigma} = \int_{t_0}^{t_1} \psi(t)C(t)\dot{q}_0(t - \sigma_0)dt;$$

3) *the condition for the delay* θ_0

$$\pi Z_{0\theta} = \int_{t_0}^{t_1} \psi(t)E(t)\dot{u}_0(t - \theta_0)dt;$$

4) *the condition for the vector* p_{00},

$$\Big(\pi Z_{0p} + (\psi_1(t_0), ..., \psi_k(t_0))\Big)p_{00} = \max_{p_0 \in P} \Big(\pi Z_{0p} + (\psi_1(t_0), ..., \psi_k(t_0))\Big)p_0;$$

5) *the condition for the initial function* $\varphi_0(t)$,

$$\int_{t_0-\tau_0}^{t_0} \psi(t + \tau_0)B(t + \tau_0)\varphi_0(t)dt = \max_{\varphi \in \Phi} \int_{t_0-\tau_0}^{t_0} \psi(t + \tau_0)B(t + \tau_0)\varphi(t)dt;$$

6) *the condition for the initial function* $g_0(t)$,

$$(\psi_{k+1}(t_0), ..., \psi_n(t_0))g_0(t_0) + \int_{t_0-\sigma_0}^{t_0} \psi(t + \sigma_0)C(t + \sigma_0)g_0(t)dt$$

$$= \max_{g \in G} \Big[(\psi_{k+1}(t_0), ..., \psi_n(t_0))g(t_0) + \int_{t_0-\sigma_0}^{t_0} \psi(t + \sigma_0)C(t + \sigma_0)g(t)dt;$$

7) *the condition for the control function* $u_0(t)$,

$$\int_{t_0}^{t_1} \psi(t)\Big[D(t)u_0(t) + E(t)u_0(t - \theta_0)\Big]dt = \max_{u \in \Omega} \int_{t_0}^{t_1} \psi(t)\Big[D(t)u(t)$$

$$+E(t)u(t - \theta_0)\Big]dt.$$

Theorem 1 on the basis of the variation formula of solution [1] will be proved by the scheme given [2, 3]. Delay optimal problems with the mixed initial condition, without optimization of delay parameter in controls are considered in [4, 5].

Now we consider the optimization problem with the integral functional

$$\dot{x}(t) = A(t)x(t) + B(t)p(t-\tau) + C(t)q(t-\sigma) + D(t)u(t)$$
$$+E(t)u(t-\theta), t \in I,$$

$$x(t) = (\varphi(t), g(t))^T, t \in [\hat{\tau}, t_0), x(t_0) = (p_0, g(t_0))^T,\ x(t_1) = x_1,$$

$$\int_{t_0}^{t_1} \Big[a^0(t)x(t) + b^0(t)p(t-\tau) + c^0(t)q(t-\sigma)$$
$$+d^0(t)u(t) + e^0(t)u(t-\theta)\Big]dt \to \min .$$

Here $a^0(t)$, $b^0(t)$, $c^0(t)$, $d^0(t)$ and $c^0(t)$ are the measurable and bounded row-vector functions with corresponding dimensions; $\varphi(t) \in C^1_\varphi$ and $g(t) \in C^1_g$ are fixed initial functions; $p_0 \in R^k_p$ and $x_1 \in R^n_x$ are fixed points.

Evidently, the above considered problem is equivalent to the following problem

$$\dot{x}^0(t) = a^0(t)x(t) + b^0(t)p(t-\tau) + c^0(t)q(t-\sigma)$$
$$+d^0(t)u(t) + e^0(t)u(t-\theta),$$
$$\dot{x}(t) = A(t)x(t) + B(t)p(t-\tau) + C(t)q(t-\sigma) + D(t)u(t)$$
$$+E(t)u(t-\theta), t \in I,$$

$$x^0(t_0) = 0,\ x(t) = (\varphi(t), g(t))^T, t \in [\hat{\tau}, t_0), x(t_0) = (p_0, g(t_0))^T,\ x(t_1) = x_1,$$

$$x^0(t_1) \to min,$$

which is a particular case of the problem (1)-(4). Therefore, Theorem 2 formulated below is a simple corollary of Theorem 1.

Let us introduce the functions

$$\hat{A}(t) = (a^0(t), A(t))^T,\ \hat{B}(t) = (b^0(t), B(t))^T,\ \hat{C}(t) = (c^0(t), C(t))^T,$$
$$\hat{D}(t) = (d^0(t), D(t))^T,\ \hat{E}(t) = (e^0(t), E(t))^T.$$

Theorem 2. *Let $(\tau_0, \sigma_0, \theta_0, u_0(t))$ be an optimal element and let $x_0(t) = (p_0(t), q_0(t))^T$ be the corresponding solution. Moreover, the function $\hat{B}(t)$ is continuous at the point $t_{00}+\tau_0$. There exists a nontrivial solution $\hat{\psi}(t) = (\psi_0(t), \psi_1(t), ..., \psi_n(t)) = (\psi_0(t), \psi(t))$, with $\psi_0(t) \equiv const \le 0$, of the equation*

$$\dot{\hat{\psi}}(t) = -\hat{\psi}(t)\hat{A}(t) - \hat{\psi}(t+\tau_0)\Big(\hat{B}(t+\tau_0), \Theta_{(n+1)\times m}\Big) - \hat{\psi}(t+\sigma_0)\Big(\Theta_{(n+1)\times k}, \hat{C}(t+\sigma_0)\Big)$$

$$\hat{\psi}(t) = 0, t > t_1$$

such that the following conditions hold:

8) *the condition for the delay τ_0*

$$\hat{\psi}(t_0+\tau_0)\hat{B}(t_0+\tau_0)[p_0 - \varphi(t_0)] + \int_{t_0}^{t_1} \hat{\psi}(t)\hat{B}(t)\dot{p}_0(t-\tau_0)dt = 0;$$

9) *the condition for the delay* σ_0

$$\int_{t_0}^{t_1} \hat{\psi}(t)\hat{C}(t)\dot{q}_0(t-\sigma_0)dt = 0;$$

10) *the condition for the delay* θ_0

$$\int_{t_0}^{t_1} \hat{\psi}(t)\hat{E}(t)\dot{u}_0(t-\theta_0)dt = 0;$$

11) *the condition for the control function* $u_0(t)$,

$$\int_{t_0}^{t_1} \hat{\psi}(t)\Big[\hat{D}(t)u_0(t)+\hat{E}(t)u_0(t-\theta_0)\Big]dt = \max_{u\in\Omega}\int_{t_0}^{t_1} \hat{\psi}(t)\Big[\hat{D}(t)u(t)$$

$$+\hat{E}(t)u(t-\theta_0)\Big]dt.$$

2. Proof of Theorem 1

On the convex set $\Pi = \mathbb{R}_+ \times W$, where $\mathbb{R}_+ = [0,\infty)$, let us define the mapping

$$Q : \Pi \to \mathbb{R}^{1+l} \tag{7}$$

by the formula

$$Q(v) = (Q^0(v), ..., Q^l(v))^T = Z(\tau,\sigma,\theta,p_0,x(t_1;w)) + (\xi,0...,0)^T, v = (\xi,w) \in \Pi.$$

It is clear that

$$Q^0(v_0) \le Q^0(v), Q^i(v) = 0, i = \overline{1,l}, \forall v \in \mathbb{R}_+ \times W_0 \subset \Pi,$$

where $v_0 = (0, w_0)$.

Thus, the point $v_0 = (0, w_0) \in \Pi$ is a critical (see [2, 3]), since $Q(z_0) \in \partial Q(\Pi)$. Moreover, the mapping (7) is continuous [3].

There exist numbers $\varepsilon_0 > 0$ and $\alpha > 0$ such that for an arbitrary $\varepsilon \in (0,\varepsilon_0)$ and

$$\delta v = (\delta\xi,\delta w) \in V_{v_0} := [0,\alpha) \times V_{w_0} \subset \Pi - v_0 = \{v - v_0 : \ \forall v \in \Pi\},$$

where

$$\delta w = (\delta\tau,\delta\sigma,\delta\theta,\delta p_0,\delta\varphi,\delta g,\delta u),$$

$$V_{w_0} = (-\alpha,\alpha) \times (-\alpha,\alpha) \times (-\alpha,\alpha) \times [P - p_0]$$

$$\times[\Phi - \varphi_0] \times [G - g_0] \times [\Omega - u_0]$$

we get

$$v_0 + \varepsilon\delta v \in \Pi.$$

On the basis of the variation formula of solution [1] we have,

$$\Delta x(t_1;\varepsilon\delta w) := x(t_1; w_0 + \varepsilon\delta w) - x_0(t_1) = \varepsilon\delta x(t_1;\delta w) + o(\varepsilon\delta w),$$

$$\forall(\varepsilon,\delta w)\in(0,\varepsilon_0)\times V_{w_0},$$

where

$$\delta x(t_1;\delta\mu)=Y(t_0;t_1)\Big((\delta p_0,\Theta_{m\times 1})^T+(\Theta_{k\times 1},\delta g(t_0))^T\Big)$$

$$-\Big\{Y(t_0+\tau_0;t_1)B(t_0+\tau_0)[p_{00}-\varphi_0(t_0)]+\int_{t_0}^{t_1}Y(s;t_1)B(s)\dot p_0(s-\tau_0)ds\Big\}\delta\tau$$

$$-\Big\{\int_{t_0}^{t_1}Y(s;t_1)C(s)\dot q_0(s-\sigma_0)ds\Big\}\delta\sigma+\int_{t_0-\tau_0}^{t_0}Y(s+\tau_0;t_1)B(s+\tau_0)\delta\varphi(s)ds$$

$$+\int_{t_{00}-\sigma_0}^{t_{00}}Y(s+\sigma_0;t)C(s+\sigma_0)\delta g(s)ds-\Big\{\int_{t_{00}}^{t}Y(s;t)E(s)\dot u_0(s-\theta_0)ds\Big\}\delta\theta$$

$$+\int_{t_{00}}^{t_1}Y(s;t_1)\Big[D(s)\delta u(s)+E(s)\delta u(s-\theta_0)\Big]ds; \qquad (8)$$

and

$$\lim_{\varepsilon\to 0}\frac{o(\varepsilon\delta w)}{\varepsilon}=0 \text{ uniformly for } \delta w\in V_{w_0};$$

$Y(t;t_1)$ is the $n\times n-$ matrix function satisfying the linear differential equation with an advanced argument

$$\frac{d}{dt}Y(t;t_1)=-Y(t;t_1)A(t)-Y(t+\tau_0;t_1)\Big(B(t+\tau_0),\Theta_{n\times m}\Big)-Y(t+\sigma_0;t_1)\Big(\Theta_{n\times k},C(t+\sigma_0)\Big)$$

and the condition

$$Y(t;t_1)=\begin{cases}\hat I \; for\ t=t_1,\\ \Theta_{n\times n}\; for\ t>t_1,\end{cases}$$

here $\hat I$ is the identity matrix.

Now we calculate a differential of the mapping (7) at the point v_0. We have,

$$Q(v_0+\varepsilon\delta v)-Q(v_0)=Z(\tau_0+\varepsilon\delta\tau,\sigma_0+\varepsilon\delta\sigma,\theta_0+\varepsilon\delta\theta,p_{00}+\varepsilon\delta p_0,x(t_1;w_0+\varepsilon\delta w))$$

$$-Z(\tau_0,\sigma_0,\theta_0,p_{00},x_0(t_1))+\varepsilon(\delta\xi,0...,0)^T,\ \varepsilon\in(0,\varepsilon_0),\delta w\in V_{w_0}.$$

We introduce the notation

$$Z[\varepsilon;s]=Z(\tau_0+\varepsilon s\delta\tau,\sigma_0+\varepsilon s\delta\sigma,\theta_0+\varepsilon s\delta\theta,p_{00}+\varepsilon s\delta p_0,x_0(t_1)+s\Delta x(t_1;\varepsilon\delta w))$$

Let us transform the difference

$$Z(\tau_0+\varepsilon\delta\tau,\sigma_0+\varepsilon\delta\sigma,\theta_0+\varepsilon\delta\theta,p_{00}+\varepsilon\delta p_0,x(t_1;w_0+\varepsilon\delta w))$$

$$-Z(\tau_0,\sigma_0,\theta_0,p_{00},x_0(t_1))=\int_0^1\frac{d}{ds}\,Z[\varepsilon;s]ds$$

$$=\int_0^1\Big[\varepsilon\Big(Z_\tau[\varepsilon;s]\delta\tau+Z_\sigma[\varepsilon;s]\delta\sigma+Z_\theta[\varepsilon;s]\delta\theta+Z_p[\varepsilon;s]\delta p_0\Big)+Z_x[\varepsilon;s]\Delta x(t_1;\varepsilon\delta w)\Big]ds$$

$$=\int_0^1\Big[\varepsilon\Big(Z_\tau[\varepsilon;s]\delta\tau+Z_\sigma[\varepsilon;s]\delta\sigma+Z_\theta[\varepsilon;s]\delta\theta+Z_p[\varepsilon;s]\delta p_0+Z_x[\varepsilon;s]\delta x(t_1;\varepsilon\delta w)\Big)$$

$$+Z_x[\varepsilon;s]o(\varepsilon\delta w)\Big]ds=\varepsilon\Big[Z_{0\tau}\delta\tau+Z_{0\sigma}\delta\sigma+Z_{0\theta}\delta\theta+Z_{0p}\delta p_0+Z_{0x}\delta x(t_1;\delta w)\Big]+\gamma(\varepsilon\delta w),$$

where

$$\gamma(\varepsilon\delta w)=\varepsilon\int_0^1\Big\{[Z_\tau[\varepsilon;s]-Z_{0\tau}]\delta\tau+[Z_\sigma[\varepsilon;s]-Z_{0\sigma}]\delta\sigma+[Z_\theta[\varepsilon;s]-Z_{0\theta}]\delta\theta$$

$$[Z_p[\varepsilon;s]-Z_{0p}]\delta p_0+[Z_x[\varepsilon;s]-Z_{0x}]\delta x(t_1;\delta w)+Z_x[\varepsilon;s]\frac{o(\varepsilon\delta w)}{\varepsilon}\Big\}ds.$$

It is easy to see that

$$\lim_{\varepsilon\to 0}[Z_\tau[\varepsilon;s]-Z_{0\tau}]=0,\lim_{\varepsilon\to 0}[Z_\sigma[\varepsilon;s]-Z_{0\sigma}]=0,\lim_{\varepsilon\to 0}[Z_\theta[\varepsilon;s]-Z_{0\theta}]=0,$$

$$\lim_{\varepsilon\to 0}[Z_p[\varepsilon;s]-Z_{0p}]=0,\lim_{\varepsilon\to 0}[Z_x[\varepsilon;s]-Z_{0x}]=0.$$

Therefore, $\gamma(\varepsilon\delta w)=o(\varepsilon\delta w)$. Thus,

$$Q(v_0+\varepsilon\delta v)-Q(v_0)=\varepsilon dQ_{v_0}(\delta v)+o(\varepsilon\delta v),$$

where $o(\varepsilon\delta v):=o(\varepsilon\delta w)$ and differential $dQ_{v_0}(\delta v)$ of the mapping (7) has the form

$$dQ_{v_0}(\delta v)=Z_{0\tau}\delta\tau+Z_{0\sigma}\delta\sigma+Z_{0\theta}\delta\theta+Z_{0p}\delta p_0+Z_{0x}\delta x(t_1;\delta w)+(\delta\xi,0,...,0)^T.$$

Due to relation (8) we get

$$dQ_{v_0}(\delta v)=\Big[Z_{0\tau}-Z_{0x}Y(t_0+\tau_0;t_1)B(t_0+\tau_0)[p_{00}-\varphi_0(t_0)]-\int_{t_0}^{t_1}Z_{0x}Y(t;t_1)B(t)\dot p_0(t-\tau_0)dt\Big]\delta\tau$$

$$\int_{t_0-\tau_0}^{t_0}Z_{0x}Y(t+\tau_0;t_1)B(t+\tau_0)\delta\varphi(t)dt+\Big[Z_{0\sigma}-\int_{t_0}^{t_1}Z_{0x}Y(t;t_1)C(t)\dot q_0(t-\sigma_0)dt\Big]\delta\sigma$$

$$+Z_{0p}\delta p_0+Z_{0x}Y(t_0;t_1(\delta p_0,\Theta_{m\times 1})^T+\Big[Z_{0x}Y(t_0;t_1)(\Theta_{k\times 1},\delta g(t_0))^T$$

$$+\int_{t_0-\sigma_0}^{t_0}Z_{0x}Y(t+\sigma_0;t_1)C(t+\sigma_0)\delta g(t)dt+\Big[Z_{0\theta}-\int_{t_0}^{t_1}Z_{0x}Y(t;t_1)E(t)\dot u_0(t-\theta_0)dt\Big]\delta\theta$$

$$+\int_{t_0}^{t_1}Z_{0x}Y(t;t_1)\Big\{D(t)\delta u(t)+E(t)\delta u(t-\theta_0)\Big\}dt+(\delta\xi,0,...,0)^T. \quad (9)$$

From the necessary condition of criticality [3, 4] it follows that: there exists a vector $\pi=(\pi_0,...,\pi_l)\neq 0$ such that

$$\pi dQ_{v_0}(\delta v)\le 0,\forall\ \delta v\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[P-p_{00}]\times[\Phi-\varphi_0] \quad (10)$$

$$\times[G-g_0]\times[\Omega-u_0]$$

Introduce the function

$$\psi(t)=\pi Z_{0x}Y(t;t_1) \quad (11)$$

as is easily seen, it satisfies equation (5) and condition (6). Taking into account (9) and (11) from inequality (10) we obtain

$$\Big[\pi Z_{0\tau}-\psi(t_0)B(t_0+\tau_0)[p_{00}-\varphi_0(t_0)]-\int_{t_0}^{t_1}\psi(t)B(t)\dot p_0(t-\tau_0)dt\Big]\delta\tau$$

$$\int_{t_0-\tau_0}^{t_0}\psi(t+\tau_0)B(t+\tau_0)\delta\varphi(t)dt+\Big[\pi Z_{0\sigma}-\int_{t_0}^{t_1}\psi(t)C(t)\dot{q}_0(t-\sigma_0)dt\Big]\delta\sigma$$
$$+\Big[\pi Z_{0p}+(\psi_1(t_0),...,\psi_k(t_0))\Big]\delta p_0+\Big[(\psi_{k+1}(t_0),...,\psi_n(t_0))\delta g(t_0)$$
$$+\int_{t_0-\sigma_0}^{t_0}\psi(t+\sigma_0)C(t+\sigma_0)\delta g(t)dt\Big]+\Big[\pi Z_{0\theta}-\int_{t_0}^{t_1}\psi(t)E(t)\dot{u}_0(t-\theta_0)dt\Big]\delta\theta$$
$$+\int_{t_0}^{t_1}\psi(t)\Big\{D(t)\delta u(t)+E(t)\delta u(t-\theta_0)\Big\}dt+(\delta\xi,0,...,0)^T\le 0. \qquad (12)$$

Let $\delta\tau=\delta\sigma=\delta\theta=0,\delta p_0=0,\delta\varphi=0,\delta g=0,\delta u=0$ in (12), then we obtain

$$\pi_0\delta\xi\le 0,\forall\delta\xi\in\mathbb{R}_+.$$

This implies $\pi_0\le 0$.

Let $\delta\xi=\delta\sigma=\delta\theta=0,\delta p_0=0,\delta\varphi=0,\delta g=0,\delta u=0$ in (12), then we have

$$\Big[\pi Z_{0\tau}-\psi(t_0+\tau_0)B(t_0+\tau_0)[p_{00}-\varphi_0(t_0)]-\int_{t_0}^{t_1}\psi(t)B(t)\dot{p}_0(t-\tau_0)dt\Big]\delta\tau\le 0,$$

taking into account that $\delta\tau\in\mathbb{R}$ we obtain condition *1)*.

Let $\delta\xi=\delta\tau=\delta\theta=0,\delta p_0=0,\delta\varphi=0,\delta g=0,\delta u=0$ in (12), then we have

$$\Big[\pi Z_{0\sigma}-\int_{t_0}^{t_1}\psi(t)C(t)\dot{q}_0(t-\sigma_0)dt\Big]\delta\sigma\le 0,$$

taking into account that $\delta\sigma\in\mathbb{R}$ we obtain condition *2)*.

Let $\delta\xi=\delta\tau=\delta\sigma=0,\delta p_0=0,\delta\varphi=0,\delta g=0,\delta u=0$ in (12), then we get

$$\Big[\pi Z_{0\theta}-\int_{t_0}^{t_1}\psi(t)E(t)\dot{u}_0(t-\theta_0)dt\Big]\delta\theta\le 0,$$

taking into account that $\delta\theta\in\mathbb{R}$ we obtain condition *3)*.

Let $\delta\xi=\delta\tau=\delta\sigma=\delta\theta=0,\delta\varphi=0,\delta g=0,\delta u=0$ in (12), then we have

$$\Big[\pi Z_{0p}+(\psi_1(t_0),...,\psi_k(t_0))\Big]\delta p_0\le 0,$$

taking into account that $\delta p_0\in P_0-p_0=\{p-p_0:\ p\in P_0\}$ we obtain condition *4)*.

Let $\delta\xi=\delta\tau=\delta\sigma=\delta\theta=0,\delta p_0=0,\delta g=0,\delta u=0$ in (12), then we have

$$\int_{t_0-\tau_0}^{t_0}\psi(t+\tau_0)B(t+\tau_0)\delta\varphi(t)dt\le 0,$$

taking into account that $\delta\varphi\in\Phi-\varphi_0$ we obtain condition *5)*.

Let $\delta\xi=\delta\tau=\delta\sigma=\delta\theta=0,\delta p_0=0,\delta\varphi=0,\delta u=0$ in (12), then we have

$$\Big[(\psi_{k+1}(t_0),...,\psi_n(t_0))\delta g(t_0)+\int_{t_0-\sigma_0}^{t_0}\psi(t+\sigma_0)C(t+\sigma_0)\delta g(t)dt\Big]\le 0,$$

taking into account that $\delta g\in G-g_0$ we obtain condition *6)*.

Let $\delta\xi = \delta\tau = \delta\sigma = \delta\theta = 0, \delta p_0 = 0, \delta\varphi = 0, \delta\varphi = 0$, in (12), then we have

$$\int_{t_0}^{t_1} \psi(t)\Big\{D(t)\delta u(t) + E(t)\delta u(t-\theta_0)\Big\}dt \le 0,$$

taking into account that $\delta u \in \Omega - u_0$ we obtain condition *7)*.

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