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ON THE EXISTENCE OF AN OPTIMAL ELEMENT FOR ONE CLASS OF A TWO-STAGE OPTIMAL PROBLEM WITH DELAYS AND THE PHASE RESTRICTIONS

Tadumadze T., Chachanidze S.

Abstract. Existence theorems of optimal element are given for the nonlinear two-stage optimal problem with the constant delays in the phase coordinates, with the general boundary conditions and the phase restrictions, with the general functional and the continuous intermediate condition. Under element we imply the collection of initial moment and delay parameters, initial function and vector, moment change of system stage, finally moment and controls.

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1. Introduction

The two stage system means that in the process of motion, at a certain instant of time not known in advance, the system can pass from one law of motion to another, and, moreover, the initial condition of the system on the second stage depends on the final state of the previous stage of the system. Such systems arise in various branches [1-4]. For example, in economics sometimes it is needed to change invested capital at some unknown moments; in engineering a controlled apparatus is to start from another controlled apparatus, which may be cosmic, ground, submarine, etc. In this work the existence theorems of optimal element are provided for the two stage optimal problem with the constant, delays in the phase coordinates of the two-stage controlled system with the general boundary conditions and the phase restrictions, with the general functional and the continuous intermediate condition. Under element we imply the collection of initial moment and delay parameters, initial function and vector, moment change of system stage, finally moment and controls. The result, obtained here, is concretized for the two stage quasi-linear and linear optimal problems and for the problem with the integral functional. Necessary conditions of optimality and existence theorems for various types of two-stage optimal problems are given in [1-8] and [9-11], respectively.

2. Statement of the two-stage optimal problem and the existence theorems

Let

$$a < t_0^1 < t_0^2 < c < \theta_1 < \theta_2 < b < t_1^1 < t_1^2 < d$$

and let

$$0 < \tau_1 < \tau_2, 0 < \sigma_1 < \sigma_2$$

be given numbers with

$$c - t_0^2 > \max\{\tau_2, \sigma_2\},\ t_1^1 - b > \sigma_2. \tag{1}$$

Let $O \subset R_x^n$ and $Y \subset R_y^m$ be open sets, $K_0 \subset O, X_0 \subset O, U \subset R_u^r$ and $V \subset R_v^k$ are compact sets. Further, let the functions

$$f(t,x_1,x_2,u)=(f^1(t,x_1,x_2,u),\dots,f^n(t,x_1,x_2,u))^T$$

and

$$g(t,y_1,y_2,v)=(g^1(t,y_1,y_2,v),\dots,g^m(t,y_1,y_2,v))^T$$

be continuous on the sets $[a,b]\times O^2\times U$ and $[c,d]\times Y^2\times V$, and continuously differentiable with respect to $x_1,x_2\in O$ and $y_1,y_2\in Y$, respectively.

Let Φ be a set of measurable initial functions $\varphi(t)\in K_0$, $t\in[\hat\tau,c]$, where $\hat\tau=a-\tau_2$; let Ω be a set of measurable control functions $u(t)\in U$, $t\in[a,b]$, and let Δ be a set of measurable control functions $v(t)\in V$, $t\in[c,d]$; suppose that

$$\psi^i(t_0,\theta,t_1,\tau,\sigma,x_0,x_1,y),\, i=\overline{0,l+\nu},$$

are continuous scaler functions on the set

$$[t_0^1,t_0^2]\times[\theta_1,\theta_2]\times[t_1^1,t_1^2]\times[\tau_1,\tau_2]\times[\sigma_1,\sigma_2]\times X_0\times O\times Y$$

and $q(t,x)\in Y$ is a continuous vector function on the set $[t_0^2,\theta_2]\times O$.

To each element

$$w=(t_0,\theta,t_1,\tau,\sigma,x_0,\varphi(\cdot),u(\cdot),v(\cdot))\in W=[t_0^1,t_0^2]\times[\theta_1,\theta_2]\times[t_1^1,t_1^2]\times$$

$$\times[\tau_1,\tau_2]\times[\sigma_1,\sigma_2]\times X_0\times\Phi\times\Omega\times\Delta$$

we assign the two-stage system of delay functional differential equations

$$\begin{cases}\dot x(t)=f(t,x(t),x(t-\tau),u(t)), & t\in[t_0,\theta],\\ \dot y(t)=g(t,y(t),y(t-\sigma),v(t)), & t\in[\theta,t_1],\end{cases}\tag{2}$$

θ is called a switching moment.

Definition 1. Let $w=(t_0,\theta,t_1,\tau,\sigma,x_0,\varphi(\cdot),u(\cdot),v(\cdot))\in W$. The pair of functions $x(t;w)$ and $y(t;w)$:

$$S(w)=\Big\{x(t)=x(t;w)\in O,\quad t\in[\hat\tau,\theta];\quad y(t)=y(t;w)\in Y,\quad t\in[\hat\sigma,t_1]\Big\},$$

where $\hat\sigma=c-\sigma_2$, is called the solution, corresponding to an element w if the initial condition

$$x(t)=\varphi(t),\quad t\in[\hat\tau,t_0),\quad x(t_0)=x_0\tag{3}$$

and the intermediate condition

$$y(t)=q(t,x(t)),\quad t\in[\hat\sigma,\theta],\tag{4}$$

are fulfilled. Moreover, the function $x(t),t\in[t_0,\theta]$, is absolutely continuous and satisfies the first equation of system (2) almost everywhere on $[t_0,\theta]$; the function $y(t)$, $t\in[\theta,t_1]$, is absolutely continuous and satisfies the second equation of system (2) almost everywhere on $[\theta,t_1]$.

We note that for any $\theta\in[\theta_1,\theta_2]$, $\sigma\in[\sigma_1,\sigma_2]$ and $t_0\in[t_0^1,t_0^2]$ we have $\theta-\sigma>c-\sigma_2=\hat\sigma>t_0$ (see (1)) i. e. the function $q(t,x(t))$ is defined on the interval $[\hat\sigma,\theta]$ (see (4)).

Definition 2. An element $w\in W$ is said to be admissible if there exists the corresponding solution $S(w)$ and the boundary condition

$$\begin{cases}\psi^i(t_0,\theta,t_1,\tau,\sigma,x_0,x(\theta),y(t_1))=0, i=\overline{1,l}\\ \psi^{l+j}(t_0,\theta,t_1,\tau,\sigma,x_0,x(\theta),y(t_1))\ge 0, j=\overline{1,\nu},\end{cases} \tag{5}$$

and the phase restriction

$$\begin{cases}\vartheta^{\varrho}(x(t))\ge 0, t\in[t_0,\theta], \varrho=\overline{1,\kappa}\\ \varsigma^{\alpha}(y(t))\ge 0, t\in[\theta,t_1], \alpha=\overline{1,\varsigma}\end{cases} \tag{6}$$

holds, where $\vartheta^{\varrho}(x), x\in O$ and $\varsigma^{\alpha}(y), y\in Y$ are continuous functions; $y(t)=y(t;w)$.

We denote the set of admissible elements by W_0. Now we consider the functional

$$J(w)=\psi^0(t_0,\theta,t_1,\tau,\sigma,x_0,y(t_1)), w\in W_0.$$

Definition 3. An element $w_0=(t_{00},\theta_0,t_{10},\tau_0,\sigma_0,x_{00},\varphi_0(\cdot),u_0(\cdot),v_0(\cdot))\in W_0$ is said to be optimal if

$$J(w_0)=\inf_{w\in W_0} J(w). \tag{7}$$

(2)-(7) is called the two-stage optimal problem with delays.

Theorem 1. *There exists an optimal element w_0 if the following conditions hold:*

1. $W_0\neq\emptyset$;

2. there exist compact sets $K_1\subset O$ and $M_1\subset Y$ such that for an arbitrary $w\in W_0$ the corresponding solution

$$S(w)=\Big\{x(t;w),\quad t\in[\hat{\tau},\theta];\quad y(t;w),\ t\in[\hat{\sigma},t_1]\Big\}$$

satisfies the conditions

$$x(t;w)\in K_1,\quad t\in[\hat{\tau},\theta];\quad y(t;w)\in M_1,\quad t\in[\hat{\sigma},t_1];$$

3. for any fixed$(t,x_1)\in[a,b]\times O$, the set

$$\Big\{f(t,x_1,x_2,u):(x_2,u)\in K_0\times U\Big\}$$

is convex.

4. for any fixed$(t,x_1,x_2)\in[a,b]\times O^2$, the set

$$\Big\{f(t,x_1,x_2,u):u\in U\Big\}$$

is convex.

5. for any fixed$(t,y_1,y_2)\in[c,d]\times Y^2$, the set

$$\Big\{g(t,y_1,y_2,v):v\in V\Big\}$$

is convex.

Let (2) has the form

$$\begin{cases}\dot{x}(t)=A(t,x(t))x(t-\tau)+B(t,x(t))u(t), t\in[t_0,\theta],\\ \dot{y}(t)=C(t,y(t),y(t-\sigma))+D(t,y(t),y(t-\sigma))v(t), t\in[\theta,t_1].\end{cases} \tag{8}$$

Theorem 2. *For the quasi-linear optimal problem (3-8)there exists an optimal element w_0 if the following conditions hold:*

6. $W_0 \neq \emptyset$;

7. there exist compact sets $K_1 \subset O$ and $M_1 \subset Y$ such that for an arbitrary $w \in W_0$ the corresponding solution

$$S(w) = \Big\{x(t;w), \quad t \in [\hat{\tau}, \theta]; \quad y(t;w), \ t \in [\hat{\sigma}, t_1]\Big\}$$

satisfies the conditions

$$x(t;w) \in K_1, \quad t \in [\hat{\tau}, \theta]; \quad y(t;w) \in M_1, \quad t \in [\hat{\sigma}, t_1];$$

8. the sets K_0, U and V are convex sets.

Let (2) have the form

$$\begin{cases} \dot{x}(t) = A_1(t)x(t) + B_1(t)x(t-\tau)) + C_1(t)u(t), t \in [t_0, \theta], \\ \dot{y}(t) = A_2(t)y(t) + B_2(t)y(t-\sigma)) + C_2(t)v(t), t \in [\theta, t_1]. \end{cases} \tag{9}$$

Theorem 3. *For the linear optimal problem (3-6, 9)there exists an optimal element w_0 if the following conditions hold:*

9. $W_0 \neq \emptyset$;

10. the sets K_0, U and V are convex sets.

Let $J(w)$ have the form

$$J(w) = \int_{t_0}^{\theta} f^0(t, x(t), x(t-\tau), u(t))dt + \int_{\theta}^{t_1} g^0(t, y(t), y(t-\sigma), v(t))dt, \tag{10}$$

where $f^0(t, x_1, x_2, u)$ and $g^0(t, y_1, y_2, v)$ are continuous scalar functions on $[a, b] \times O^2 \times U$ and $[c, d] \times Y^2 \times V$, respectively.

We introduce the notations

$$F = (f^0, f)^T, G = (g^0, g)^T.$$

Theorem 4. *There exists an optimal element w_0 if the following conditions hold:*

11. $W_0 \neq \emptyset$;

12. there exist compact sets $K_1 \subset O$ and $M_1 \subset Y$ such that for an arbitrary $w \in W_0$ the corresponding solution

$$S(w) = \Big\{x(t;w), \quad t \in [\hat{\tau}, \theta]; \quad y(t;w), \ t \in [\hat{\sigma}, t_1]\Big\}$$

satisfies the conditions

$$x(t;w) \in K_1, \quad t \in [\hat{\tau}, \theta]; \quad y(t;w) \in M_1, \quad t \in [\hat{\sigma}, t_1];$$

13. for any fixed$(t, x_1) \in [a, b] \times O$, the set

$$\Big\{F(t, x_1, x_2, u) : (x_2, u) \in K_0 \times U\Big\}$$

is convex.

14. for any fixed$(t, x_1, x_2) \in [a, b] \times O^2$, the set

$$\Big\{F(t,x_1,x_2,u) : u \in U\Big\}$$

is convex.

15. for any fixed$(t,y_1,y_2) \in [c,d] \times Y^2$, the set

$$\Big\{G(t,y_1,y_2,v) : v \in V\Big\}$$

is convex.

R E F E R E N C E S

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Authors' addresses:

T. Tadumadze
I. Javakhishvili Tbilisi State University
Department of Mathematics &
I. Vekua Institute of Applied Mathematics
2, University St., Tbilisi 0186
Georgia
E-mail: tamaz.tadumadze@tsu.ge

S. Chachanidze
Department of Mathematics
I. Javakhishvili Tbilisi State University
13, University St., Tbilisi 0186
Georgia
E-mail: salome.chachanidze2014@ens.tsu.edu.ge