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EFFECTIVE SOLUTION OF THE ONE NONLOCAL PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURE IN A CIRCULAR DOMAIN

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Abstract. In the paper we consider the boundary value problem of statics of the linear theory of elastic mixture in a circular domain, when on the boundary of the domain the partial diplacement vectors satisfy the conditions of the Dirichlet and the Neumann problems respectively, and the rotor vector satisfies the condition of A. Bitsadze's nonlocal problem for harmonic vector-function in a circle. The problem can be reduced in same domain to the of the Dirichlet and the Neumann problems for equation of the Poisson.

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1. Introduction

The basic two-dimensional classical and nonlocal (nonclassical) boundary value problems of statics of the linear theory of elastic mixtures are studied in [1, 2, 4, 5] and also by many other authors.

In the work [3] A. Bitsadze considered the nonlocal boundary value problem in a circle for harmonic and bi-harmonic equations.

In the work in a circle for the homogeneous equation of statics of the linear theory of elastic mixture we consider the nonlocal boundary value problem, which is of this type of work [3] by A. Bitsadze.

In the work it is shown that the solution of the considering problem is reduced to the solution of the Dirichlet and Neumann problems in a circle for the Poisson equation.

2. Some auxiliary formulas and operators

In the two-dimensional case, the basic homogeneous equations of statics of the elastic mixture theory have the form [1]

$$a_1 \triangle u' + b_1 graddivu' + c \triangle u'' + dgraddivu'' = 0,$$

$$c \triangle u' + dgraddivu' + a_2 \triangle u'' + b_2 graddivu'' = 0, \tag{2.1}$$

where $u' = (u_1, u_2)^T$ and $u'' = (u_3, u_4)^T$ are partial displacements, a_k, b_k, $(k = 1, 2)$ c, d are the known constants, characterizing the physical properties of a mixture, and at that

$$a_1 = \mu_1 - \lambda_5, \quad a_2 = \mu_2 - \lambda_5, \quad c = \mu_3 + \lambda_5, \quad b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2 \frac{\rho_2}{\rho},$$

$$b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2 \frac{\rho_1}{\rho}, \quad \alpha_2 = \lambda_3 - \lambda_4, \ \ \rho = \rho_1 + \rho_2,$$

$$d = \mu_3 + \lambda_3 - \lambda_5 - \alpha_2 \frac{\rho_1}{\rho} \equiv \mu_3 + \lambda_4 - \lambda_5 + \alpha_2 \frac{\rho_2}{\rho}.$$

where $\mu_1, \mu_2, \mu_3,\ \lambda_p,\ p=\overline{1,5}$ are elastic constants, ρ_1 and ρ_2 are partial densities. The above constants, satisfying the definite conditions (inequalities [1]).

Using the identity

$$\Delta u' = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}\theta' + \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}\omega', \quad \theta^{'} = divu^{'}, \quad \omega^{'} = rotu^{'},$$

$$\Delta u'' = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}\theta'' + \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}\omega'', \theta^{''} = divu^{''}, \quad \omega^{''} = rotu^{''},$$

we can rewrite (2.1) as follows:

$$a\Delta u' + c_0\Delta u'' - b_1\begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}\omega' - d\begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}\omega'' = 0,$$

$$c_0\Delta u' + b\Delta u'' - d\begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}\omega' - b_2\begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \\ \frac{\partial}{\partial x_1} \end{pmatrix}\omega'' = 0, \tag{2.2}$$

where $a = a_1 + b_1, \quad b = a_2 + b_2, \quad c_0 = c + d.$

From (2.1) and (2.2) elementary calculations yield:

$$\Delta u' = H(x) = \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}\left(\frac{bb_1 - c_0 d}{d_1}\omega' + \frac{bd - c_0 b_2}{d_1}\omega''\right) =$$

$$\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \\ \frac{\partial}{\partial x_2} \end{pmatrix}\left(\frac{cd - a_1 b_1}{d_2}\theta' + \frac{cb_2 - a_2 d}{d_2}\theta''\right), \tag{2.3}$$

$$\Delta u'' = Q(x) = \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}\left(\frac{ad - c_0 b_1}{d_1}\omega' + \frac{ab_2 - c_0 d}{d_1}\omega''\right) =$$

$$\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \\ \frac{\partial}{\partial x_2} \end{pmatrix}\left(\frac{cb_1 - a_1 d}{d_2}\theta' + \frac{cd - a_1 b_2}{d_2}\theta''\right), \tag{2.4}$$

where $d_1 = ab - c_0^2 > 0, \quad d_2 = a_1 a_2 - c^2 > 0.$

Let D^0 be a circle $x_1^2 + x_2^2 < 1$ and $L = \{x : |x| = 1\}$. In what follows we assume $U \in C^3(D^0) \cap C^2(D^0 \cup L), \quad U = (u^{'}, u^{''})^T = (u_1, u_2, u_3, u_4)^T.$

Finally note that from (2.1) we'll have

$$\Delta\omega(x) = 0, \quad \omega = (\omega', \omega'')^T, ; \quad x \in D^0, \tag{2.5}$$

$$\Delta\theta(x) = 0, \quad \theta = (\theta', \theta'')^T, \quad x \in D^0. \tag{2.6}$$

By $TU = M_1\frac{\partial U}{\partial n(x)} + M_2\frac{\partial U}{\partial S(x)} + M_3 U$, we denote the stress vector, where [1]

$$M_1 = \begin{bmatrix} a & 0 & c_0 & 0 \\ 0 & a & 0 & c_0 \\ c_0 & 0 & b & 0 \\ 0 & c_0 & 0 & b \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & a - 2\mu_1 & 0 & c_0 - 2\mu_3 \\ 2\mu_1 - a & 0 & 2\mu_3 - c_0 & 0 \\ 0 & c_0 - 2\mu_3 & 0 & b - 2\mu_2 \\ 2\mu_3 - c_0 & 0 & 2\mu_2 - b & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} -b_1 n_2 \frac{\partial}{\partial x_2} & b_1 n_2 \frac{\partial}{\partial x_1} & -d n_2 \frac{\partial}{\partial x_2} & d n_2 \frac{\partial}{\partial x_1} \\ b_1 n_1 \frac{\partial}{\partial x_2} & -b_1 n_1 \frac{\partial}{\partial x_1} & d n_1 \frac{\partial}{\partial x_2} & -d n_1 \frac{\partial}{\partial x_1} \\ -d n_2 \frac{\partial}{\partial x_2} & d n_2 \frac{\partial}{\partial x_1} & -b_2 n_2 \frac{\partial}{\partial x_2} & b_2 n_2 \frac{\partial}{\partial x_1} \\ d n_1 \frac{\partial}{\partial x_2} & -d n_1 \frac{\partial}{\partial x_1} & b_2 n_1 \frac{\partial}{\partial x_2} & -b_2 n_1 \frac{\partial}{\partial x_1} \end{bmatrix},$$

$\frac{\partial}{\partial n(x)} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2}$, $\frac{\partial}{\partial S(x)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}$, $n = (n_1, n_2)^T$ is a unit vector.

Using the condition [1] $\int_L TU ds = 0$, we get

$$\int_L \frac{\partial u'(y)}{\partial n(y)} ds - \int_{D^0} H(x) dx = 0, \quad \int_L \frac{\partial u''(y)}{\partial n(y)} ds - \int_{D^0} Q(x) dx = 0. \tag{2.7}$$

3. Statement of the boundary value problem and the uniqueness theorem

In the present work we consider the following boundary value problem. Find in the domain D^0 a vector-function $U(x) = (u_1, u_2, u_3, u_4)^T$, which belongs to the class $C^3(D^0) \cap C^2(D^0 \cup L)$, is a solution of equations (2.2) and satisfies the following conditions:

$$(M)^+_{f,\varphi,h}: \quad u'(y) = f(y), \frac{\partial u''(y)}{\partial n(y)} = g(y), \quad y \in L,$$

$$\omega(e^{i\varphi_0}) - \omega(\delta e^{i\varphi_0}) = h(e^{i\varphi_0}), \quad 0 < \delta < 1, \quad 0 \le \varphi \le 2\pi, \tag{3.1}$$

where $f = (f_1, f_2)^T$,$g = (g_1, g_2)^T$ and $h = (h_1, h_2)^T$ are the real given vector-functions on the boundary L and belongs to the class $C^{1,\alpha}(L)$, $0 < \alpha < 1$, also $\int_0^{2\pi} h(e^{i\varphi_0}) = 0$.

The following assertion is true.

Theorem 3.1. *The general solution of the problem $[M]^+_{0,0,0}$ is represented by the formula $U_0 = (0, 0, \gamma, \beta)^T$, where γ and β are arbitrary real constants.*

Proof. The general solution of the nonlocal problem

$$\Delta\omega_0(x) = 0, \quad x \in D^0, \quad \omega_0(e^{i\varphi_0}) - \omega_0(\delta e^{i\varphi_0}) = 0, \quad 0 < \delta < 1, \quad 0 \le \varphi_0 \le 2\pi,$$

is represented by the formula [3]

$$\omega_0(x) = (\omega_0', \omega_0'')^T = (\gamma_1, \gamma_2)^T = const, \quad x \in D^0,$$

therefore from (2.3) and (2.4) when $\omega' = \omega_0'$ and $\omega'' = \omega_0''$ we obtain

$$H(x) = Q(x) = 0, \quad x \in D^0. \tag{3.2}$$

By virtue of the above said it follows (see (2.3.), (2.4), (3.1) and (3.2)) that our problem $[M]^+_{0,0,0}$ is reduced to the Dirichlet and the Neuman boundary value problems for the Laplace equation

$$\Delta u_0'(x) = 0, \quad x \in D^0, \quad u_0'(y) = 0, \quad y \in L, \tag{3.3}$$

$$\Delta u_0''(x) = 0, \quad x \in D^0, \quad \frac{\partial u_0''(y)}{\partial n(y)} = 0, \quad y \in L. \tag{3.4}$$

Finally note that [6], since the general solutions of problems (3.3) and (3.4) are

$u_0^{'}(x) = (u_{01}, u_{02})^T = 0$ and $u_0^{''}(x) = (u_{03}, u_{04})^T = const, \quad x \in D^0$ respectively therefore we can conclude that the Theorem 3.1. is true.

4. Solution of the problem $[M]^+_{f,g,h}$

Let us consider the following nonlocal problem

$$\Delta\omega(x) = 0, \quad x \in D^0, \quad \omega(e^{i\varphi_0}) - \omega(\delta e^{i\varphi_0}) = h(e^{i\varphi_0}), \tag{4.1}$$

$$0 < \delta < 1, \quad h \in C^{1,\alpha}, \quad (0 \le \varphi_0 \le 2\pi), \quad 0 < \alpha < 1, \quad \int_0^{2\pi} h(e^{i\varphi_0})d\varphi_0 = 0.$$

Note that the solution of problem (4.1) has the form [3]

$$\omega(x) = \omega(r, \varphi) = C_0 + \sum_{k=1}^{\infty} \frac{r^k}{1-\delta^k}(C_k \cos k\varphi + D_k \sin k\varphi), \tag{4.2}$$

where $r = \sqrt{x_1^2 + x_2^2}$, $\quad (x_1, x_2) \in D^0$, $\quad C_k = (C_{k1}, C_{k2})^T$ and $D_k = (D_{k1}, D_{k2})^T$ are the Fourier coefficients of the vector-function $h(e^{i\varphi_0})$, $0 \le \varphi_0 \le 2\pi$, and $C_0 = (C_{01}, C_{02})^T$ is an arbitrary real constant vector.

Owing to (4.2) we can conclude that $H(x)$ and $Q(x)$ (see (2.3) and (2.4)) are known vector-functions and belong to the class $C^1(D^0) \bigcap C(D^0 \bigcup L)$.

By virtue of the above said from (2,3), (2,4) and (3.1)it follows that our problem $[M]^+_{f,g,h}$ is reduced to the Dirichlet and Neumann boundary value problems for the Poisson equation:

$$\Delta u^{'}(x) = H(x), \quad x \in D^0, \quad u^{'}(y) = f(y), \quad y \in L, \tag{4.3}$$

$$\Delta u^{''}(x) = Q(x), \quad x \in D^0, \quad \frac{\partial u^{''}(y)}{\partial n(y)} = g(y), \quad y \in L. \tag{4.4}$$

Further note that owing to the second formula (2.7) the necessary condition (see[6])

$$\int_L g(y)ds - \int_{D^0} Q(x)dx = 0$$

of solvability (4.4) problem is fulfilled.

Taking into account the obtained results, we'll have that the problem $[M]^+_{f,g,h}$ is solved and the solution is represented by means of the solutions of problems (4.3) and (4.4) .

A solution of the problem $[M]^+_{f,g,h}$ is given by the formula

$$U(x) = U(r, \varphi) = (u^{'}, u^{''})^T,$$

where

$$u^{'}(x) = u^{'}(r, \varphi) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 G_{(1)}(r, \varphi; R, \varphi_0)H(R, \varphi_0)RdRd\varphi_0$$
$$+\frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)f(\varphi_0)d\varphi_0}{1 - 2r\cos(\varphi - \varphi_\varphi) + r^2},$$

$$u^{''}(x) = u^{''}(r, \varphi) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 G_{(2)}(r, \varphi; R, \varphi_0)Q(R, \varphi_0)RdRd\varphi_0$$
$$-\frac{1}{\pi}\int_0^{2\pi} ln\sqrt{1 - 2r\cos(\varphi - \varphi_0) + r^2}g(\varphi_0)d\varphi_0.$$

Here $G_{(1)}(x, y^0)$ and $G_{(2)}(x, y^0)$ are the Green functions of the Dirichlet and the Neumann problems respectively in a circle $x_1^2 + x_2^2 < 1$ for the Laplace equation.

Further note that:

$$G_{(1)}(x, y^0) = ln \frac{\sqrt{(y_1^0 - x_1)^2 + (y_2^0 - x_2)^2}}{r\sqrt{(y_1^0 - \frac{x_1}{r^2})^2 + (y_2^0 - \frac{x_2}{r^2})^2}},$$

$$G_2(x, y^0) = \ln \frac{r}{R}\sqrt{(y_1^0 - x_1)^2 + (y_2^0 - x_2)^2}\sqrt{(y_1^0 - \frac{x_1}{r^2})^2 + (y_2^0 - \frac{x_2}{r^2})^2},$$

$$r = \sqrt{x_1^2 + x_2^2}, \quad (x_1, x_2) \in D^0, \quad x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi, \quad 0 \leq \varphi \leq 2\pi,$$

$$R = \sqrt{y_1^{0^2} + y_2^{0^2}}, \quad (y_1^0, y_2^0) \in D^0, \quad y_1^0 = R\cos\varphi_0, \quad y_2^0 = R\sin\varphi_0, \quad 0 \leq \varphi_0 \leq 2\pi,$$

Remark 4.1. The following BVP can be solved in a similar way:

Find in the domain D^0 a vector $U = (u^{'}, u^{''})^T = (u_1, u_2, u_3, u_4)^T$, belongs to the class $C^3(D^0) \bigcap C^2(D^0 \bigcup L)$, satisfying the equations (2.1) and the following boundary conditions:

$$u^{'}(y) = f(y), \quad \frac{\partial u^{''}(y)}{\partial n(y)} = g(y), \quad y \in L,$$

$$\Theta(e^{i\varphi_0}) - \Theta(\delta e^{i\varphi_0}) = h(e^{i\varphi_0}),$$

$$0 < \delta < 1, \quad 0 \leq \varphi_0 \leq 2\pi.$$

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