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ON THE EXISTENCE OF AN OPTIMAL ELEMENT FOR ONE CLASS OF NEUTRAL OPTIMAL PROBLEM WITH THE PHASE RESTRICTIONS

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Abstract. For an optimal problem involving quasi-linear neutral differential equation with the general boundary conditions and the phase restrictions, existence theorems of optimal element are provided. Under element we imply the collection of initial and final moments, delay parameters, initial vector, initial functions and control.

Keywords and phrases: Neutral optimal problem, optimal element, existence theorem, phase restrictions.

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1. Introduction

Neutral controlled differential equation is a mathematical model of such controlled dynamical system whose behavior depends on the prehistory of the state of the system and on its velocity (derivative of the trajectory) at a given moment of time. Such mathematical models arise in different areas of natural sciences and economics [1-3]. To illustrate this, we consider the simplest model of economic growth. Let $N(t)$ be a quantity of a product produced at the moment t which is expressed in money units.The fundamental principle of the economic growth has the form

$$N(t) = C(t) + I(t), \tag{1}$$

where $C(t)$ is a quantity of money for various requirements and $I(t)$ is a quantity-induced investment. We consider the case where the functions $C(t)$ and $I(t)$ have the form

$$C(t) = u_1(t)N(t), \tag{2}$$

$$I(t) = u_2(t)N(t-\tau) + u_3(t)\dot{N}(t) + u_4(t)\dot{N}(t-\tau)$$

$$+\alpha_1\ddot{N}(t) + \alpha_2\ddot{N}(t-\sigma), \tag{3}$$

where $u_i(t) \in [u_i, v_i], i = \overline{1,4}$ are control functions; $u_i > v_i > 0, \alpha_1 > 0$ and $\alpha_2 > 0$ are given numbers; $\tau > 0$ and $\sigma > 0$ are the so-called delay parameters. Formula (3) shows that the value of investment at the moment t depends: on the quantity of money at the moment $t-\tau$ (in the past); on the velocity (production current) at the moments t and $t-\tau$; on the acceleration at the moments t and $t-\sigma$. From formulas (1)-(3) we get the equation

$$\ddot{N}(t) = \frac{1-u_1(t)}{\alpha_1}N(t) - \frac{u_2(t)}{\alpha_1}N(t-\tau) - \frac{u_3(t)}{\alpha_1}\dot{N}(t)$$

$$-\frac{u_4(t)}{\alpha_1}\dot{N}(t-\tau) - \frac{\alpha_2}{\alpha_1}\ddot{N}(t-\sigma),$$

which is equivalent to the following neutral controlled functional differential equation:

$$\begin{cases}\dot{x}_1(t)=x_2(t),\\ \dot{x}_2(t)=\frac{1-u_1(t)}{\alpha_1}x_1(t)-\frac{u_2(t)}{\alpha_1}x_1(t-\tau)-\frac{u_3(t)}{\alpha_1}x_2(t)\\ -\frac{u_4(t)}{\alpha_1}x_2(t-\tau)-\frac{\alpha_2}{\alpha_1}\dot{x}_2(t-\sigma),\end{cases} \tag{4}$$

where $x_1(t)=N(t)$.

Let $\tau_2>\tau_1>0, \sigma_2>\sigma_1>0$ and let $\vartheta>0$ be given numbers and let Ω_0 be a set of measurable control functions

$$u(t)=(u_1(t),u_2(t),u_3(t),u_4(t))\in[u_1,v_1]\times[u_2,v_2]\times[u_3,v_3]$$
$$\times[u_4,v_4], t\in[0,\vartheta].$$

To each element

$$v=(\tau,\sigma,u(\cdot))\in V=[\tau_1,\tau_2]\times[\sigma_1,\sigma_2]\times\Omega_0$$

we assign equation (4) with the initial condition

$$x_1(t)=\varphi_1(t), t\in[\tilde{\tau},0]; x_2(t)=\varphi_2(t), t\in[\tilde{\tau},0), \tag{5}$$

where $\varphi_1(t)\geq 0$ and $\varphi_2(t)$ are fixed continuously differentiable initial functions, $\tilde{\tau}=-\max(\tau_2,\sigma_2)$.

Definition 1. Let $v=(\tau,\sigma,u(\cdot))\in V$. A function

$$x(t;v)=(x_1(t;v),x_2(t;v)), t\in[\tilde{\tau},\vartheta]$$

is called a solution corresponding to the element v, if it satisfies condition (5) and is absolutely continuous on the interval $[\tilde{\tau},\vartheta]$ and satisfies Eq. (4) almost everywhere on $[0,\vartheta]$.

Definition 2. An element $v\in V$ is said to be admissible if there exists the corresponding solution $x(t;v)$ satisfying the condition

$$x_1(\vartheta;v)\geq 0. \tag{6}$$

We denote the set of admissible elements by V_0.

Definition 3. An element $v_0=(\tau_0,\sigma_0,u_0(\cdot))\in V_0$ is said to be optimal if

$$q^0(v_0)=\inf_{v\in V_0}q^0(v), \tag{7}$$

where $q^0(v)=-x_1(\vartheta;v)$.

(4)-(7) is called the neutral optimal problem of economic growth. In the paper the existence theorems of optimal element are formulated for the quasi-linear neutral optimal problem with the general boundary conditions, the phase restrictions and functional. Under element we imply the collection of initial and finally moments, delay parameters, initial vector,initial functions and control. Theorems of existence for optimal problems involving various functional differential equations are given in [4-15].

2. Statement of the neutral optimal problem and the existence theorems

Let R_x^n be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$, where T means transpose. Let $a<t_{01}<t_{02}<t_{11}<t_{12}<b$ be given numbers with $t_{11}-t_{02}>\max\{\tau_2,\sigma_2\}$;

suppose that $O \subset R_x^n$ is an open set and $U \subset R_u^r$ is a compact set, the function

$$f(t,x,y,u) = (f^1(t,x,y,u),...,f^n(t,x,y,u))^T$$

is continuous on the set $I \times O^2 \times U$ and continuously differentiable with respect to x and y, where $I = [a,b]$; further, let Φ and Δ be sets of measurable initial functions $\varphi(t) \in K_0, t \in [\hat{\tau}, t_{02}]$ and $\varsigma(t) \in K_1, t \in [\hat{\tau}, t_{02}]$, respectively, where $\hat{\tau} = a - \max\{\tau_2, \sigma_2\}, K_0 \subset O$ is a compact set, $K_1 \subset R_x^n$ is a convex and compact set; let Ω be a set of measurable control function $u(t) \in U, t \in I$ and let

$$q^i(t_0, t_1, \tau, \sigma, x_0, x_1), i = \overline{0, l+k}$$

be continuous scalar functions on the set

$$[t_{01}, t_{02}] \times [t_{11}, t_{12}] \times [\tau_1, \tau_2] \times [\sigma_1, \sigma_2] \times X_0 \times O,$$

where $X_0 \subset O$ is a compact set.

To each element

$$w = (t_0, t_1, \tau, \sigma, x_0, \varphi(\cdot), \varsigma(\cdot), u(\cdot)) \in W = [t_{01}, t_{02}] \times [t_{11}, t_{12}] \times [\tau_1, \tau_2] \times [\sigma_1, \sigma_2] \times$$

$$\times X_0 \times \Phi \times \Delta \times \Omega$$

we assign the quasi-linear (linear with respect to prehistory of the velocity) neutral differential equation

$$\dot{x}(t) = A(t)\dot{x}(t-\sigma) + f(t, x(t), x(t-\tau), u(t)), t \in [t_0, t_1] \tag{8}$$

with the initial conditions

$$\begin{cases} x(t) = \varphi(t), t \in [\hat{\tau}, t_0), x(t_0) = x_0, \\ \dot{x}(t) = \varsigma(t), t \in [\hat{\tau}, t_0), \end{cases} \tag{9}$$

where $A(t), t \in I$ is a given $n \times n$ -dimensional continuous matrix function.

Remark. The symbol $\dot{x}(t)$ on the interval $[\hat{\tau}, t_0)$ is not connected with derivative of the function $\varphi(t)$.

Definition 4. Let $w = (t_0, t_1, \tau, \sigma, x_0, \varphi(\cdot), \varsigma(\cdot), u(\cdot)) \in W$. A function

$$x(t) = x(t; w) \in O, t \in [\hat{\tau}, t_1],$$

is called a solution, corresponding to the element w, if it satisfies condition (9) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies Eq. (9) almost everywhere on $[t_0, t_1]$.

Definition 5. An element $w \in W$ is said to be admissible if there exists the corresponding solution $x(t) = x(t; w)$ satisfying the boundary conditions

$$\begin{cases} q^i(t_0, t_1, \tau, \sigma, x_0, x(t_1)) = 0, i = \overline{1, l}, \\ q^{l+j}(t_0, t_1, \tau, \sigma, x_0, x(t_1)) \geq 0, j = \overline{l+1, l+k} \end{cases} \tag{10}$$

and the phase restrictions

$$\Theta^\varrho(x(t)) \geq 0, t \in [t_0, t_1], \varrho = \overline{1, m}, \tag{11}$$

where $\Theta^\varrho(x), x \in O, \varrho = \overline{1, m}$ are given continuous functions.

We denote the set of admissible elements by W_0. Now we consider the functional

$$J(w) = q^0(t_0, t_1, \tau, \sigma, x_0, x(t_1))$$

where $x(t) = x(t; w)$.

Definition 6. An element $w_0 = (t_{00}, t_{10}, \tau_0, \sigma_0, x_{00}, \varphi_0(\cdot), \varsigma_0(\cdot), u_0(\cdot)) \in W_0$ is said to be optimal if

$$J(w_0) = \inf_{w \in W_0} J(w). \tag{12}$$

(8)-(12) is called the quasi-linear neutral optimal problem.

Theorem 1. *There exists an optimal element w_0 if the following conditions hold:*
a) $W_0 \neq \emptyset$;
b) there exists a compact set $K_2 \subset O$ such that for an arbitrary $w \in W_0$

$$x(t; w) \in K_2, t \in [\hat{\tau}, t_1];$$

c) for any fixed $(t, x) \in I \times O$ the set

$$\{f(t, x, y, u) : (y, u) \in K_0 \times U\}$$

is convex;
d) for any fixed $(t, x, y) \in I \times O^2$ the set

$$\{f(t, x, y, u) : u \in U\}$$

is convex.

Theorem 1 is proved by the scheme given in [14-15].

Theorem 2. *There exists an optimal element w_0 if the following conditions hold:*
e) $f(t, x, y, u) = B(t, x)y + C(t, x)u$;
f) $W_0 \neq \emptyset$;
g) *there exists a compact set $K_2 \subset O$ such that for an arbitrary $w \in W_0$*

$$x(t; w) \in K_2, t \in [\hat{\tau}, t_1];$$

h) *K_0 and U are convex sets.*

Theorem 3. *Let the initial function $\varphi(t)$ be fixed (see (9))and the conditions a), b), d) hold. Then there exists an optimal element.*

Finally, we note that from Theorem 3 it follows that for the neutral optimal problem (4)-(7) there exists an optimal element v_0.

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