Seminar of I. Vekua Institute of Applied Mathematics
REPORTS, Vol. 46, 2020

REPRESENTATION OF THE SNELL ENVELOPE AS THE FUTURE SUPREMUM PROCESS

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Abstract. For a right continuous (with left limits) process X we consider its Snell envelope Y, that is the smallest supermartingale bounding X from above, with the Doob–Meyer decomposition $Y = M - B$. Here M is the uniformly integrable martingale and B is a nondecreasing predictable process. We establish that Y is indistinguishable from the process $M+C$, where C is the so-called future supremum of the difference $(X-M)$. From the latter result we obtain the dual representation of the value of the optimal stopping problem due to Rogers [3] and Haugh and Kogan [4].

Keywords and phrases: Future supremum process, optimal stopping, dual representation.

AMS subject classification (2010): 60G40, 60G44, 91B24.60G40, 60G44, 91B24.

1. Introduction

Let $[0,T]$ be a finite time interval and let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space with a filtration $(\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual conditions. Consider a stochastic process $X = (X_t)_{0\le t\le T}$ adapted to the given filtration $(\mathcal{F}_t)_{0\le t\le T}$ with trajectories $X_t(\omega)$, $0 \le t \le T$, which are right continuous with left limits and which satisfy the following condition: there exists some nonnegative uniformly integrable martingale U such that

$$|X| \le U. \tag{1}$$

It is well known (see Thompson [1]) that in this case there exists a smallest supermartingale $Y = (Y_t)_{0\le t\le T}$ greater or equal then X, which is called the Snell envelope of X and which possesses the following representation as the value process of the optimal stopping problem

$$Y_t = \operatorname*{ess\,sup}_{\tau_t} \mathbf{E}(X_{\tau_t} \mid \mathcal{F}_t), \quad 0 \le t \le T,$$

where supremum is taken over all $(\mathcal{F}_t)$-stopping times τ_t with $t \le \tau_t \le T$.

Let us write the Doob–Meyer decomposition for the supermartingale Y

$$Y_t = M_t - B_t, \quad 0 \le t \le T, \tag{2}$$

where M is a uniformly integrable martingale and B is a predicable nondecreasing integrable process with $B_0 = 0$.

Consider the process $(X_t - M_t)_{0\le t\le T}$ with right continuous paths and let us introduce the so-called future supremum process

$$C_t = \sup_{t\le s\le T} (X_s - M_s), \quad 0 \le t \le T.$$

It is clear that the trajectories of the process C are nonincreasing and therefore have left and right hand limits. Now we will show that the right-continuity of the process $(X - M)$ implies the right continuity of the process C.

Lemma. *The future supremum process* $C = (C_t)_{0\le t\le T}$ *is right continuous, that is*

$$C_{t+} = C_t, \quad 0 \le t < T. \tag{3}$$

Proof. We have from the definition of the process C

$$C_t = \max\Big(\sup_{t\le s<t+\delta}(X_s - M_s), C_{t+\delta}\Big), \quad 0 \le t < t+\delta \le T.$$

We pass to limit $\delta \downarrow 0$ and get

$$C_t = \max(X_t - M_t, C_{t+}), \quad 0 \le t < T.$$

Fix t (and $\omega \in \Omega$), there are two cases

(a) $C_t > X_t - M_t$,

(b) $C_t = X_t - M_t$.

In the first case we obtain $C_t = C_{t+}$.
In the second case we have

$$X_t - M_t = C_t \ge C_{t+} \ge C_{t+\delta} \ge X_{t+\delta} - M_{t+\delta}.$$

If we pass to limit $\delta \downarrow 0$ in the last chain of inequalities, we shall have

$$X_t - M_t = C_t \ge C_{t+} \ge X_t - M_t$$

and hence we get the right-continuity (3) of the process C.

2. Formulation and the proof of the main result

We recall at first the fundamental martingale property of the Snell envelope Y (see, for example, Shashiashvili [2, Theorem 1])

$$Y_t = \mathbf{E}(\mathbf{Y}_{\tau_t^\varepsilon} \mid \mathcal{F}_\mathbf{t}), \quad \mathbf{0 \le t \le T}, \tag{4}$$

where for arbitrary $\varepsilon > 0$ the stopping times τ_t^ε are defined as

$$\tau_t^\varepsilon = \inf\big\{s \ge t:\ X_s \ge Y_s - \varepsilon\big\} \wedge T, \quad 0 \le t \le T.$$

From equality (4) and the Doob-Meyer decomposition (2) of the Snell envelope Y, we have

$$B_t = B_{\tau_t^\varepsilon} \ \ (P\text{-a.s.}), \quad 0 \le t \le T, \quad \varepsilon > 0. \tag{5}$$

The following theorem is the basic result of this article

Theorem. *Let the right continuous with left-hand limits adapted stochastic process* X *satisfy condition* (1). *Then its Snell envelope* Y *is indistinguishable from the process* $M + C$, *that is*

$$Y_t = \sup_{t\le s\le T}(X_s - (M_s - M_t)) \ \text{ for all } 0 \le t \le T. \tag{6}$$

Proof. Take arbitrary t, s with $t \leq s \leq T$. We have

$$Y_s - X_s + B_s \geq B_s \geq B_t \text{ as } Y_s - X_s \geq 0,$$

hence we get

$$\inf_{t \leq s \leq T} (Y_s - X_s + B_s) \geq B_t, \ \ 0 \leq t \leq T \ \ (P\text{-a.s.}). \tag{7}$$

Consider the following random variable

$$Y_{\tau_t^\varepsilon} - X_{\tau_t^\varepsilon} + B_{\tau_t^\varepsilon}.$$

We have from equality (5) $B_{\tau_t^\varepsilon} = B_t$ and from the right-continuity of the process $(Y - X)$ we obtain

$$Y_{\tau_t^\varepsilon} - X_{\tau_t^\varepsilon} \leq \varepsilon,$$

therefore we can write

$$Y_{\tau_t^\varepsilon} - X_{\tau_t^\varepsilon} + B_{\tau_t^\varepsilon} \leq \varepsilon + B_t,$$

from which it follows that

$$\inf_{t \leq s \leq T} (Y_s - X_s + B_s) \leq \varepsilon + B_t.$$

Tending ε to zero we get

$$\inf_{t \leq s \leq T} (Y_s - X_s + B_s) \leq B_t,$$

which together with (7) gives us the important equality

$$\inf_{t \leq s \leq T} (Y_s - X_s + B_s) = B_t,$$

which is equivalent to the following equality

$$Y_t = M_t + C_t \ \ (P\text{-a.s.}) \text{ for fixed } t, \ \ 0 \leq t \leq T.$$

But the process Y, M and C are right continuous and then it is well known, that the process Y and $M + C$ are indistinguishable.

Rogers [3] and, independently, Haugh and Kogan [4] introduced a "dual" way to price American options based on simulating the path of the option payoff. Their basic theoretical result establishes the following dual representation of the value process Y (the Snell envelope) of the optimal stopping problem.

Proposition. *Let us assume that the stochastic process X satisfies the integrability condition*

$$\sup_{0 \leq t \leq T} |X_t| \in L^p \ \ \textit{for some} \ \ p > 1.$$

Then the following dual representation is valid

$$Y_t = \inf_{N \in H^1} \mathbf{E}\Big(\sup_{\mathbf{t} \leq \mathbf{s} \leq \mathbf{T}} (\mathbf{X_s} - (\mathbf{N_s} - \mathbf{N_t})) \mid \mathcal{F}_\mathbf{t} \Big), \tag{8}$$

where H^1 is the space of martingales N, for which

$$\sup_{0 \leq t \leq T} |N_t| \in L^1.$$

We note that from our the Theorem we easily get the Proposition. Indeed we have

$$\mathbf{E}(\mathbf{X}_{\tau_\mathbf{t}} \mid \mathcal{F}_\mathbf{t}) = \mathbf{E}\big((\mathbf{X}_{\tau_\mathbf{t}} - (\mathbf{N}_{\tau_\mathbf{t}} - \mathbf{N}_\mathbf{t})) \mid \mathcal{F}_\mathbf{t}\big) \leq \mathbf{E}\Big(\sup_{\mathbf{t}\leq\mathbf{s}\leq\mathbf{T}}(\mathbf{X}_\mathbf{s} - (\mathbf{N}_\mathbf{s} - \mathbf{N}_\mathbf{t})) \mid \mathcal{F}_\mathbf{t}\Big).$$

On the other hand, taking the conditional expectation with respect to $\mathcal{F}_t$ in the equality (6) we get

$$Y_t = \mathbf{E}\Big(\sup_{\mathbf{t}\leq\mathbf{s}\leq\mathbf{T}}(\mathbf{X}_\mathbf{s} - (\mathbf{M}_\mathbf{s} - \mathbf{M}_\mathbf{t})) \mid \mathcal{F}_\mathbf{t}\Big),$$

thus the infimum in (8) is achieved by the martingale $N = M$.

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Received 10.09.2020; accepted 17.11.2020

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