Seminar of I. Vekua Institute of Applied Mathematics
REPORTS, Vol. 46, 2020

SOME BASIC PROBLEMS OF THE PLANE THEORY OF ELASTICITY FOR MATERIALS WITH VOIDS.

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Abstract. In the present paper we consider the materials with voids. The two dimensional system of equations, corresponding to a plane deformation case, is written in a complex form and its general solution is presented with the use of two analytic functions of a complex variable and a solution of the Helmholtz equation. The boundary value problems are solved for a circle and the plane with a circular hole.

Keywords and phrases: Materials with voids, boundary value problems.

AMS subject classification (2010): 74F05, 74G10.

1. Introduction

This paper is conserved with plane problems in the theory of linear elastic materials with voids. Goodman and Cowin [1] presented a theory for granular materials with interstitial voids and used the formal argument of continuum mechanics. Nunziato and Cowin [2] presented a nonlinear theory for the behavior of porous solids with (single) voids or vacuous pores, where the matrix material is elastic and the interstices are void of material. Further, Cowin and Nunziato [3] considered the linear theory of elastic materials with voids. Ieşan [4] extended this theory and presented a linear theory of thermoelastic materials with voids. The mathematical theories for materials with single voids are extensively investigated by several authors and the basic results may be found in the books of Ciarletta and Ieşan [5], Ieşan [6], Straughan [7] and references therein.

In [8-14] some results of the 2D and 3D theories of elasticity for materials with voids are given.

The present paper deals with the plane strain problem for linear elastic materials with voids. The governing system of equations of the plane strain is rewritten in the complex form and its general solution is represented by means of two analytic functions of the complex variable and a solution of the Helmholtz equation. The constructed general solution enables us to solve analytically a problem for a circle and a problem for the infinite plane with a circular hole.

2. Basic equations

Let $x = (x_1; x_2; x_3)$ be a point of the Euclidean three dimensional space R^3. In what follows we consider an isotropic and homogeneous elastic solid with voids, occupying a region of $\Omega \in R^3$. The governing equations of the theory of elastic materials with voids can be expressed in the following form [3]:

• Equations of equilibrium

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \Phi_i = 0, \quad i, j = 1, 2, 3, \tag{1}$$

$$\frac{\partial h_i}{\partial x_i} + g + \Psi = 0, \quad i = 1, 2, 3, \tag{2}$$

where σ_{ij} is the symmetric stress tensor, Φ_i are the volume force components, h_i is the equilibrated stress vector, g is the intrinsic equilibrated body force and Ψ is the extrinsic equilibrated body force.

• Constitutive equations

$$\begin{aligned} &\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + \beta\phi\delta_{ij}, \quad i, j, k = 1, 2, 3, \\ &h_i = \nu \frac{\partial \phi}{\partial x_i}, \quad i = 1, 2, 3, \\ &g = -\xi\phi - \beta e_{kk}, \quad k = 1, 2, 3, \end{aligned} \tag{3}$$

where λ and μ are the Lamé constants, ν, β and ξ are the constants characterizing the body porosity; δ_{ij} is the Kronecker delta; $\phi := \nu - \nu_0$ is the change of the volume fraction from the matrix reference volume fraction ν_0 (clearly, the bulk density $\rho = \nu\gamma$, $0 < \nu \leq 1$, here γ is the matrix density and ρ is the mass density); e_{ij} is the strain tensor and

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{4}$$

where u_i, $i = 1, 2, 3$ are the components of the displacement vector.

The constitutive equations also meet some other conditions, following from physical considerations

$$\begin{aligned} &\mu > 0, \qquad \nu > 0, \qquad \xi > 0, \\ &3\lambda + 2\mu > 0, \qquad (3\lambda + 2\mu)\xi > 3\beta^2. \end{aligned} \tag{5}$$

From the basic three-dimensional equations we obtain the basic equations for the case of plane strain. Let Ω be a sufficiently long cylindrical body with generatrix parallel to the Ox_3-axis. Denote by V the crosssection of this cylindrical body, thus $V \subset R^2$. In the case of plane deformation $u_3 = 0$ while the functions u_1, u_2 and ϕ do not depend on the coordinate x_3 [16].

As it follows from formulas (3) and (4), in the case of plane strain

$$\sigma_{k3} = \sigma_{3k} = 0, \quad h_3 = 0, \quad k = 1, 2.$$

Assuming $\Phi_i \equiv 0$ and $\Psi \equiv 0$. Therefore the system of equilibrium equations (1), (2) takes the form

$$\begin{aligned} &\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} = 0, \\ &\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0, \\ &\frac{\partial h_1}{\partial x_1} + \frac{\partial h_2}{\partial x_2} + g = 0. \end{aligned} \tag{6}$$

Now, Relations (3) are rewritten as

$$\begin{aligned} &\sigma_{11} = \lambda\theta + 2\mu\frac{\partial u_1}{\partial x_1} + \beta\phi, \\ &\sigma_{22} = \lambda\theta + 2\mu\frac{\partial u_2}{\partial x_2} + \beta\phi, \\ &\sigma_{12} = \sigma_{21} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \\ &\sigma_{33} = \sigma(\sigma_{11} + \sigma_{22}), \\ &h_k = \nu\frac{\partial \phi}{\partial x_k}, \quad k = 1, 2, \\ &g = -\xi\phi - \beta\theta, \end{aligned} \tag{7}$$

where σ is the Poisson ratio, $\theta = \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2}$.

If relations (7) are substituted into system (6) then we obtain the following system of governing equations of statics with respect to the functions u_1, u_2 and ϕ

$$\begin{aligned} &\mu\Delta u_k + (\lambda + \mu)\frac{\partial \theta}{\partial x_k} + \beta\frac{\partial \phi}{\partial x_k} = 0, \quad k = 1, 2 \\ &(\nu\Delta - \xi)\Delta\phi - \beta\theta = 0. \end{aligned} \tag{8}$$

Note that Δ is the two-dimensional Laplace operator.

On the plane Ox_1x_2, we introduce the complex variable $z = x_1 + ix_2 = re^{i\vartheta}$, $(i^2 = -1)$ and the operators $\partial_z = 0.5\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)$, $\partial_{\bar z} = 0.5\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$, $\bar z = x_1 - ix_2$, and $\Delta = 4\partial_z\partial_{\bar z}$.

To write system (6) in the complex form, we multiplied the second equation of this system by i and sum up with the first equation

$$\begin{aligned} &\partial_z(\sigma_{11} - \sigma_{22} + 2i\sigma_{12}) + \partial_{\bar z}(\sigma_{11} + \sigma_{22}) = 0, \\ &\partial_z h_+ + \partial_{\bar z}\bar h_+ + g = 0, \end{aligned} \tag{9}$$

where we rewrite $h_+ = h_1 + ih_2$ and formulas (7) as follows

$$\begin{aligned} &\sigma_{11} - \sigma_{22} + 2i\sigma_{12} = 4\mu\partial_{\bar z}u_+, \\ &\sigma_{11} + \sigma_{22} = 2(\lambda + \mu)\theta + 2\beta\phi, \\ &h_+ = 2\nu\partial_{\bar z}\phi, \\ &g = -\xi\phi - \beta\theta, \\ &\theta = \partial_z u_+ + \partial_{\bar z}\bar u_+, \quad u_+ = u_1 + iu_2. \end{aligned} \tag{10}$$

Substituting relations (10) into system (9), we rewrite system (8) in the complex form

$$\begin{aligned} &2\mu\partial_{\bar z}\partial_z u_+ + (\lambda + \mu)\partial_{\bar z}\theta + \beta\partial_{\bar z}\phi = 0, \\ &(\nu\Delta - \xi)\phi - \beta\theta = 0. \end{aligned} \tag{11}$$

Theorem (see [15]). *The general solution of system (11) is represented as follows:*

$$2\mu u_+ = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - \frac{4\alpha\beta\mu}{\xi(\lambda+2\mu)-\beta^2}\partial_{\bar z}\chi(z,\bar z), \tag{12}$$

$$\phi = \chi(z,\bar z) - \frac{\beta}{\xi(\lambda+\mu)-\beta^2}(\varphi'(z)+\overline{\varphi'(z)}), \tag{13}$$

where $\kappa = \dfrac{\xi(\lambda+3\mu)-\beta^2}{\xi(\lambda+\mu)-\beta^2}$, $\varphi(z)$ *and* $\psi(z)$ *are arbitrary analytic functions of a complex variable* z *in the domain* V, $\chi(z,\bar z)$ *is an arbitrary solution of the Helmholtz equation*

$$\Delta\chi(z,\bar z) - \gamma^2\chi(z,\bar z) = 0, \quad \gamma^2 = \frac{\xi(\lambda+2\mu)-\beta^2}{\alpha(\lambda+2\mu)}.$$

From (10) we have

$$\begin{aligned}
&\sigma_{11}-\sigma_{22}+2i\sigma_{12} = -2z\overline{\varphi''(z)} - 2\overline{\psi'(z)} - \frac{8\alpha\beta\mu}{\xi(\lambda+2\mu)-\beta^2}\partial_{\bar z}\partial_{\bar z}\chi(z,\bar z),\\
&\sigma_{11}+\sigma_{22} = \frac{2\xi(\lambda+2\mu)^2-2(\lambda+3\mu)\beta^2}{(\lambda+2\mu)(\xi(\lambda+2\mu)-\beta^2)}\left(\varphi'(z)+\overline{\varphi'(z)}\right) + \frac{2\mu\beta}{\lambda+2\mu}\chi(z,\bar z),\\
&h_+ = 2\alpha\partial_{\bar z}\chi(z,\bar z) - \frac{2\alpha\beta}{\xi(\lambda+\mu)-\beta^2}\overline{\varphi''(z)},\\
&g = \left(\frac{\beta^2}{\lambda+2\mu}-\xi\right)\chi(z,\bar z) - \frac{\beta\mu(\xi(\lambda+2\mu)-\beta^2)}{(\lambda+\mu)(\lambda+2\mu)(\xi(\lambda+\mu)-\beta^2)}\left(\varphi'(z)+\overline{\varphi'(z)}\right).
\end{aligned} \tag{14}$$

Assume that mutually perpendicular unit vectors $\mathbf{l}$ and $\mathbf{s}$ are such that

$$\mathbf{l}\times\mathbf{s} = \mathbf{e}_3,$$

where $\mathbf{e}_3$ is the unit vector, directed along the x_3-axis. The vector $\mathbf{l}$ forms the angle α with the positive direction of the x_1-axis. Then the displacement components $u_l = \mathbf{u}\cdot\mathbf{l}$, $u_s = \mathbf{u}\cdot\mathbf{s}$ as well as the stress and moment stress components, acting on an area of arbitrary orientation are expressed by the formulas

$$\begin{aligned}
&u_l + iu_s = e^{-i\alpha}u_+,\\
&\sigma_{ll}+i\sigma_{ls} = \frac{1}{2}\left[\sigma_{11}+\sigma_{22}+(\sigma_{11}-\sigma_{22}+2i\sigma_{12})e^{-2i\alpha}\right],\\
&h_l = \frac{1}{2}\left[h_+e^{-i\alpha}+\bar h_+e^{i\alpha}\right].
\end{aligned} \tag{15}$$

We can analytically solve the class of plane boundary value problems for both-finite and infinite domains.

3. The boundary value problem for a circle

Let us consider the elastic circle with voids, bounded by the circumference of radius R (Fig. 1). The origin of coordinates is at the center of the circle.

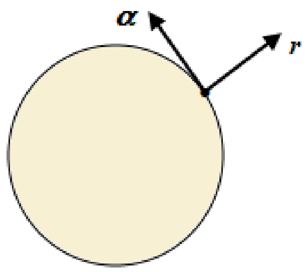

Fig. 1

On the circumference we consider the following boundary value problem

$$\sigma_{rr}+i\sigma_{r\alpha}=A+iB,\quad \phi=C,\quad \text{on } r=R, \tag{16}$$

where A, B and C are sufficiently smooth functions.

Substituting the formulas (14) into (15) we have

$$\begin{aligned}\sigma_{rr}+i\sigma_{r\alpha}&=\kappa_1\left(\varphi'(z)+\overline{\varphi'(z)}\right)+\kappa_2\chi(z,\bar z)\\ &-\left[z\overline{\varphi''(z)}+\overline{\psi'(z)}+\kappa_3\partial_{\bar z}\partial_{\bar z}\chi(z,\bar z)\right]e^{-2i\alpha},\end{aligned} \tag{17}$$

where

$$\kappa_1=\frac{2\xi(\lambda+2\mu)^2-2(\lambda+3\mu)\beta^2}{2(\lambda+2\mu)(\xi(\lambda+2\mu)-\beta^2)},\quad \kappa_2=\frac{\mu\beta}{\lambda+2\mu},\quad \kappa_3=\frac{4\alpha\beta\mu}{\xi(\lambda+2\mu)-\beta^2}.$$

The analytic functions $\varphi'(z)$, $\psi'(z)$ and the metaharmonic function $\chi(z,\bar z)$ are represented as the following series

$$\varphi'(z)=\sum_{n=0}^{\infty}a_nz^n,\quad \psi'(z)=\sum_{n=0}^{\infty}b_nz^n,\quad \chi(z,\bar z)=\sum_{-\infty}^{\infty}\alpha_nI_n(\gamma r)e^{in\alpha}, \tag{18}$$

where $I_n(\gamma r)$ is the modified Bessel function of the first kind of n-th order.

Substituting (18) in (17), taking into account the boundary conditions (16) and assuming that the series converge on the circumference $r=R$, one finds

$$\begin{aligned}&\sum_{n=0}^{\infty}R^n\left(\kappa_1a_ne^{in\alpha}+(\kappa_1-n)\bar a_ne^{-in\alpha}\right)-\sum_{n=0}^{\infty}R^n\bar b_ne^{-i(n+2)\alpha}\\ &+\sum_{-\infty}^{\infty}\left[\kappa_2I_n(\gamma R)-\frac{\kappa_3\gamma^2}{4}I_{n+2}(\gamma R)\right]\alpha_ne^{in\alpha}=A+iB,\end{aligned} \tag{19}$$

$$\sum_{-\infty}^{\infty}\alpha_nI_n(\gamma R)e^{in\alpha}-\kappa_4\sum_{n=0}^{\infty}R^n\left(a_ne^{in\alpha}+\bar a_ne^{-in\alpha}\right)=C, \tag{20}$$

where

$$\kappa_4=\frac{\beta}{\xi(\lambda+\mu)-\beta^2}.$$

Expand the function $A+iB$ and C, given on $r=R$, in a complex Fourier series

$$A+iB=\sum_{-\infty}^{\infty}A_n e^{in\alpha},\quad C=\sum_{-\infty}^{\infty}C_n e^{in\alpha}.$$

Comparing in (19), (20) the coefficients of $e^{i0\alpha}$ we have (it is also assumed that a_0 is a real value [26])

$$\begin{aligned}&2\kappa_1 a_0+\left(\kappa_2 I_0(\gamma R)-\frac{\kappa_3\gamma^2}{4}I_2(\gamma R)\right)\alpha_0=A_0,\\&-2\kappa_4 a_0+I_0(\gamma R)\alpha_0=C_0.\end{aligned}\tag{21}$$

In order for the problem to have a solution, the following condition must be met

$$A_0=\bar{A}_0.$$

From Eqs. (19) we determine the coefficients a_0 and α_0

$$a_0=\frac{\Delta_1}{\Delta},\quad \alpha_0=\frac{\Delta_2}{\Delta},$$

where

$$\Delta=2\kappa_1 I_0(\gamma R)+2\kappa_4\left(\kappa_2 I_0(\gamma R)-\frac{\kappa_3\gamma^2}{4}I_2(\gamma R)\right),$$

$$\Delta_1=I_0(\gamma R)A_0-\left(\kappa_2 I_0(\gamma R)-\frac{\kappa_3\gamma^2}{4}I_2(\gamma R)\right)C_0,$$

$$\Delta_2=2\kappa_1 C_0+2\kappa_4 A_0.$$

Comparing in (19), (20) the coefficients of $e^{in\alpha}$ $(n\neq 0)$ we have

$$\begin{aligned}&\kappa_1 R^n a_n+\left(\kappa_2 I_n(\gamma R)-\frac{\kappa_3\gamma^2}{4}I_{n+2}(\gamma R)\right)\alpha_n=A_n,\\&-\kappa_4 R^n a_n+I_n(\gamma R)\alpha_n=C_n,\end{aligned}\tag{22}$$

$$R^n(\kappa_1-n)\bar{a}_n-R^{n-2}\bar{b}_{n-2+}\left(\kappa_2 I_n(\gamma R)-\frac{\kappa_3\gamma^2}{4}I_{n-2}(\gamma R)\right)\alpha_{-n}=A_{-n}.\tag{23}$$

From (20) one finds

$$a_n=\frac{\overline{\Delta}_1}{\overline{\Delta}},\quad \alpha_n=\frac{\overline{\Delta}_2}{\overline{\Delta}},$$

where

$$\overline{\Delta}=\kappa_1 R^n I_n(\gamma R)+\kappa_4\left(\kappa_2 I_n(\gamma R)-\frac{\kappa_3\gamma^2}{4}I_{n+2}(\gamma R)\right),$$

$$\Delta_1=I_n(\gamma R)A_n-\left(\kappa_2 I_n(\gamma R)-\frac{\kappa_3\gamma^2}{4}I_{n+2}(\gamma R)\right)C_n,$$

$$\Delta_2=\kappa_1 R^n C_n+\kappa_4 R^n A_n.$$

From (21)

$$b_n = R^2(\kappa_1 - n)a_{n+2} + \left(\kappa_2 I_{n+2}(\gamma R) - \frac{\kappa_3\gamma^2}{4} I_n(\gamma R)\right)\frac{\alpha_n}{R^n} - \frac{\bar{A}_{-n-2}}{R^n}.$$

It is easy to prove the absolute and uniform convergence of the series, obtained in the circle (including the contours) when the functions, set on the boundaries, have sufficient smoothness.

4. The problem for the infinite plane with a circular hole

Now let us have an infinite plane with a circular hole (Fig. 2). Assume that the origin of coordinates is at the center of the hole of radius R.

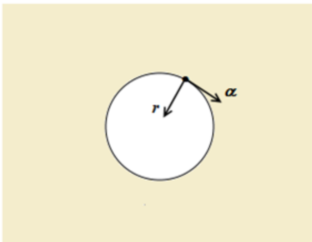

Fig. 2. The infinite plane with a circular hole

On the circle we consider the following boundary value problem

$$\sigma_{rr} + i\sigma_{r\alpha} = M + iN, \quad \phi = K, \quad \text{on } r = R, \tag{24}$$

where M, N and K are sufficiently smooth functions.

Conditions at infinity are

$$\sigma_{11}^{\infty} = \Gamma_1, \quad \sigma_{22}^{\infty} = \Gamma_2, \quad \sigma_{12}^{\infty} = \sigma_{21}^{\infty} = \Gamma_3, \quad \phi = \Gamma_4, \tag{25}$$

where Γ_1, Γ_2, Γ_3, Γ_4 are the constants.

In this case the analytic functions $\varphi'(z)$, $\psi'(z)$ and the metaharmonic functions $\chi(z,\bar{z})$ are represented as a series

$$\varphi'(z) = \sum_{n=0}^{\infty} a_n z^{-n}, \quad \psi'(z) = \sum_{n=0}^{\infty} b_n z^{-n}, \quad \chi(z,\bar{z}) = \sum_{-\infty}^{\infty} \alpha_n K_n(\gamma r)e^{in\alpha}, \tag{26}$$

where $K_n(\gamma r)$ is the modified Bessel function of the second kind of n-th order.

Substituting (26) in (17), (13) taking into account the boundary conditions (14) and assuming that the series converge on the circumference $r = R$, one finds

$$\begin{gathered}\sum_{n=0}^{\infty} \frac{1}{R^n}\left(\kappa_1 a_n e^{-in\alpha} + (\kappa_1 + n)\bar{a}_n e^{in\alpha}\right) - \bar{b}_0 e^{-2i\alpha} - \frac{\bar{b}_1}{R}e^{-i\alpha} - \sum_{n=0}^{\infty}\frac{\bar{b}_{n+2}}{R^{n+2}}e^{in\alpha} \\ + \sum_{-\infty}^{\infty}\left[\kappa_2 K_n(\gamma R) - \frac{\kappa_3\gamma^2}{4}K_{n+2}(\gamma R)\right]\alpha_n e^{in\alpha} = M + iN,\end{gathered} \tag{27}$$

$$\sum_{-\infty}^{\infty}\alpha_n K_n(\gamma R)e^{in\alpha}-\kappa_4\sum_{n=0}^{\infty}\frac{1}{R^n}\left(a_n e^{-in\alpha}+\bar{a}_n e^{in\alpha}\right)=K. \tag{28}$$

Expand the function $M+iN$ and K, given on $r=R$, in a complex Fourier series

$$M+iN=\sum_{-\infty}^{\infty}A_n e^{in\alpha},\quad K=\sum_{-\infty}^{\infty}C_n e^{in\alpha}. \tag{29}$$

Due to the fact that $\chi(z,\bar{z})$ and K are real functions, we have

$$\alpha_n=\overline{\alpha}_{-n},\quad C_n=C_{-n}.$$

It is known that [16]

$$a_0=\Gamma,\quad b_0=\Gamma', \tag{30}$$

where Γ, Γ' are known quantities, specifying the stress distribution at infinity (It is also assumed that a_0 is a real value [16]). As follows from formulas (13), (14) and conditions (25)

$$\mathrm{Re}\Gamma=\frac{S_1+S_2}{2\kappa_1}=-\frac{S_4}{2\kappa_4},\quad \mathrm{Re}\Gamma'=\frac{S_2-S_1}{2},\quad \mathrm{Im}\Gamma'=S_3.$$

We use the condition of single-valuedness of the displacements which in the present case is expressed as

$$\kappa_1 a_1+\bar{b}_1=0. \tag{31}$$

After introducing (29) into (27), (28), and comparing the coefficients of $e^{in\alpha}$, we have

$$2\kappa_1 a_0-\frac{1}{R^2}\bar{b}_2+\left(\kappa_2 K_0(\gamma R)+\frac{\kappa_3\gamma^2}{4}K_2(\gamma R)\right)\alpha_0=A_0, \tag{32}$$

$$\frac{\kappa_1}{R}a_1-\frac{1}{R}\bar{b}_1+\left(\kappa_2 K_{-1}(\gamma R)+\frac{\kappa_3\gamma^2}{4}K_1(\gamma R)\right)\alpha_{-1}=A_{-1}, \tag{33}$$

$$\frac{\kappa_1}{R^2}a_2-\bar{b}_0+\left(\kappa_2 K_{-2}(\gamma R)+\frac{\kappa_3\gamma^2}{4}K_0(\gamma R)\right)\alpha_{-2}=A_{-2}, \tag{34}$$

$$\frac{\kappa_1}{R^n}a_n+\left(\kappa_2 K_n(\gamma R)+\frac{\kappa_3\gamma^2}{4}K_{n+2}(\gamma R)\right)\alpha_{-n}=A_{-n},\quad n\geq 3, \tag{35}$$

$$\frac{\kappa_1+n}{R^n}\bar{a}_n-\frac{1}{R^{n+2}}\bar{b}_{n+2}+\left(\kappa_2 K_n(\gamma R)+\frac{\kappa_3\gamma^2}{4}K_{n+2}(\gamma R)\right)\alpha_n=A_n,\quad n\geq 1, \tag{36}$$

$$K_0(\gamma R)\alpha_0-2\kappa_4 a_0=C_0, \tag{37}$$

$$K_n(\gamma R)\alpha_n-\kappa_4\bar{a}_n=C_n. \tag{38}$$

The coefficients a_n, b_n and α_n are found by solving (30)-(38).

It is easy to prove the absolute and uniform convergence of the series, obtained in the infinite plane with a circular hole (including the contours), when the functions set on the boundaries have sufficient smoothness.

4. The problem for a circular ring

In this section, we consider a boundary value problem for a concentric circular ring with radius R_1 and R_2 (Fig. 3).

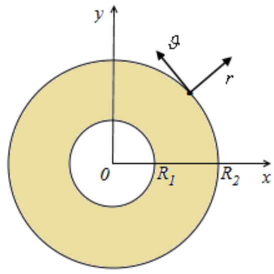

Fig. 3. The circular ring

We consider the following problem

$$\sigma_{rr} + i\sigma_{r\vartheta} = \begin{cases} \sum\limits_{-\infty}^{+\infty} A'_n e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} A''_n e^{in\vartheta}, & |z| = R_2, \end{cases} \tag{39}$$

$$\phi = \begin{cases} \sum\limits_{-\infty}^{+\infty} C'_n e^{in\vartheta}, & |z| = R_1, \\ \sum\limits_{-\infty}^{+\infty} C''_n e^{in\vartheta}, & |z| = R_2. \end{cases} \tag{40}$$

The analytic functions $\varphi'(z)$, $\psi'(z)$ and the metaharmonic functions $\chi(z,\bar{z})$ are represented as a series

$$\begin{gathered} \varphi'(z) = \delta \ln z + \sum_{-\infty}^{+\infty} a_n z^n, \quad \psi'(z) = \sum_{-\infty}^{+\infty} b_n z^n, \\ \chi(z,\bar{z}) = \sum_{-\infty}^{+\infty} \left(\alpha_n I_n(\gamma r) + \beta_n K_n(\gamma r)\right) e^{in\vartheta}. \end{gathered} \tag{41}$$

Analogous to the above problems we can find all coefficients of (41).

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Received 12.03.2020; revised 10.07.2020; accepted 15.08.2020.

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