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THE BVPs FOR TRANSVERSELY ISOTROPIC HALF-PLANE WITH DOUBLE POROSITY

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Abstract. This paper is concerned with the study of the 2D boundary value problems for transversely isotropic elastic half-plane with double porosity. Explicitly is solved the basic BVPs for half-plane. For finding explicit solutions of the basic BVPs the potential method and the theory of Fredholm integral equations are used. The Poisson type formulas are constructed.

Keywords and phrases: Double porosity, explicit solution, Fredholm type integral equations.

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Introduction

In most of cases naturally or artificially constructed solids are not completely filled with matter. In nearly every body there are empty interspaces which are called pores. In some bodies they are immediately visible, in others the pores are recognized only with a magnifier. For example, the human skin has a larger number of pores. Wood and bricks, if covered with water, absorb the liquid in their pores and increase their weight. Early studies reported that bone tissue could be assumed to be transversely isotropic, and in reality orthotropy most closely describes mechanical anisotropy of bone and cancellous bone is considered as a porous material. Porous media theories play an important role in many branches of engineering, including material science, the petroleum industry, chemical engineering, and soil mechanics, as well as biomechanics.

The physical and mathematical foundations of the theory of consolidation for elastic materials with so-called double porosity were first presented by Aifantis and co-workers in the papers [1]-[3] (see [1]-[3] and the references cited therein). They gave detailed physical interpretations of the phenomenological coefficients appearing in the double porosity theory. They also solved several representative boundary value problems, uniqueness and variational principles were established for the equations of double porosity, and provided a related finite element to consider the numerical solution of Aifantis' equations of double porosity. The basic results and the historical information on the theory of porous media were summarized by Boer [4]. The cross-coupled effect in porous media with double porosity was studied in [5,6]. The phenomenological equations of the quasi-static theory for double porous media are established in [7], where a method to calculate the relevant coefficients is also presented. The materials with double porosity are of interest in mechanics of bones [8].

In the last years many authors have investigated different types of problems of the 2-dimensional and 3-dimensional theories of elasticity for materials with double porosity, publishing a large number of papers(For example, some results can be seen in the works [9-24] and references therein). There the explicit solutions on some BVPs in the form of series and in quadratures are given in a form useful to engineering practice.

The above models are all based on the assumption of isotropy, whereas most rocks are characterized by anisotropy of various degrees. Transverse isotropy is an important type of anisotropy in geophysical applications and mechanics of bones. Therefore, research on the behavior of anisotropic dual-porosity media is very important. In [25] fully coupled dual porosity model for anisotropic formulations is considered. In [26] is given the deformation of transversely isotropic porous elastic circular cylinder. The non-linear theory and the linear theory of porous materials with voids is described by Nunziato and Cowin in [27-28]. The history of development of porous body mechanics, the main results and the sphere of their application are set forth in detail in the monographs [29-30].

This paper is concerned with the study of the 2D boundary value problems for transversely isotropic elastic half-plane with double porosity. Effectively are solved the basic BVPs for the half-plane. For finding explicit solutions of the basic BVPs the potential method and the theory of Fredholm integral equations are used. The Poisson type formulas are constructed.

1. Basic equations. Boundary value problems

We say that a body is subject to a plane deformation if the component u_2 of the displacements vector $\mathbf{u}(u_1, u_2, u_3)$ vanish and the other components are functions only of the variables x_1, x_3. In what follows we denote by R^2_+ the upper half-plane $x_3 > 0$. Clearly, the boundary of R^2_+ is Ox_1-exes and we denote it by S.

Let

$$\mathbf{x} := (x_1, x_3) \in R^2_+, \quad \partial_{\mathbf{x}} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3} \right).$$

We assume the domain R^2_+ to be filled with transversely isotropic materials with double porosity. $\mathbf{n}(0,1)$ is an unite normal vector.

The governing homogeneous system of equations in the linear theory of elasticity for transversely isotropic materials with double porosity can be written as follows [25]

$$\mathbf{C}(\partial x)\mathbf{u} = \boldsymbol{\alpha'}\mathrm{grad}p_1 + \boldsymbol{\alpha''}\mathrm{grad}p_2, \qquad \mathbf{u} = (u_1, u_3), \tag{1}$$

$$\begin{cases} \dfrac{k'_{11}}{\mu}\dfrac{\partial^2 p_1}{\partial x_1^2} + \dfrac{k'_{33}}{\mu}\dfrac{\partial^2 p_1}{\partial x_3^2} + \gamma(p_1 - p_2) = 0, \\[2ex] \dfrac{k''_{11}}{\mu}\dfrac{\partial^2 p_2}{\partial x_1^2} + \dfrac{k''_{33}}{\mu}\dfrac{\partial^2 p_2}{\partial x_3^2} - \gamma(p_1 - p_2) = 0, \end{cases} \tag{2}$$

where

$$\mathbf{C}(\partial x) = \|C_{pq}(\partial x)\|_{2x2},$$

$$C_{11}(\partial x) = c_{11}\frac{\partial^2}{\partial x_1^2} + c_{44}\frac{\partial^2}{\partial x_3^2}, \quad C_{22}(\partial x) = c_{44}\frac{\partial^2}{\partial x_1^2} + c_{33}\frac{\partial^2}{\partial x_3^2},$$

$$C_{21}(\partial x) = C_{12}(\partial x) = (c_{13} + c_{44})\frac{\partial^2}{\partial x_1 \partial x_3},$$

$$\boldsymbol{\alpha'} := -\begin{pmatrix} \alpha'_{11} & 0 \\ 0 & \alpha'_{33} \end{pmatrix}, \quad \boldsymbol{\alpha''} := -\begin{pmatrix} \alpha''_{11} & 0 \\ 0 & \alpha''_{33} \end{pmatrix}.$$

$\mathbf{u} = (u_1, u_3)$ is a displacement vector, p_1 and p_2 are the pore and fissure fluid pressures respectively. c_{pq} are Hooke's coefficients, α'_{ij} and α''_{ij} are the Biot coefficients.

Definition 1. A vector-function $\mathbf{U}(\mathbf{x}) = (u_1, u_3, p_1, p_2)$ defined in the domain R^2_+ is called regular if it has integrable continuous second order derivatives in R^2_+, $\mathbf{U}$ itself and its first derivatives are continuously extendable at every point of the boundary of R^2_+ and satisfies the following conditions at infinity:

$$\mathbf{U}(x) = O(|x|^{-1}), \quad \frac{\partial U_l(x)}{\partial x_k} = O(|x|^{-2}), \quad |x| >> 1.$$

The basic BVPs are formulated as follows: find a regular solution $\mathbf{U}$ in R^2_+, of the equations (1)-(2), if on the boundary S one of the following conditions are given:

Problem I. The displacement vector and the fluid pressures are given on S:

$$\mathbf{u}^+(z) = \mathbf{f}(z_1), \quad p_1^+(z) = f_3(z_1), \quad p_2^+(z) = f_4(z_1), \quad z_1 \in S.$$

Problem II. The stress vector and the normal derivatives of the pressure functions are given on S :

$$\tau_{13} = c_{44}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = f_1(z_1), \quad \tau_{33} = c_{13}\frac{\partial u_1}{\partial x_1} + c_{33}\frac{\partial u_3}{\partial x_3} + (\alpha'_{33}p_1 + \alpha''_{33}p_2) = f_2(z_1),$$

$$\left(\frac{\partial p_1}{\partial x_3}\right)^+ = f_3(z_1), \left(\frac{\partial p_2}{\partial x_3}\right)^+ = f_4(z_1), \quad z_1 \in S.$$

Note that BVPs for the system (2), which contain p_1 and p_2 can be investigated separately. Then if supposing p_j as known, we can study BVPs for the system (1) with respect to $\mathbf{u}$. By combining the obtained results, we arrive at explicit solutions of BVPs for system (1)-(2).

Let us assume that p_j are known functions and search the solution of the following non-homogeneous equation

$$\mathbf{C}(\partial x)\mathbf{u} = \boldsymbol{\alpha'}\mathrm{grad}p_1 + \boldsymbol{\alpha''}\mathrm{grad}p_2, \tag{3}$$

general solution of which has the following form

$$\mathbf{u}(\mathbf{x}) = \mathbf{V}(\mathbf{x}) + \mathbf{u}_0(\mathbf{x}),$$

where $\mathbf{V}(\mathbf{x})$ is a general solution of the equation

$$\mathbf{C}(\partial x)\mathbf{V}(\mathbf{x}) = 0$$

and $\mathbf{u}_0(\mathbf{x})$ is a particular solution of the non-homogeneous equation (3)

$$\mathbf{u}_0(\mathbf{x}) = -\frac{1}{2\pi}\iint\limits_{R^2_+}\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})\left[\boldsymbol{\alpha'}\mathrm{grad}p_1 + \boldsymbol{\alpha''}\mathrm{grad}p_2\right]d\tau_y, \tag{4}$$

$\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})$ is a matrix of fundamental solution of the equation

$$\mathbf{C}(\partial x)\mathbf{u}(\mathbf{x}) = 0,$$

$$\begin{cases} \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}) = 2Im\sum\limits_{k=2}^{3}\|A_{lm}^{(k)}(\partial x)\|_{2x2}\ln\sigma_k, \\ A_{11}^{(k)} = \frac{(-1)^k}{\sqrt{a_k}}i(c_{44} - a_k c_{33})n, \quad A_{12}^{(k)} = A_{21}^{(k)} = (-1)^k(c_{13}+c_{44})n, \\ A_{22}^{(k)} = \frac{(-1)^k}{\sqrt{a_k}}i(c_{11} - a_k c_{44})n, \quad n^{-1} = (a_2 - a_3)c_{44}c_{33}, \\ \sigma_k = z_k - \zeta_k, \quad \zeta_k = y_1 + i\sqrt{a_k}y_3, \quad z_k = x_1 + i\sqrt{a_k}x_3, \end{cases} \tag{5}$$

$a_k,\ \ k=2,3$ are the positive roots of a characteristic equation

$$c_{33}c_{44}a^2+[(c_{13}+c_{44})^2-c_{11}c_{33}-c_{44}^2]a+c_{11}c_{44}=0.$$

In (4) $\boldsymbol{\alpha}'\mathrm{grad}p_1+\boldsymbol{\alpha}''\mathrm{grad}p_2$ is a continuous vector in R_+^2 along with its first derivatives and satisfies the following conditions at infinity

$$\boldsymbol{\alpha}'\mathrm{grad}p_1+\boldsymbol{\alpha}''\mathrm{grad}p_2=O(|x|^{-1-\alpha}),\ \ \alpha>0.$$

Thus, we reduce the solution of BVP of the theory of poroelastisity to the solution of the BVP of elasticity for the equation of the transversely isotropic elastic body

$$\begin{cases}\mathbf{C}(\partial x)\mathbf{V}(\mathbf{x})=0,\\[2ex] \mathbf{V}^+(z)=\mathbf{f}^+(z)-\mathbf{u}_0^+(z)=\mathbf{F}^+(z),\quad z\in S.\end{cases}\tag{6}$$

First of all we will construct a fundamental matrix of solutions for equations (2). Let us rewrite the system of equations (2) in the following form

$$\begin{cases}(k_1\Delta_4+\gamma)p_1-\gamma p_2=0,\\[2ex] (k_2\Delta_4+\gamma)p_2-\gamma p_1=0,\end{cases}\tag{7}$$

where

$$k_1=\frac{k'_{33}}{\mu},\quad k_2=\frac{k''_{33}}{\mu},$$

$$\Delta_4=a_4\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_3^2},\quad a_4=\frac{k'_{11}}{k'_{33}}=\frac{k''_{11}}{k''_{33}}.$$

We look for the functions p_j in the form

$$\begin{pmatrix}p_1\\ p_2\end{pmatrix}=\begin{pmatrix}k_2\Delta_4+\gamma & \gamma\\ \gamma & k_1\Delta_4+\gamma\end{pmatrix}\boldsymbol{\psi},\tag{8}$$

where the vector $\boldsymbol{\psi}(x)$ is a fundamental solution of the scalar equation

$$\Delta_4(\Delta_4+\lambda^2)\psi=0,\quad \lambda^2=\frac{\gamma(k_1+k_2)}{k_1k_2}>0,\quad \boldsymbol{\psi}=\frac{\varphi_4-\varphi_5}{\lambda^2},$$

$$\Delta_4\varphi_4=0,\quad (\Delta_4+\lambda^2)\varphi_5=0,\quad \varphi_4=\ln r_4,\quad r_4^2=x_1^2+a_4x_3^2,$$

$\varphi_5=K_0(\lambda r_4)$ is a modified Hankel's function of the first kind of order zero

$$K_0(z)=J_0(z)\left(\ln\frac{z}{2}+C\right)+\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{z}{2}\right)^{2n}\left(\frac{1}{n}+....1\right),$$

$$J_0(z)=\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{z}{2}\right)^{2n}.$$

From (8) it follows that the fundamental matrix of solution for equation (2) reads as

$$\mathbf{\Gamma}^{(1)}(\mathbf{x}-\mathbf{y})=\frac{1}{k_1k_2}\begin{pmatrix}k_2\varphi_5+\frac{\gamma}{\lambda^2}[\varphi_4-\varphi_5] & \frac{\gamma}{\lambda^2}[\varphi_4-\varphi_5]\\[2ex] \frac{\gamma}{\lambda^2}[\varphi_4-\varphi_5] & k_1\varphi_5+\frac{\gamma}{\lambda^2}[\varphi_4-\varphi_5]\end{pmatrix},\tag{9}$$

$$r_4^2 = (x_1 - y_1)^2 + a_4 x_3^2.$$

From (9), it is easy to show, that columns and rows of the matrix $\mathbf{\Gamma}^{(1)}$ are solutions of the equation (7) with respect to $\mathbf{x}$, for any $\mathbf{x} \neq \mathbf{y}$.

2. Matrix of singular solution for equation $\mathbf{C}(\partial x)\mathbf{u} = 0$

Let's write now the expression of the stress vector, which acts on elements with the normal $(0, 1)$. Denoting the stress vector By $\mathbf{T}(\partial x, n)\mathbf{u}$, we obtain

$$\mathbf{T}(\partial x, \mathbf{n})\mathbf{u} := \begin{pmatrix} c_{44}\dfrac{\partial}{\partial x_3} & c_{44}\dfrac{\partial}{\partial x_1} \\ \\ c_{13}\dfrac{\partial}{\partial x_1} & c_{33}\dfrac{\partial}{\partial x_3} \end{pmatrix} \mathbf{u}.$$

Let us consider the generalized stress vector

$$\mathbf{P}(\partial x, \mathbf{n})\mathbf{u} := \mathbf{T}(\partial x, n)\mathbf{u} + \begin{pmatrix} 0 & \chi_1 \\ \\ \chi_2 & 0 \end{pmatrix} \frac{\partial \mathbf{u}}{\partial s},$$

where

$$\frac{\partial}{\partial s} = n_3 \frac{\partial}{\partial x_1} - n_1 \frac{\partial}{\partial x_3},$$

χ_1 and χ_2 are arbitrary constants.

Taking into account (5), let us calculate the matrix $\mathbf{P}(\partial x, \mathbf{n})\mathbf{\Gamma}(\mathbf{x} - \mathbf{y})$. After simple calculations we get

$$\mathbf{P}(\partial x, \mathbf{n})\mathbf{\Gamma}(\mathbf{x} - \mathbf{y}) := 2Im \sum_{k=2}^{3} \begin{pmatrix} q_{11}^{(k)} & q_{12}^{(k)} \\ \\ q_{21}^{(k)} & q_{22}^{(k)} \end{pmatrix} \frac{\partial \ln \sigma_k}{\partial s_x} + \begin{pmatrix} 0 & \chi_1 \\ \\ \chi_2 & 0 \end{pmatrix} \frac{\partial}{\partial s_x} \Gamma(\mathbf{x} - \mathbf{y}), \qquad (10)$$

where

$$q_{11}^{(k)} = \frac{(-1)^k (c_{33} a_k + c_{13})}{c_{33}(a_2 - a_3)}, \qquad q_{12}^{(k)} = \frac{(-1)^k i (c_{13} a_k + c_{11})}{c_{33}(a_2 - a_3)\sqrt{a_k}},$$

$$q_{21}^{(k)} = \frac{(-1)^k i (c_{13} + c_{33} a_k)}{c_{33}(a_2 - a_3)\sqrt{a_k}}, \qquad q_{22}^{(k)} = \frac{(-1)^k (c_{11} + c_{13} a_k)}{c_{33}(a_2 - a_3) a_k}.$$

Consider now the matrix $\mathbf{P}^*(\partial x, \mathbf{n})$ which is obtained from (10) by transposition of the columns and rows and the variables $\mathbf{x}$ and $\mathbf{y}$

$$\begin{aligned} \mathbf{P}^*(\mathbf{y} - \mathbf{x}) := &-2Im \sum_{k=2}^{3} \begin{pmatrix} q_{11}^{(k)} & q_{21}^{(k)} \\ \\ q_{12}^{(k)} & q_{22}^{(k)} \end{pmatrix} \frac{\partial \ln \sigma_k}{\partial s_y} \\ &-2Im \sum_{k=2}^{3} \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ \\ A_{12}^{(k)} & A_{22}^{(k)} \end{pmatrix} \begin{pmatrix} 0 & \chi_2 \\ \\ \chi_1 & 0 \end{pmatrix} \frac{\partial \ln \sigma_k}{\partial s_y}. \end{aligned} \qquad (11)$$

In (11) the function $\dfrac{\partial \ln \sigma_k}{\partial s_y}$ is a singular kernel on S, which is integrable in the sense of the principal Cauchy value.

The following theorem is true.

Theorem 1. *Every column of the matrix* $\boldsymbol{P}^{*}(\mathbf{x}\text{-}\mathbf{y})$, *for any values* χ_1 and χ_2, *considered as a vector, is a solution of the equation* $\boldsymbol{C}(\partial x)\boldsymbol{u}(\boldsymbol{x}) = 0$ *at every point* $\mathbf{x}$ if $\mathbf{x} \neq \mathbf{y}$.

Definition 2. The vector

$$\mathbf{w}(\mathbf{x}) = \frac{1}{\pi}\int\limits_{S} \boldsymbol{P}^{*}(\mathbf{x}\text{-}\mathbf{y})\mathbf{h}(y)dy$$

is called a double layer potential.

Definition 3. The vector

$$\mathbf{v}(\mathbf{x}) = \frac{1}{\pi}\int\limits_{S} \boldsymbol{\Gamma}(\mathbf{x}\text{-}\mathbf{y})\mathbf{h}(y)dy$$

is called a single layer potential.

Theorem 2. *If S is a Lyapunow curve, $\boldsymbol{h} \in C^{1,\alpha}(S)$, $\alpha > 0$, then the function $\boldsymbol{w} \in C^{0,\alpha}(S)$ and*

$$\boldsymbol{w}^{\pm}(\boldsymbol{x}) = \pm\boldsymbol{h} + \frac{1}{\pi}\int\limits_{S} \boldsymbol{P}^{*}(\boldsymbol{x}\text{-}\boldsymbol{y})\boldsymbol{h}(y)dy.$$

Theorem 3. *If S is a Lyapunow curve, $\boldsymbol{h} \in C^{0,\alpha}(S)$, $\alpha > 0$, then the function $\boldsymbol{v} \in C^{0,\alpha}(S)$ and*

$$[\boldsymbol{T}\boldsymbol{v}]^{\pm}(\boldsymbol{x}) = \mp\boldsymbol{h} + \frac{1}{\pi}\int\limits_{S} \boldsymbol{T}(\boldsymbol{\partial x}, \boldsymbol{n})\boldsymbol{\Gamma}(\boldsymbol{x}\text{-}\boldsymbol{y})\boldsymbol{h}(y)dy.$$

The kernel, which will be obtained from $\boldsymbol{P}^{*}(\mathbf{x}\text{-}\mathbf{y})$, for

$$\chi_1 = \frac{c_{44}(c_{13}+c_{44})}{c_{44}+c_{33}\sqrt{a_2a_3}} \quad \textit{and} \quad \chi_2 = \chi_1\sqrt{a_2a_3},$$

will be called the kernel $\boldsymbol{N}^{*}(\mathbf{x}\text{-}\mathbf{y})$, and anable to obtain the Fredholm integral equation of the second kind for the first BVP, where

$$\boldsymbol{N}^{*}(\partial x) = 2Im\sum_{k=2}^{3}\|N_{pq}^{(k)}(\partial x)\|_{2x2}\frac{1}{t-z_k},$$

$$N_{11}^{(k)} = (-1)^k d(c_{33}a_k - c_{44})\sqrt{a_2a_3a_k^{-1}}, \quad N_{21}^{(k)} = i(-1)^k d(c_{13}+c_{44}),$$

$$N_{12}^{(k)} = \sqrt{a_2a_3}N_{21}^{(k)}, \quad N_{22}^{(k)} = (-1)^k d(c_{44}a_k - c_{11})\sqrt{a_k^{-1}}, \tag{12}$$

$$d^{-1} = (\sqrt{a_2}-\sqrt{a_3})(c_{44}+c_{33}\sqrt{a_2a_3}).$$

From (12) we find

$$\sum_{k=2}^{3}N_{11}^{(k)} = 1, \quad \sum_{k=2}^{3}N_{22}^{(k)} = 1, \quad \sum_{k=2}^{3}N_{12}^{(k)} = 0.$$

3. Solution of the Problem I in the Domain R^2_+

Let us consider the BVP (6). We look for a solution of equation (2) with boundary conditions $p_1^+ = f_3(z_1),\ p_2^+ = f_4(z_1)$ in the form of the double layer potential

$$\begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix} = \frac{1}{\pi}\int\limits_S \frac{\partial}{\partial x_3}\mathbf{\Gamma}^{(1)}(y-x)\mathbf{g}(y)dy, \tag{13}$$

where $\mathbf{g}(g_1, g_2)$ is a two-dimensional unknown vector. For determining it we obtain the following Fredholm integral equation of the second kind

$$-\begin{pmatrix} k_2 & 0 \\ 0 & k_1 \end{pmatrix}\mathbf{g}(z_1) + \frac{1}{\pi}\int\limits_S \frac{\partial}{\partial x_3}\mathbf{\Gamma}^{(1)}(y-z)\mathbf{g}(y)dy = \begin{pmatrix} f_3(z_1) \\ f_4(z_1) \end{pmatrix}. \tag{14}$$

Taking into account that $\dfrac{\partial}{\partial x_3}\mathbf{\Gamma}^{(1)}(y-x) = 0$ for $x_3 = 0$, from equation (14) we have

$$\mathbf{g}(z_1) = -\frac{1}{k_1 k_2}\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}\begin{pmatrix} f_3(z_1) \\ f_4(z_1) \end{pmatrix}. \tag{15}$$

Using (15), (13) takes the form

$$\begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix} = -\frac{1}{\pi k_1 k_2}\int\limits_S \frac{\partial}{\partial x_3}\mathbf{\Gamma}^{(1)}(y-x)\begin{pmatrix} k_1 f_3(y) \\ k_2 f_4(y) \end{pmatrix}dy. \tag{16}$$

We look for a solution of the equation $\mathbf{C}(\partial x)\mathbf{V} = 0$ with the boundary condition $(\mathbf{V}^+ = \mathbf{F})$, in the domain R^2_+ in the form of the double layer potential

$$\mathbf{V}(\mathbf{x}) = \frac{1}{\pi}Im\sum_{k=2}^{3}\|N_{pq}^{(k)}(\partial x)\|_{2x2}\int\limits_S \frac{\mathbf{g}(t)}{t-z_k}dt, \tag{17}$$

where $\mathbf{g}(t)$ is an unknown real vector-function.To determine it we obtain the following Fredholm integral equation

$$\mathbf{g}(t_0) + \frac{1}{\pi}Im\sum_{k=2}^{3}\|N_{pq}^{(k)}(\partial x)\|_{2x2}\int\limits_S \frac{\mathbf{g}(t)}{t-t_0}dt = \mathbf{F}(t_0). \tag{18}$$

Taking into account the fact that

$$\sum_{k=2}^{3} N_{11}^{(k)} = 1, \quad \sum_{k=2}^{3} N_{22}^{(k)} = 1, \quad \sum_{k=2}^{3} N_{12}^{(k)} = 0, \quad \sum_{k=2}^{3} N_{21}^{(k)} = 0,$$

from (18) we obtain $\mathbf{g}(t_0) = \mathbf{F}(t_0)$ and (17) takes the form

$$\mathbf{V}(\mathbf{x}) = \frac{1}{\pi}Im\sum_{k=2}^{3}\|N_{pq}^{(k)}(\partial x)\|_{2x2}\int\limits_S \frac{\mathbf{F}(t)}{t-z_k}dt. \tag{19}$$

Formulas (16) and (19) are analogues of Poisson's type formulas for the solution of Problem I in the domain R^2_+.

For the regularity of the solution of the first boundary value problem it is sufficient that $\mathbf{F}, f_j \in C^{1,\alpha}$, $\mathbf{F}, f_j = \dfrac{c_1}{|x|^{1+\beta}}$, $j = 3, 4$, $\beta > 0$, for large $|x|$, where $c_1 = const$.

4. Solution of the Problem II in the Domain R^2_+

We look for a solution of equation (2) with boundary conditions $\left(\frac{\partial p_1}{\partial x_3}\right)^+ = f_3(z_1)$, $\left(\frac{\partial p_2}{\partial x_3}\right)^+ = f_4(z_1)$ in the form of a single layer potential

$$\begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix} = \frac{1}{\pi}\int_S \mathbf{\Gamma}^{(1)}(x-y)\mathbf{q}(y)dy. \tag{20}$$

Passing to the limit as $x \to z \in S$ for $\mathbf{q}(y)$ we obtain the following Fredholm integral equation of the second kind

$$\begin{pmatrix} k_2 & 0 \\ 0 & k_1 \end{pmatrix}\mathbf{q}(z_1) + \frac{1}{\pi}\int_S \frac{\partial}{\partial z_3}\mathbf{\Gamma}^{(1)}(z_1-y)\mathbf{q}(y)dy = \begin{pmatrix} f_3(z_1) \\ f_4(z_1) \end{pmatrix}.$$

Taking into account that $\dfrac{\partial}{\partial z_3}\mathbf{\Gamma}^{(1)}(y-z_1) = 0$, $for \quad z_3 = 0$, from the last equation we have

$$\mathbf{q}(z_1) = \frac{1}{k_1k_2}\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}\begin{pmatrix} f_3(z_1) \\ f_4(z_1) \end{pmatrix}$$

and (20) takes the form

$$\begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix} = \frac{1}{\pi k_1k_2}\int_S \mathbf{\Gamma}^{(1)}(x-y)\begin{pmatrix} k_1f_3(y) \\ k_2f_4(y) \end{pmatrix}dy.$$

We look for a solution of equation $\mathbf{C}(\partial x)\mathbf{V} = 0$ with the boundary condition $(\mathbf{T}(\partial n)\mathbf{V})^+ = \mathbf{F}(z_1)$ in the domain R^2_+ as a single layer potential of the second kind

$$\mathbf{V}(\mathbf{x}) = \frac{1}{\pi}Re\sum_{k=2}^{3}\|L^{(k)}_{pq}(\partial x)\|_{2x2}\int_S \ln(t-z_k)\mathbf{h}(t)dt, \tag{21}$$

where

$$L^{(k)}_{11} = (-1)^k(c_{13}+a_kc_{33})n, \quad L^{(k)}_{12} = (-1)^k i(c_{13}+a_kc_{33})\sqrt{a_2a_3a_k^{-1}}n,$$

$$L^{(k)}_{21} = (-1)^k i(c_{11}+a_kc_{13})\sqrt{a_k^{-1}}n, \quad L^{(k)}_{22} = (-1)^{k+1} i(c_{11}+a_kc_{13})a_k^{-1}\sqrt{a_2a_3}n,$$

$$n^{-1} = (\sqrt{a_2}-\sqrt{a_3})(c_{11}c_{33}-c_{13}^2), \quad z_k = x_1 + i\sqrt{a_k}\,x_3,$$

$\mathbf{h}$ is an unknown real vector.

From (21) for the stress vector we obtain

$$\mathbf{T}(\partial\mathbf{x},\mathbf{n})\mathbf{V}(\mathbf{x}) = \frac{1}{\pi}Re\sum_{k=2}^{3}\mathbf{M}^{(k)}\int_S \frac{\mathbf{h}(t)dt}{t-z_k},$$

where

$$\mathbf{M}^{(k)} = \frac{(-1)^k}{(\sqrt{a_2} - \sqrt{a_3})} \begin{pmatrix} -i\sqrt{a_k} & \sqrt{a_2 a_3} \\ 1 & i\sqrt{a_2 a_3 a_k^{-1}} \end{pmatrix}.$$

Taking into account the properties of the vector **TV** and boundary condition $(\mathbf{TV})^+ = \mathbf{F}(z_1), \quad z_1 \in S$, for the unknown density $\mathbf{h}(y)$, we obtain the following Fredholm integral equation of the second kind

$$-\mathbf{h}(z_1) + \frac{1}{\pi} Re \sum_{k=2}^{3} \mathbf{M}^{(k)} \int_S \frac{\mathbf{h}(t)}{t - z_1} dt = \mathbf{F}(z_1).$$

The solution to the integral equation exists if the principal vector $\int_S \mathbf{F}(y)dy$ and the principal moment $\int_S yF_2(y)dy$ of external stresses are equal to zero. Note that $Re \sum_{k=2}^{3} \mathbf{M}^{(k)} = 0, \ for \ z_3 = 0$. Then $\mathbf{h}(z_1) = -\mathbf{F}(z_1)$. Substituting $\mathbf{h}(t)$ into (21), we get a solution of the second BVP (the Poisson type formula), provided the principal vector and the principal moment of external stresses are equal to zero.

Therefore, we have the following Poisson type formula for the solution of the second BVP

$$\mathbf{V}(\mathbf{x}) = -\frac{1}{\pi} Re \sum_{k=2}^{3} \| L_{pq}^{(k)}(\partial x) \|_{2x2} \int_S \ln(t - z_k) \mathbf{F}(t) dt.$$

For the regularity of the solution $\mathbf{V}(\mathbf{x})$ it is sufficient that $\mathbf{F}(t) \in C^{0,\alpha}(S), \quad \alpha > 0,$ and $\mathbf{F}(t) = O(|t|^{-1-\beta}), \quad \beta > 0$, for large $|t|$.

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