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THE VARIATION FORMULA OF SOLUTION FOR THE LINEAR CONTROLLED DIFFERENTIAL EQUATION CONSIDERING THE MIXED INITIAL CONDITION AND PERTURBATION OF DELAYS

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Abstract. For the linear controlled differential equation with constant delays in the phase coordinates and controls the variation formula of solution is established, which is the linear representation of the main part of solution increment with respect to perturbation of initial data. Under initial data we mean the collection of the initial moment, delay parameters, the initial vector, the initial and control functions. In the formula, effects of perturbation of delay parameters and control function, and of the mixed initial condition are revealed.

Keywords and phrases: Variation formula, linear delay controlled differential equation, mixed initial condition.

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Let $\mathbb{R}_x^n$ be the n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$, where T denotes transposition. Let $p \in \mathbb{R}_k^p$ and $q \in \mathbb{R}_m^q$, with $k + m = n$ and $x = (p, q)^T$. Furthermore, let $0 < \tau_1 < \tau_2, 0 < \sigma_1 < \sigma_2, 0 < \theta_1 < \theta_2$ be given numbers.

Let $I = [a, b], I_1 = [\hat{\tau}, b]$ and $I_2 = [a - \theta_2, b]$, where $\hat{\tau} = a - \max\{\tau_2, \sigma_2\}$; denote by C_φ the space of continuous functions

$$\varphi : I_1 \to R_p^k$$

and by C_g^1 the space of continuous differentiable functions

$$g : I_1 \to R_q^m.$$

Next, denote by AC_u the space of absolutely continuous control functions

$$u : I_2 \to R_u^r.$$

To any element

$$\mu = (t_0, \tau, \sigma, \theta, \varphi, g, u) \in \Lambda = (a, b) \times (\tau_1, \tau_2) \times (\sigma_1, \sigma_2) \times (\theta_1, \theta_2)$$
$$\times C_\varphi \times C_g^1 \times AC_u$$

we assign the controlled delay differential equation

$$\dot{x}(t) = (\dot{p}(t), \dot{q}(t))^T = A(t)x(t) + B(t)p(t - \tau) + C(t)q(t - \sigma)$$
$$+D(t)u(t) + E(t)u(t - \theta), t \in (t_0, b) \quad (1)$$

with the mixed initial condition

$$x(t) = (\varphi(t), g(t))^T, t \in [\hat{\tau}, t_0), x(t_0) = (p_0, g(t_0))^T. \quad (2)$$

Here $A(t)$ and $B(t)$ are the integrable matrix functions with dimensions $n \times n$ and $n \times k$, respectively; $C(t)$ is the integrable matrix function with dimension $n \times m$; $D(t)$ and $E(t)$ are the integrable matrix functions with dimension $n \times r$.

The condition (2) is said to be the mixed initial condition, because it consists of two parts: the first part is

$$p(t) = \varphi(t), t \in [\hat{\tau}, t_0), p(t_0) = p_0,$$

the discontinuous part, since in general $p(t_0) \neq p_0$ (discontinuity at the initial moment may be related to the instant change in a dynamic process, for example, changes of investment, environment and etc); the second part is

$$q(t) = g(t), t \in [\hat{\tau}, t_0],$$

the continuous part, since always $q(t_0) = g(t_0)$.

Definition. Let $\mu = (t_0, \tau, \sigma, \theta, \varphi, g, u) \in \Lambda$. A function $x(t) = x(t;\mu), t \in [\hat{\tau}, b]$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element μ and defined on the interval $[\hat{\tau}, t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, b]$ and satisfies equation (1) almost everywhere on $[t_0, b]$.

From the linearity of equation (1) it follows that for every element $\mu \in \Lambda$ there exists a corresponding solution. Let $\mu_0 = (t_{00}, \tau_0, \sigma_0, p_{00}, \varphi_0, g_0, u_0) \in \Lambda$ be a given element. Now we introduce the set of variations

$$V = \Big\{ \delta\mu_0 = (\delta t_0, \delta\tau, \delta\sigma, \delta p_0, \delta\varphi, \delta g_0, \delta u) : \ |\delta t_0| \leq \alpha, |\delta\tau_0| \leq \alpha, |\delta\sigma| \leq \alpha,$$

$$\delta\varphi = \sum_{i=1}^{\nu} \lambda_i \delta\varphi_i, \delta g = \sum_{i=1}^{\nu} \lambda_i \delta g_i, \delta u = \sum_{i=1}^{\nu} \lambda_i \delta u_i, |\lambda_i| \leq \alpha, i = \overline{1,\nu} \Big\},$$

where

$$\delta\varphi_i \in C_\varphi - \varphi_0, \delta g_i \in C_g^1 - g_0, \delta u_i \in AC_u - u_0, i = \overline{1,\nu},$$

are fixed functions and $\alpha > 0$ is a given number.

There exists a number $\varepsilon_1 > 0$ such that for arbitrary $(t, \delta\mu) \in (0, \varepsilon_1) \times V$ we have $\mu_0 + \varepsilon\delta\mu \in V$.

Let us define the increment of the solution $x_0(t) = x(t;\mu_0)$:

$$\Delta x(t;\varepsilon\delta\mu) = x(t;\mu_0 + \varepsilon\delta\mu) - x_0(t), (t, \varepsilon, \delta\mu) \in [\hat{\tau}, b] \times (0, \varepsilon_1) \times V.$$

Theorem. *Let the following conditions hold:*
1) $t_{00} + \tau_0 < b$;
2) the function $\varphi_0(t)$ *is absolutely continuous and* $\dot{\varphi}_0(t)$ *is bounded;*
3) the functions $A(t)$, $B(t)$, $C(t)$, $D(t)$, *and* $E(t)$ *are continuous at the point* t_{00};
4) the function $B(t)$ *is continuous at the point* $t_{00} + \tau_0$.
Then for arbitrary $(t, \varepsilon, \delta\mu) \in [b - \delta, b] \times (0, \varepsilon_1) \times V$, with $\delta > 0$ and $b - \delta > t_{00} + \tau_0$, *we have*

$$\Delta x(t;\varepsilon\delta\mu) = \varepsilon\delta x(t;\delta\mu) + o(t;\varepsilon\delta\mu),$$

where

$$\delta x(t;\delta\mu) = Y(t_{00};t)\Big(\delta p_0, \delta g(t_{00})\Big)^T + \Big\{ Y(t_{00};t)\Big[\Big(\Theta_{k\times 1}, \dot{g}_0(t_{00})\Big)^T - f_0\Big]$$

$$-Y(t_{00} + \tau_0;t) f_1 \Big\} \delta t_0 - \Big\{ Y(t_{00} + \tau_0;t) f_1 + \int_{t_{00}}^{t} Y(s;t) B(s) \dot{p}_0(s - \tau_0) ds \Big\} \delta\tau$$

$$-\Big\{ \int_{t_{00}}^{t} Y(s;t) C(s) \dot{q}_0(s - \sigma_0) ds \Big\} \delta\sigma + \int_{t_{00}-\tau_0}^{t_{00}} Y(s + \tau_0;t) B(s + \tau_0) \delta\varphi(s) ds$$

$$+\int_{t_{00}-\sigma_0}^{t_{00}} Y(s+\sigma_0;t)C(s+\sigma_0)\delta g(s)ds - \Big\{\int_{t_{00}}^{t} Y(s;t)E(s)\dot{u}_0(s-\theta_0)ds\Big\}\delta\theta$$

$$+\int_{t_{00}}^{t} Y(s;t)\Big[D(s)\delta u(s)+E(s)\delta u(s-\theta_0)\Big]ds; \qquad (3)$$

$$\lim_{\varepsilon\to 0}\frac{o(t;\varepsilon\delta\mu)}{\varepsilon}=0$$

uniformly for $(t,\delta\mu)\in[b-\delta,b]\times V$;

$$f_0 = A(t_{00})x_0(t_{00}) + B(t_{00})p_0(t_{00}-\tau_0) + C(t_{00})q_0(t_{00}-\sigma_0)$$

$$+D(t_{00})u_0(t_{00}) + E(t_{00})u_0(t_0-\theta_{00}),$$

$$f_1 = B(t_{00}+\tau_0)[p_{00}-\varphi_0(t_{00})];$$

$\Theta_{k\times 1}$ *is the* $k\times 1$ *zero matrix,* $Y(s:t)$ *is the* $n\times n$ *matrix function satisfying on the interval* $[t_{00},t]$ *the equation*

$$\frac{\partial}{\partial s}Y(s;t) = -Y(s;t)A(s) - \Big(Y(s+\tau_0;t)B(s+\tau_0), Y(s+\sigma_0;t)C(s+\sigma_0)\Big)$$

and the condition

$$Y(s;t)=\begin{cases} E \ \text{ for } \ s=t,\\ \Theta_{n\times n} \ \text{ for } \ s>t.\end{cases}$$

Here, E *is the* $n\times n$ *identity matrix.*

The Theorem is proved by the scheme given in [1,2].

Some Comments. The function $\delta x(t;\delta\mu)$ is called the variation of the solution $x_0(t)$ on the interval $[b-\delta,b]$ and the expression (3) is called the variation formula.

The addend

$$Y(t_{00};t)\Big(\delta p_0,\delta g(t_{00})\Big)^T + \Big\{Y(t_{00};t)\Big[\Big(\Theta_{k\times 1},\dot{g}_0(t_{00})\Big)^T - f_0\Big] - Y(t_{00}+\tau_0;t)f_1\Big\}\delta t_0$$

is the effect of the mixed initial condition (2) under perturbations of initial moment t_{00}, initial vector p_{00} and function $g_0(t)$.

The expression

$$-\Big\{Y(t_{00}+\tau_0;t)f_1 + \int_{t_{00}}^{t} Y(s;t)B(s)\dot{p}_0(s-\tau_0)ds\Big\}\delta\tau - \Big\{\int_{t_{00}}^{t} Y(s;t)C(s)\dot{q}_0(s-\sigma_0)ds\Big\}\delta\sigma$$

is the effect of perturbation of the delays τ_0,σ_0 and the mixed initial condition (2).

The expression

$$-\Big\{\int_{t_{00}}^{t} Y(s;t)E(s)\dot{u}_0(s-\theta_0)ds\Big\}\delta\theta$$

is the effect of perturbation of the delay θ_0.

The addend

$$\int_{t_{00}-\tau_0}^{t_{00}} Y(s+\tau_0;t)B(s+\tau_0)\delta\varphi(s)ds + \int_{t_{00}-\sigma_0}^{t_{00}} Y(s+\sigma_0;t)C(s+\sigma_0)\delta g(s)ds$$

$$+\int_{t_{00}}^{t} Y(s;t)\Big[D(s)\delta u(s)+E(s)\delta u(s-\theta_0)\Big]ds$$

is the effect of perturbations of initial functions $\varphi_0(t), g_0(t)$ and the control function $u_0(t)$.

Finally, we note that the variation formulas of solutions for various classes controlled differential equations with delay are given in [3-9].

R E F E R E N C E S

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