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SOLUTION OF BOUNDARY VALUE PROBLEMS OF ELASTOSTATICS FOR AN ELASTIC POROUS CIRCULAR RING WITH VOIDS

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Abstract. The boundary value problems of elastostatics for a porous circular ring with voids are considered. The general solution of the system of equations is represented by harmonic, biharmonic and metaharmonic functions. Explicit solutions of problems are obtained in the form of series.The conditions are established that ensure absolute and uniform convergence of these series.

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Dedicated to my teacher professor Michael Basheleishvili on the occasion of his 90^{th} *birthday anniversary.*

1. Introduction

In this paper we study boundary problems for elastic materials with empty pores. The foundations of the linear theory of elastic materials with voids were first proposed by Cowin and Nunziato [1]. Such materials include, in particular, rocks and soils, granulated and some other manufactured porous materials.Problems of elasticity for materials with voids were investigated by many authors. The history of development of porous body mechanics, the main results and the sphere of their application are set forth in detail in the monographs [2-5]. The generalization of the theory of elasticity and thermoelasticity for materials with double void pores belongs to Iesan and Quintanilla [6].

For applications, it is especially important to construct the solutions of boundary value problems in an explicit form because such solutions enable us to effectively perform quantitative analysis of the investigated problem. Questions related to this topic are considered, for example, in [7-18], where the explicit solutions of static boundary value problems of porous elasticity are constructed for the specific fluid-saturated media with double porosity. The boundary problems of elastostatics for a porous circular ring with voids are considered.

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2. Basic equations and boundary value problems

Let us assume that the isotropic elastic circular ring, with center at the origin, is bounded by the circumferences S_1 and S_2 with the radius R_1 and R_2, respectively; $R_1 < R_2$.

The basic system of equations of the theory of elastostatics for porous material with voids can be written in the form [1]:

$$\begin{cases} \mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} + \beta grad\varphi = 0, \\ \alpha\Delta\varphi - \xi\varphi - \beta div\mathbf{u} = 0, \end{cases} \tag{1}$$

where $\mathbf{u} = \mathbf{u}(u_1, u_2)$ is the displacement vector in a solid, φ is a change with respect to the pore area; λ and μ are the Lamé constants; α, β and ξ are the constants, characterizing the body porosity.

Let us now formulate the boundary value problems.

Find, a regular vector $\mathbf{U} = (u_1, u_2, \varphi)$, $(\mathbf{U} \in C^1(\overline{D}) \cap C^2(D), \overline{D} = D \cup S_1 \cup S_2)$ satisfies in the ring D a system of equations (1) and on the circumferences S_1 and S_2 the boundary conditions:

Problem I:

$$\left[\begin{array}{lll} \mathbf{u}^-(z) = \mathbf{f}^-(z), & \varphi^-(z) = f_3^-(z), & z \in S_1, \\ \mathbf{u}^+(z) = \mathbf{f}^+(z), & \varphi^+(z) = f_3^+(z), & z \in S_2; \end{array} \right. \tag{2}$$

Problem II:

$$\left[\begin{array}{ll} \mathbf{R}(\partial_\mathbf{z}, \mathbf{n})\mathbf{U}(\mathbf{z})^- = \mathbf{f}^-(\mathbf{z}), & \dfrac{\partial\varphi(\mathbf{z})}{\partial\mathbf{n}}^- = \mathbf{f}_3^-(\mathbf{z}), \\ \mathbf{R}(\partial_\mathbf{z}, \mathbf{n})\mathbf{U}(\mathbf{z})^+ = \mathbf{f}^+(\mathbf{z}), & \dfrac{\partial\varphi(\mathbf{z})}{\partial\mathbf{n}}^+ = \mathbf{f}_3^+(\mathbf{z}), \end{array} \right. \tag{3}$$

where $\mathbf{f}^\mp(\mathbf{z}) = (f_1^\mp(\mathbf{z}), f_2^\mp(\mathbf{z})), f_3^\mp(\mathbf{z})$ are the given functions on the circumferences S_1 and S_2 ;

$$\mathbf{R}\,(\partial_\mathbf{x}, \mathbf{n})\,\mathbf{U}(\mathbf{x}) = (\mathbf{P}\,(\partial_\mathbf{x}, \mathbf{n})\,\mathbf{U}(\mathbf{x}), \alpha\frac{\partial\varphi(\mathbf{x})}{\partial\mathbf{n}})$$

is the stress vector in the theory of elasticity for porous bodies with voids [1],

$$\mathbf{P}\,(\partial_\mathbf{x}, \mathbf{n})\,\mathbf{U}(\mathbf{x}) = \mathbf{T}\,(\partial_\mathbf{x}, \mathbf{n})\,\mathbf{u}(\mathbf{x}) + \beta\mathbf{n}\varphi(\mathbf{x}),$$

$\mathbf{T}\,(\partial_\mathbf{x}, \mathbf{n})\,\mathbf{u}(\mathbf{x}) = \mu\partial_\mathbf{n}\mathbf{u}(\mathbf{x}) + \lambda\mathbf{n} div\mathbf{u}(\mathbf{x}) + \mu\sum\limits_{i=1}^{2} n_i(\mathbf{x}) gradu_i(\mathbf{x})$ is the stress vector in the classical theory of elasticity.

3. General representations of solution of a system of equations

The solution of system (1) are written in the form

$$\left[\begin{array}{l} u(\mathbf{x}) = c_0\mathbf{u}^1(\mathbf{x}) + c_1\mathbf{u}^2(\mathbf{x}), \\ \varphi(\mathbf{x}) = \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}), \end{array} \right. \tag{4}$$

where φ_1 is a harmonic function, $\Delta\varphi_1 = 0$, and φ_2 is a metaharmonic function with the parameter s_1^2, $\quad (\Delta + s_1^2)\varphi_2 = 0; \quad s_1 = i\sqrt{\dfrac{\mu_0\xi - \beta^2}{\mu_0\alpha}} = is_0, \quad i = \sqrt{-1},$

$$\lambda > 0, \quad \mu > 0, \quad \alpha > 0, \quad \mu_0\xi > \beta^2; \tag{5}$$

c_0 and c_1 are the unknown constants. A general solution $\mathbf{u}^1 = (u_1^1, u_2^1)$ of the homogeneous equation, corresponding to the nonhomogeneous equation $(1)_1$ with respect, is represented as follows

$$\mathbf{u}(\mathbf{z}) = grad[\Phi_1(\mathbf{x}) + \Phi_2(\mathbf{x})] + rot\Phi_3(\mathbf{x}), \tag{6}$$

where the functions $\Phi_2(\mathbf{x})$ and $\Phi_3(\mathbf{x})$ are related to each other by the equality

$$\mu_0 grad\Delta\Phi_2(\mathbf{x}) + \mu rot\Delta\Phi_3(\mathbf{x}) = 0; \tag{7}$$

$\Delta\Phi_1(\mathbf{x}) = 0$, $\Delta\Delta\Phi_2(\mathbf{x}) = 0$, $\Delta\Delta\Phi_2(\mathbf{x}) = 0$; $\Phi_1(\mathbf{x})$, $\Phi_2(\mathbf{x})$, $\Phi_3(\mathbf{x})$- are scalar functions, $rot = \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right)$.
$\mathbf{u}^2 = (u_1^2, u_2^2)$ is one of the particular solutions of equation $(1)_1$:

$$\mathbf{u}^2(\mathbf{z}) = -\frac{\beta}{\mu_0} grad(-\frac{1}{s_1^2}\varphi_2 + \varphi_0), \tag{8}$$

where φ_0 is chosen such that $\Delta\varphi_0 = \varphi_1$. It is obvious that φ_0 is a biharmonic function: $\Delta\Delta\varphi_0 = \Delta\varphi_1 = 0$. For simplicity, the function is chosen such that $\varphi_1 = div\mathbf{u}^1 \equiv \Delta\Phi_2$. Then we can take $\varphi_0 = \Phi_2$. Let us calculate the values of the coefficients c_0 and c_1 in representation (4). We apply the operator div to the first equality in (4) and compare the obtained expression with $div\mathbf{u}$ defined by equation (1_2). We obtain

$$c_0 = \frac{\mu_0\xi - \beta^2}{\mu_0\beta}, \quad c_1 = 1.$$

By an immediate verification we make sure that representations (4) satisfy equations $(1)_1$ and $(1)_2$.

4. Solution of the problems

Let us rewrite representations (5) in terms of polar coordinates r and ψ as normal and tangential components

$$\left[\begin{array}{l} u_n = \partial_r(c_0\Phi_1 + c_3\Phi_2 + c_4\varphi_2) - c_0\frac{1}{r}\partial_\psi\Phi_3, \\ u_s = \frac{1}{r}\partial_\psi(c_0\Phi_1 + c_3\Phi_2 + c_4\varphi_2) + c_0\partial_r\Phi_3, \\ \varphi = \varphi_1 + \varphi_2, \end{array}\right. \tag{9}$$

where

$$c_3 = -\frac{\xi}{\beta}, \quad c_4 = \frac{\beta}{\mu_0 s_1^2}, \quad r^2 = x_1^2 + x_2^2.$$

Harmonic, biharmonic and metaharmonic functions in a circular ring can be represented as follows[19-21]:

$$\left[\begin{array}{l}\varphi_1(\mathbf{x}) = \ln r X_{01} + \sum\limits_{m=1}^{\infty}[\left(\dfrac{r}{R_2}\right)^m (\mathbf{X}_{m1}\cdot\nu_m(\psi)) + \\ \left(\dfrac{R_1}{r}\right)^m (\mathbf{X}_{m2}\cdot\nu_m(\psi))], \\ \Phi_2(\mathbf{x}) = \ln r X_{01} + \dfrac{R_2^2}{4}\sum\limits_{m=2}^{\infty}[\dfrac{1}{m+1}\left(\dfrac{r}{R_2}\right)^{m+2}(\mathbf{X}_{m1}\cdot\nu_m(\psi)) \\ +\dfrac{1}{1-m}\left(\dfrac{R_1}{r}\right)^{m-2}(\mathbf{X}_{m2}\cdot\nu_m(\psi))] + \dfrac{1}{2}\left(\dfrac{r}{R_2}\right)^2\mathbf{X}_{02}, \\ \Phi_3(\mathbf{x}) = \ln r X_{01} - \dfrac{R_2^2\mu_0}{4\mu}\sum\limits_{m=0}^{\infty}[\dfrac{1}{m+1}\left(\dfrac{r}{R_2}\right)^{m+2}(\mathbf{X}_{m1}\cdot\mathbf{s}_m(\psi)) \\ +\dfrac{1}{1-m}\left(\dfrac{R_1}{r}\right)^{m-2}(\mathbf{X}_{m2}\cdot\mathbf{s}_m(\psi))] + \dfrac{1}{2}\left(\dfrac{r}{R_2}\right)^2\mathbf{X}_{01}, \\ \Phi_1(\mathbf{x}) = \ln r X_{05} + \sum\limits_{m=1}^{\infty}[\left(\dfrac{r}{R_2}\right)^m (\mathbf{X}_{m5}\cdot\nu_m(\psi)) + \\ \left(\dfrac{R_1}{r}\right)^m (X_{m6}\cdot\nu_m(\psi))], \\ \varphi_2(\mathbf{x}) = \sum\limits_{m=0}^{\infty}[I_m(\lambda_0 r)(\mathbf{X}_{m3}\cdot\nu_m(\psi)) + K_m(\lambda_0 r)(\mathbf{X}_{m4}\cdot\nu_m(\psi))],\end{array}\right. \tag{10}$$

where I_m and K_m are Bessel's and modified Hankel's functions of an imaginary argument, respectively; $\mathbf{X}_{mi}$ is the unknown two-component constant vector, $\nu_m(\psi) = (\cos m\psi, \sin m\psi), \mathbf{s}_m(\psi) = (-\sin m\psi, \cos m\psi), i = 1,2,3,4,5,6; \quad \mathbf{x} = (r,\psi), \quad \mathbf{x}\in D$.
We rewrite conditions (2) in the tangential and normal components:

$$u_n^{\mp}(\mathbf{z}) = f_n^{\mp}, \quad u_s^{\mp}(\mathbf{z}) = f_s^{\mp}, \quad \varphi^{\mp}(\mathbf{z}) = f_3^{\mp}(\mathbf{z}). \tag{11}$$

Expand the functions $f_n^{\mp}, f_s^{\mp}$ and $f_3^{\mp}$; in the Fourier series, whose Fourier coefficients are: $\boldsymbol{\alpha}_m^{\mp} = (\alpha_{m1}^{\mp}, \alpha_{m2}^{\mp}), \quad \boldsymbol{\beta}_m^{\mp} = (\beta_{m1}^{\mp}, \beta_{m2}^{\mp}), \quad \boldsymbol{\gamma}_m^{\mp} = (\gamma_{m1}^{\mp}, \gamma_{m2}^{\mp})$. We substitute (9) into (10) and then the obtained expression into (11). Passing to the limit, as $r\to R_1$ and $r\to R_2$ for the unknowns $\mathbf{X}_{mi}$ we obtain a system of algebraic equations:

m=0

$$\left\{\begin{array}{l}\dfrac{c_3}{R_1}X_{01} + \dfrac{c_3}{R_2}R_1X_{02} + c_4\lambda_0 I'_0(\lambda_0R_1)X_{03} + c_4\lambda_0K'_0(\lambda_0R_1)X_{04} \\ +\frac{c_0}{R_1}X_{05} = \dfrac{\boldsymbol{\alpha}_0^-}{2}, \\ \dfrac{c_3}{4R_2}X_{01} + c_3X_{02} + c_4\lambda_0 I'_0(\lambda_0R_2)X_{03} + c_4\lambda_0K'_0(\lambda_0R_2)X_{04} \\ +\frac{c_0}{R_2}X_{05} = \dfrac{\boldsymbol{\alpha}_0^+}{2}, \\ \dfrac{c_0}{R_1}X_{01} + \dfrac{R_1}{R_2}X_{06} = \dfrac{\beta_0^-}{2}, \\ \dfrac{c_0}{R_2}X_{01} + X_{06} = \dfrac{\boldsymbol{\beta}_0^+}{2} \\ \ln R_1X_{01} + I_0(\lambda_0R_1)X_{03} + K_0(\lambda_0R_1)X_{04} = \dfrac{\boldsymbol{\gamma}_0^-}{2}, \\ \ln R_2X_{01} + I_0(\lambda_0R_2)X_{03} + K_0(\lambda_0R_2)X_{04} = \dfrac{\boldsymbol{\gamma}_0^+}{2};\end{array}\right. \tag{12}$$

m=2,3,...

$$\begin{cases} A_1(R_1)\mathbf{X}_{m1}+A_2(R_1)\mathbf{X}_{m2}+A_3(R_1)\mathbf{X}_{m3}+A_4(R_1)\mathbf{X}_{m4}+A_5(R_1)\mathbf{X}_{m5}\\ +A_6(R_1)\mathbf{X}_{m6}=\boldsymbol{\alpha}_m^-,\\ A_1(R_2)\mathbf{X}_{m1}+A_2(R_2)\mathbf{X}_{m2}+A_3(R_2)\mathbf{X}_{m3}+A_4(R_2)\mathbf{X}_{m4}+A_5(R_2)\mathbf{X}_{m5}\\ +A_6(R_2)\mathbf{X}_{m6}=\boldsymbol{\alpha}_m^+,\\ B_1(R_1)\mathbf{X}_{m1}+B_2(R_1)\mathbf{X}_{m2}+B_3(R_1)\mathbf{X}_{m3}+B_4(R_1)\mathbf{X}_{m4}+B_5(R_1)\mathbf{X}_{m5}\\ +B_6(R_1)\mathbf{X}_{m6}=\boldsymbol{\beta}_m^-,\\ B_1(R_2\mathbf{X}_{m1}+B_2(R_2)\mathbf{X}_{m2}+B_3(R_2)\mathbf{X}_{m3}+B_4(R_2)\mathbf{X}_{m4}+B_5(R_2)\mathbf{X}_{m5}\\ +B_6(R_2)\mathbf{X}_{m6}=\boldsymbol{\beta}_m^+,\\ \left(\frac{R_1}{R_2}\right)^m\mathbf{X}_{m1}+\mathbf{X}_{m2}+I_m(\lambda_0R_1)\mathbf{X}_{m3}\\ +K_m(\lambda_0R_1)\mathbf{X}_{m4}=\boldsymbol{\gamma}_m^-,\\ \mathbf{X}_{m1}+\left(\frac{R_1}{R_2}\right)^m\mathbf{X}_{m2}+I_m(\lambda_0R_2)\mathbf{X}_{m3}\\ +K_m(\lambda_0R_2)\mathbf{X}_{m4}=\boldsymbol{\gamma}_m^+, \end{cases} \tag{13}$$

where

$$A_1(r)=\frac{1}{4(m+1)}\Big[c_3(m+2)r+\frac{c_0m\mu_0R_2^2}{\mu}\left(\frac{r}{R_2}\right)^{m+2}\Big],$$
$$A_2(r)=\tfrac{R_1^2}{4(m+1)}\Big[\frac{c_0m\mu_0}{\mu}-c_3(m-2)\Big]\left(\frac{R_1}{R_2}\right)^m,$$
$$A_3(r)=c_4\lambda_0I'_m(\lambda_0r),\quad A_4(r)=c_4\lambda_0K'_m(\lambda_0r),\quad A_5(r)=\frac{c_0m}{R_2}\left(\frac{r}{R_2}\right)^{m-1},$$
$$A_6(r)=-\frac{c_0m}{R_1}\left(\frac{R_1}{r}\right)^{m+1},$$
$$B_1(r)=\frac{R_2}{4(m+1)}\Big[\frac{c_3mR_2}{r}-\frac{c_0\mu_0(m+2)}{\mu}\Big]\left(\frac{r}{R_2}\right)^{m+2},$$
$$B_2(r)=\frac{R_1^2}{4(1-m)r}\Big[c_3m-\frac{c_0(m-2)\mu_0}{\mu}\left(\frac{R_1}{r}\right)^{m-2}\Big],$$
$$B_3(r)=\frac{c_4m}{r}I_m(\lambda_0r),\quad B_4(r)=\frac{c_4m}{r}K_m(\lambda_0r),$$
$$B_5(r)=\frac{c_0m}{r}\left(\frac{r}{R_2}\right)^m,\quad B_6(r)=\frac{c_0m}{r}\left(\frac{R_1}{r}\right)^m.$$

We substitute the solutions of systems (12) and (13) $\mathbf{X}_{mi}$ in (10) and then in formulas (6) and (8). Then taking into account $\varphi_0=\Phi_2$ we get the solution to problem I. Applying representations (10) and (4), we can solve problem II by a similar method.

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