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VARIATION FORMULAS OF SOLUTIONS FOR THE CONTROLLED FUNCTIONAL DIFFERENTIAL EQUATIONS CONSIDERING DELAY PARAMETERS PERTURBATIONS AND OPTIMAL CONTROL PROBLEMS

Tadumadze T., Shavadze T.

Abstract. For the nonlinear controlled functional differential equations with delays in the phase coordinates and controls considering the discontinuous (continuous) initial condition the local (global) variation formulas of solutions are obtained. For the optimization problems with general boundary conditions and functional the necessary optimality conditions are proved: for the initial and final moments in the form of inequalities and equalities; for delays containing in the phase coordinates and for the initial vector in the form of equalities; for the initial and control functions in the form of the integral maximum principle.

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1. Introduction

As is known, real economical, biological, physical and majority of processes contain an information about their behavior in the past, i. e., the processes that contain effects with delayed action and which are described by functional differential equations with delays [1-3]. In the present work the controlled functional differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_s), u(t), u(t-\theta_1)$$
$$, ..., u(t-\theta_k)),\ x(t) \in \mathbb{R}^n \tag{1.1}$$

as well as with the discontinuous initial condition

$$x(t) = \varphi(t), t < t_0,\ x(t_0) = x_0 \tag{1.2}$$

and with the continuous initial condition

$$x(t) = \varphi(t), t \le t_0 \tag{1.3}$$

is considered

Condition (1.2) is called a discontinuous initial condition since, in general, $x(t_0) \neq x_0$. Discontinuity at the initial moment may be related to the instant change in a dynamical process (changes of investment, environment and so on). Condition (1.3) is called a continuous initial condition since, always, $x(t_0) = \varphi(t_0)$.

In section 2, the local variation formula of solution is obtained, that is, a linear representation of variation of the solution of problem (1.1)-(1.2) in the neighborhood of the right end of the main interval with respect to perturbations of the initial data. In the section 3 the global variation formula of solution is obtained for problem (1.1)-(1.3). In the variation formulas the effects of perturbations of the initial moment and several delays and also the effects of discontinuous and continuous initial conditions are detected. The variation formula plays a basic role in proving the necessary conditions of optimality [4-9] and in the sensitivity analysis of mathematical models [10]. Moreover, the variation formula allows one to get an approximate solution of the perturbed equation [11]. The term "variation formula of solution" has been introduced by R. V. Gamkrelidze and proved in [4] for the ordinary differential equation. The effects of perturbation of the initial moment and the discontinuous initial condition in the variation formulas for the first time were revealed in [12] for the delay differential equation. The variation formulas for various classes of controlled functional differential equations are derived in [8,13-19].

In sections 4 for the optimization problem with the discontinuous initial conditions with general boundary conditions and functional the necessary conditions of optimality are proved: for the initial and final moments in the form of inequalities and equalities; for delays containing in the phase coordinates and for the initial vector in the form of equalities; for the initial and control functions in the form of the integral maximum principle. In sections 5 for the optimization problem with the continuous initial conditions the necessary conditions of optimality are proved.

Optimal control problems for various classes of functional differential equations are investigated in [5-8,16,20].

2. Local variation formulas of solution for the controlled functional differential equation with the discontinuous initial condition

Let $O \subset \mathbb{R}^n$ and $U_0 \subset \mathbb{R}^r$ be open sets. Let $h_{i2} > h_{i1} > 0, i = \overline{1,s}$ and $\theta_k > \cdots > \theta_1 > 0$ be given numbers and the n-dimensional function $f(t, x, x_1, ..., x_s, u, u_1, ..., u_k)$ satisfies the following conditions:

(a) for almost all fixed $t \in I = [a, b]$ the function $f(t, \cdot) : O^{1+s} \times U_0^{1+k} \to \mathbb{R}^n$ is continuously differentiable;

(b) for each fixed $(x, x_1, ..., x_s, u, u_1, ..., u_k) \in O^{1+s} \times U_0^k$ the functions

$$f(t, x, x_1, \cdots, x_s, u, u_1, ..., u_k), f_x(t, \cdot), f_{x_i}(t, \cdot), i = \overline{1,s}$$

and

$$f_u(t, \cdot), f_{u_i}(t, \cdot), i = \overline{1,k}$$

are measurable on I;

(c) for compact sets $K \subset O$ and $U \subset U_0$ there exists a function $m_{K,U}(t) \in L_1(I, [0, \infty))$ such that

$$| f(t, x, x_1, ..., x_s, u, u_1, ..., u_k) | + | f_x(t, \cdot) | + \sum_{i=1}^{s} | f_{x_i}(t, \cdot) |$$

$$+ | f_u(t, \cdot) | + \sum_{i=1}^{k} | f_{u_i}(t, \cdot) | \leq m_{K,U}(t)$$

for all $(x, x_1, ..., x_s, u, u_1, ..., u_k) \in K^{1+s} \times U^{1+k}$ and for almost all $t \in I$.

Furthermore, Φ is the set of continuous initial functions $\varphi : I_1 = [\hat{\tau}, b] \to O, \hat{\tau} = a - \max\{h_{12}, \cdots, h_{s2}\}$ and let Ω be a set of control measurable functions $u(t), t \in I_2 = [a - \theta_k, b]$ satisfying the conditions: the set $clu(I_2) \subset U_0$ and it is compact in $\mathbb{R}^r$.

To each element

$$\mu = (t_0, \tau_1, ..., \tau_s, \ x_0, \varphi, u) \in \Lambda_1 = [a, b) \times [h_{11}, h_{12}] \times$$

$$\cdots \times [h_{s1}, h_{s2}] \times O \times \Phi \times \Omega$$

we assign the delay controlled functional differential equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau_1), ..., x(t-\tau_s), u(t), u(t-\theta_1)$$
$$, ..., u(t-\theta_k)) \tag{2.1}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), t < t_0, x(t_0) = x_0. \tag{2.2}$$

Condition (2.2) is called discontinuous because, in general, $x(t_0) \neq \varphi(t_0)$.

Definition 2.1. Let $\mu = (t_0, \tau_1, ..., \tau_s, \; x_0, \varphi, u) \in \Lambda_1$. A function $x(t) = x(t; \mu) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b]$ is called a solution of equation (2.1) with the initial condition (2.2) or the solution corresponding to μ and defined on the interval $[\hat{\tau}, t_1]$ if it satisfies condition (2.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (2.1) almost everywhere (a.e.) on $[t_0, t_1]$.

Let $\mu_0 = (t_{00}, \tau_{10}, \cdots, \tau_{s0}, x_{00}, \varphi_0, u_0) \in \Lambda_1$ be a fixed element and let $x_0(t)$ be the solution corresponding to μ_0 and defined on the interval $[\hat{\tau}, t_{10}]$, where $t_{00}, t_{10} \in (a, b), t_{00} < t_{10}$ and $\tau_{i0} \in (h_{i1}, h_{i2}), i = \overline{1, s}$.

Let us introduce the set of variation:

$$V_1 = \Big\{ \delta\mu = (\delta t_0, \delta\tau_1, ..., \delta\tau_s, \delta x_0, \delta\varphi, \delta u) : \; \delta t_0 \in (a, b) - t_{00},$$
$$\delta\tau_i \in (h_{i1}, h_{i2}) - \tau_{i0}, i = \overline{1, s}, \delta x_0 \in O - x_{00}, \delta\varphi = \sum_{i=1}^{m} \lambda_i \delta\varphi_i,$$
$$\delta\varphi_i \in \Phi - \varphi_0, \; i = \overline{1, m}, \delta u \in \Omega - u_0, | \delta t_0 | \leq \alpha, | \delta\tau_i | \leq \alpha, i = \overline{1, s},$$
$$| \delta x_0 | \leq \alpha, | \lambda_i | \leq \alpha, i = \overline{1, m}, \; ||\delta u|| \leq \alpha \Big\},$$

where $\alpha > 0$ is a fixed number, $(a, b) - t_{00} = \{\delta t_0 = t_0 - t_{00} : t_0 \in (a, b)\}$ and $||\delta u|| = \sup \big\{ |\delta u(t)| : t \in I_2 \big\}$.

There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times V_1$ we have $\mu_0 + \varepsilon\delta\mu \in \Lambda_1$, and the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to it [16].

By the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\hat{\tau}, t_{10} + \delta_1]$. Therefore, we can assume that the solution $x_0(t)$ is defined on the whole interval $[\hat{\tau}, t_{10} + \delta_1]$.

Now we introduce the increment of the solution $x_0(t) := x(t; \mu_0)$,

$$\Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \forall (\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times V_1.$$

Theorem 2.1. *Let the following conditions hold:*

2.1) $\tau_{s0} > \cdots > \tau_{10}$ and $t_{00} + \tau_{s0} < t_{10}$;

2.2) *the function $\varphi_0(t)$ is absolutely continuous and $\dot{\varphi}_0(t), t \in I_1$ is bounded;*

2.3) *the function $f(w, u, u_1, ..., u_k)$, where $w = (t, x, x_1, ..., x_s) \in I \times O^{1+s}$ is bounded on*

$I\times O^{1+s}\times U_0^{1+k}$;
2.4) *there exists the finite limit*

$$\lim_{w\to w_0} f(w,u_0(t),u_0(t-\theta_1),...,u_0(t-\theta_k))=f^-, w\in(a,t_{00}]\times O^{1+s},$$

where $w_0=(t_{00},x_{00},\varphi_0(t_{00}-\tau_{10}),...,\varphi_0(t_{00}-\tau_{s0}))$;
2.5) *there exist the finite limits*

$$\lim_{(w_{1i},w_{2i})\to(w_{1i}^0,w_{2i}^0)}[f(w_{1i},u_0(t),u_0(t-\theta_1),...,u_0(t-\theta_k))$$

$$-f(w_{2i},u_0(t),u_0(t-\theta_1),...,u_0(t-\theta_k))]=f_i,$$

where $w_{1i},w_{2i}\in(a,b)\times O^{1+s}, i=\overline{1,s}$,

$$w_{1i}^0=(t_{00}+\tau_{i0},x_0(t_{00}+\tau_{i0}),x_0(t_{00}+\tau_{i0}-\tau_{10}),\cdots,x_0(t_{00}+\tau_{i0}-\tau_{i-10}),$$

$$x_{00},x_0(t_{00}+\tau_{i0}-\tau_{i+10}),...,x_0(t_{00}+\tau_{i0}-\tau_{s0})),$$

$$w_{2i}^0=(t_{00}+\tau_{i0},x_0(t_{00}+\tau_{i0}),x_0(t_{00}+\tau_{i0}-\tau_{10}),...,x_0(t_{00}+\tau_{i0}-\tau_{i-10}),$$

$$\varphi_0(t_{00}),x_0(t_{00}+\tau_{i0}-\tau_{i+10}),...,x_0(t_{00}+\tau_{i0}-\tau_{s0})).$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$ *with* $t_{10}-\delta_2>t_{00}+\tau_{s0}$ *such that for arbitrary* $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V_1^-$, *where* $V_1^-=\{\delta\mu\in V_1:\delta t_0\le 0\}$, *we have*

$$\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x(t;\delta\mu)+o(t;\varepsilon\delta\mu). \tag{2.3}$$

where

$$\delta x(t;\delta\mu)=-Y(t_{00};t)f^-\delta t_0+\beta_1(t;\delta\mu), \tag{2.4}$$

$$\beta_1(t;\delta\mu)=Y(t_{00};t)\delta x_0-\Big[\sum_{i=1}^s Y(t_{00}+\tau_{i0};t)f_i\Big]\delta t_0-\sum_{i=1}^s\Big[Y(t_{00}+\tau_{i0};t)f_i$$

$$+\int_{t_{00}}^t Y(\xi;t)f_{x_i}[\xi]\dot x_0(\xi-\tau_{i0})d\xi\Big]\delta\tau_i$$

$$+\sum_{i=1}^s\int_{t_{00}-\tau_{i0}}^t Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)d\xi$$

$$+\int_{t_{00}}^t Y(\xi;t)\Big[f_u[\xi]\delta u(\xi)+\sum_{i=1}^k f_{u_i}[\xi]\delta u(\xi-\theta_i)\Big]d\xi;$$

here it is assumed that

$$\int_{t_{00}}^t Y(\xi;t)f_{x_i}[\xi]\dot x_0(\xi-\tau_{i0})d\xi=\int_{t_{00}}^{t_{00}+\tau_{i0}} Y(\xi;t)f_{x_i}[\xi]\dot\varphi_0(\xi-\tau_{i0})d\xi$$

$$+\int_{t_{00}+\tau_{i0}}^t Y(\xi;t)f_{x_i}[\xi]\dot x_0(\xi-\tau_{i0})d\xi.$$

Next, $Y(\xi;t)$ *is the* $n\times n$*-matrix function, satisfying the equation*

$$Y_\xi(\xi;t)=-Y(\xi;t)f_x[\xi]-\sum_{i=1}^s Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}],\xi\in[t_{00},t] \tag{2.5}$$

and the condition

$$Y(\xi;t)=\begin{cases} H\ for\ \xi=t,\\ \Theta\ for\ \xi>t, \end{cases} \tag{2.6}$$

where H *is the identity matrix and* Θ *is the zero matrix;* $f_{x_i}[\xi]=f_{x_i}(\xi,x_0(\xi),x_0(\xi-\tau_{10}),...,x_0(\xi-\tau_{s0}),u_0(\xi),u_0(\xi-\theta_1),...,u_0(\xi-\theta_k))$,

$$\lim_{\varepsilon\to 0} o(t;\varepsilon\delta\mu)/\varepsilon=0 \text{ uniformly for } (t,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times V_1^-.$$

Some comments. The function $\delta x(t;\delta\mu)$ is called the first variation of the solution $x_0(t), t\in[t_{00}-\delta_2,t_{00}+\delta_2]$. The expression (2.4) is called the local variation formula of solution.

The addend

$$-\Big[Y(t_{00};t)f^-+\sum_{i=1}^{s}Y(t_{00}+\tau_{i0};t)f_i\Big]\delta t_0$$

in formula (2.4) is the effect of the discontinuous initial condition (2.2) and perturbation of the initial moment t_{00}.

The addend

$$-\sum_{i=1}^{s}\Big[Y(t_{00}+\tau_{i0};t)f_i+\int_{t_{00}}^{t}Y(\xi;t)f_{x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})d\xi\Big]\delta\tau_i$$

in formula (2.4) is the effect of the discontinuous initial condition (2.2) and perturbations of the delays $\tau_{i0}, i=\overline{1,s}$.

The expression

$$Y(t_{00};t)\delta x_0+\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)d\xi$$

in formula (2.4) is the effect of perturbations of the initial vector x_{00} and the initial function $\varphi_0(t)$.

The addend

$$\int_{t_{00}}^{t}Y(\xi;t)\Big[f_u[\xi]\delta u(\xi)+\sum_{i=1}^{m}f_{u_i}[\xi]\delta u(\xi-\theta_i)\Big]d\xi$$

is the effect of perturbation of the control function $u_0(t)$.

It is easy to see that (2.4) can be represented by the following form:

$$\delta x(t;\delta\mu)=\delta x_0(t;\delta\mu)+\sum_{i=1}^{s}\delta x_i(t;\delta\mu) \tag{2.7}$$

where

$$\delta x_0(t;\delta\mu)=Y(t_{00};t)(\delta x_0-f^-\delta t_0)+\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)d\xi$$

$$+\int_{t_{00}}^{t}Y(\xi;t)\Big[f_u[\xi]\delta u(\xi)+\sum_{i=1}^{m}f_{u_i}[\xi]\delta u(\xi-\theta_i)$$

$$-\sum_{i=1}^{s} f_{x_i}[\xi]\dot{x_0}(\xi-\tau_{i0})\delta\tau_i\Big]d\xi,$$

and

$$\delta x_i(t;\delta\mu)=-Y(t_{00}+\tau_{i0};t)f_i(\delta t_0+\delta\tau_i), i=\overline{1,s}.$$

Using of the Cauchy formula [16], we can conclude that the function

$$\delta x_0(t)=\begin{cases}\delta\varphi(t), t\in[\hat{\tau},t_{00}),\\ \delta x_0(t;\delta\mu), t\in[t_{00},t_{10}+\delta_2]\end{cases}$$

satisfies the following equation

$$\dot{\delta x}(t)=f_x[t]\delta x(t)+\sum_{i=1}^{s} f_x[t]\delta x(t-\tau_{i0})+f_u[t]\delta u(t)+\sum_{i=1}^{m} f_{u_i}[t]\delta u(t-\theta_i)$$

$$-\sum_{i=1}^{s} f_{x_i}[t]\dot{x_0}(t-\tau_{i0})\delta\tau_i \tag{2.8}$$

with the initial condition

$$\delta x(t)=\delta\varphi(t), t\in[\hat{\tau},t_{00}), \delta x(t_{00})=\delta x_{00}-f^-\delta t_0;$$

and the function

$$\delta x_i(t)=\begin{cases}0, t\in[\hat{\tau},t_{00}+\tau_{i0}),\\ \delta x_i(t;\delta\mu), t\in[t_{00}+\tau_{i0},t_{10}+\delta_2]\end{cases}$$

satisfies the following equation

$$\dot{\delta x}(t)=f_x[t]\delta x(t)+\sum_{i=1}^{s} f_{x_i}[t]\delta x(t-\tau_{i0}) \tag{2.9}$$

with the initial condition

$$\delta x(t)=0, t\in[\hat{\tau},t_{00}+\tau_{i0}), \delta x(t_{00}+\tau_{i0})=-f_i(\delta t_0+\delta\tau_i).$$

Therefore, the local first variation of solution can be calculated by two ways: first-find the matrix function $Y(\xi;t)$ (see (2.5) and (2,6)), second- find the solutions of $s+1$ linear equations (see (2.7)-(2.9)).

Theorem 2.2. *Let the conditions 2.1)-2.3) and 2.5) of the Theorem 2.1 hold. Moreover, there exists the finite limit*

$$\lim_{w\to w_0} f(w,u_0(t),u_0(t-\theta_1),...,u_0(t-\theta_k))=f^+,$$

$$w\in[t_{00},b)\times O^{1+s}. \tag{2.10}$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$, with $t_{10}-\delta_2>t_{00}+\tau_{s0}$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V_2^+$, where $V_2^+=\{\delta\mu\in V:\delta t_0\geq 0\}$, the formula (2.3) holds, where

$$\delta x(t;\delta\mu)=-Y(t_{00};t)f^+\delta t_0+\beta(t;\delta\mu).$$

Theorem 2.3. *Let the conditions 2.1)-2.5) of Theorem 2.1 and condition* (2.10) *hold. Moreover,*

$$f^- = f^+ := \hat{f}.$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$, *with* $t_{10} - \delta_2 > t_{00} + \tau_0$ *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V$, *formula (2.3) holds, where*

$$\delta x(t; \delta\mu) = -Y(t_{00}; t)\hat{f}\delta t_0 + \beta(t; \delta\mu).$$

Theorems 2.1 and 2.2 are proved by the scheme given in [16]. Theorem 2.3 is a corollary to Theorems 2.1 and 2.2. For the controlled functional differential equation without delay in controls the analogous local variation formulas are proved in [18].

3. Global variation formulas of solution for the controlled functional differential equation with the continuous initial condition

To each element $\varrho = (t_0, \tau_1, \cdots, \tau_s, \varphi, u) \in \Lambda_2 = [a, b) \times [h_{11}, h_{12}] \times \cdots \times [h_{s1}, h_{s2}] \times \Phi \times \Omega$ we assign the delay controlled functional differential equation (2.1) with the continuous initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0]. \tag{3.1}$$

Definition 3.1. Let $\varrho = (t_0, \tau_1, ..., \tau_s, \varphi, u) \in \Lambda_2$. A function $x(t) = x(t; \varrho) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b]$ is called a solution of equation (2.1) with the initial condition (3.1) or the solution corresponding to ϱ and defined on the interval $[\hat{\tau}, t_1]$ if it satisfies condition (3.1) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (2.1) a. e. on $[t_0, t_1]$.

Let us introduce the set of variation:

$$V_2 = \Big\{\delta\varrho = (\delta t_0, \delta\tau_1, ..., \delta\tau_s, \delta\varphi, \delta u) : \ \delta t_0 \in (a, b) - t_{00},$$

$$\delta\tau_i \in (h_{i1}, h_{i2}) - \tau_{i0}, i = \overline{1, s}, \delta\varphi = \sum_{i=1}^{m} \lambda_i \delta\varphi_i,$$

$$\delta\varphi_i \in \Phi - \varphi_0, \ i = \overline{1, m}, \delta u \in \Omega - u_0, |\ \delta t_0\ | \le \alpha, |\ \delta\tau_i\ | \le \alpha, i = \overline{1, s},$$

$$|\ \lambda_i\ | \le \alpha, i = \overline{1, m}, \ ||\delta u|| \le \alpha\Big\}.$$

Let $x_0(t)$ be a solution corresponding to the element $\varrho_0 = (t_{00}, \tau_{10}, ..., \tau_{s0}, \varphi_0, u_0) \in \Lambda_2$ and defined on the interval $[\hat{\tau}, t_{10}]$, where $t_{00}, t_{10} \in (a, b), t_{00} < t_{10}$ and $\tau_{i0} \in (h_{i1}, h_{i2}), i = \overline{1, s}$.

There exist the numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\varrho) \in (0, \varepsilon_1) \times V_2$ we have $\varrho_0 + \varepsilon\delta\varrho \in \Lambda_2$, and the solution $x(t; \varrho_0 + \varepsilon\delta\varrho)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to it [16].

Due to the uniqueness, the solution $x(t; \varrho_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\hat{\tau}, t_{10} + \delta_1]$. Therefore, we cane assume that the solution $x_0(t)$ is defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$.

Let us define the increment of the solution $x_0(t) := x(t; \varrho_0)$,

$$\Delta x(t; \varepsilon\delta\varrho) = x(t; \varrho_0 + \varepsilon\delta\varrho) - x_0(t),$$

$$\forall (t, \varepsilon, \delta\varrho) \in [\hat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times V_2.$$

Theorem 3.1. *Let the function* $\varphi_0(t), t \in I_1$ *be absolutely continuous. Let the functions* $\dot{\varphi}_0(t)$ *and* $f(w, u, u_1, ..., u_k), (w, u, u_1, ..., u_k) \in I \times O^{1+s} \times U_0^{1+k}$ *be bounded, where* $w = (t, x, x_1, ..., x_s)$*. Moreover, there exist the finite limits*

$$\lim_{t \to t_{00}-} \dot{\varphi}_0(t) = \dot{\varphi}_0^-,$$

$$\lim_{w \to w_0} f(w, u_0(t), u_0(t-\theta_1), ..., u_0(t-\theta_k)) = f^-, w \in (a, t_{00}] \times O^{1+s},$$

where $w_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(t_{00} - \tau_{10}), ..., \varphi_0(t_{00} - \tau_{s0}))$*;*
Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *with* $t_{10} - \delta_2 > t_{00} + \tau_{s0}$ *such that for arbitrary* $(t, \varepsilon, \delta\varrho) \in [t_{10}, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V_2^-$*, where* $V_2^- = \{\delta\varrho \in V_2 : \delta t_0 \le 0\}$*, we have*

$$\Delta x(t; \varepsilon\delta\varrho) = \varepsilon\delta x(t; \delta\varrho) + o(t; \varepsilon\delta\varrho). \tag{3.2}$$

where

$$\delta x(t; \delta\varrho) = Y(t_{00}; t)\Big[\dot{\varphi}_0^- - f^-\Big]\delta t_0 + \beta_2(t; \delta\varrho),$$

$$\beta_2(t; \delta\varrho) = Y(t_{00}; t)\delta\varphi(t_{00}) + \sum_{i=1}^{s} \int_{t_{00}-\tau_{i0}}^{t} Y(s + \tau_{i0}; t) f_{x_i}[s + \tau_{i0}]\delta\varphi(s)ds$$

$$- \int_{t_{00}}^{t} Y(s; t)\Big[\sum_{i=1}^{s} f_{x_i}[s]\dot{x}_0(s - \tau_{i0})\delta\tau_i\Big]ds$$

$$+ \int_{t_{00}}^{t} Y(s; t)\Big[f_u[s]\delta u(s) + \sum_{i=1}^{k} f_{u_i}[s]\delta u(s - \theta_i)\Big]ds,$$

where $Y(s; t)$ *is the* $n \times n$*-matrix function satisfying the equation (2.5) and the condition (2.6).*

Theorem 3.2. *Let the function* $\varphi_0(t), t \in I_1$ *be absolutely continuous and* $f(w, u, u_1, ..., u_k)$*,* $(w, u, u_1, ..., u_k) \in I \times O^{1+s} \times U_0^{1+k}$ *be bounded. Moreover, there exist the finite limits*

$$\lim_{t \to t_{00}+} \dot{\varphi}_0(t) = \dot{\varphi}_0^+,$$

$$\lim_{w \to w_0} f(w, u_0(t), u_0(t-\theta_1), ..., u_0(t-\theta_k)) = f^+, w \in [t_{00}, b) \times O^{1+s},$$

Then for each $\hat{t_0} \in (t_{00}, t_{10})$*, there exist numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that for arbitrary* $(t, \varepsilon, \delta\varrho) \in [t_{00}, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V_2^+$*, where* $V_2^+ = \{\delta\varrho \in V_2 : \delta t_0 \ge 0\}$*, the formula (3.2) holds, where*

$$\delta x(t; \delta\varrho) = Y(t_{00}; t)\Big[\dot{\varphi}_0^+ - f^+\Big]\delta t_0 + \beta(t; \delta\varrho).$$

Theorem 3.3. *Let the assumptions of Theorems 3.1 and 3.2 be fulfilled. Moreover,*

$$\dot{\varphi}_0^- - f^- = \dot{\varphi}_0^+ - f^+ = \widetilde{f}.$$

Then for each $\hat{t_0} \in (t_{00}, t_{10})$*, there exist numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that for arbitrary* $(t, \varepsilon, \delta\varrho) \in [\hat{t_0}, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V_2$*, the formula (2.4) holds, where*

$$\delta x(t; \delta\varrho) = Y(t_{00}; t)\widetilde{f}\delta t_0 + \beta_2(t; \delta\varrho).$$

Theorems 3.1 and 3.2 are proved by the scheme given in [16]. Theorem 3.3 is a corollary to Theorems 3.1 and 3.2. For the controlled functional differential equation without delay in controls the analogous global variation formulas are proved in [19].

4. Optimal control problem with the discontinuous initial condition

Let $U \subset \mathbb{R}^r$ be a convex compact set. Assume that the n-dimensional function $f(t, x, x_1, ..., x_s, u, u_1, ..., u_k)$ is continuous on the set $I \times O^{1+s} \times U^{1+k}$ and continuously differentiable with respect to $(x, x_1, ..., x_s, u, u_1, ..., u_k) \in O^{1+s} \times U^{1+k}$; moreover,there exists a number $M > 0$ such that

$$| f(t, x, x_1, ..., x_s, u) | + | f_x(t, x, \cdot) | + \sum_{i=1}^{s} | f_{x_i}(t, x, \cdot) | + | f_u(t, x, \cdot) |$$

$$+ \sum_{i=1}^{k} | f_{u_i}(t, x, \cdot) | \leq M$$

for all $(t, x, x_1, ..., x_s, u, u_1, ..., u_k) \in I \times O^{1+s} \times U^{1+k}$.

Furthermore, let Φ_1 be the set of continuous initial functions $\varphi(t) \in N, t \in I_1$, where $N \subset O$ is a convex compact set; Ω_1 is the set of measurable functions $u(t) \in U, t \in I_2; X_0 \subset O$ is a convex compact set.

To each element

$$\nu = (t_0, t_1, \tau_1, ..., \tau_s,\ x_0, \varphi, u) \in A_1 = I \times I \times [h_{11}, h_{12}] \times ... \times [h_{s1}, h_{s2}]$$

$$\times X_0 \times \Phi_1 \times \Omega_1$$

we assign the delay controlled functional differential equation

$$\dot{x}(t) = f(t, x(t), x(t - \tau_1), ..., x(t - \tau_s), u(t), u(t - \theta_1), ..., \tag{4.1}$$

$$u(t - \theta_k)), t \in [t_0, t_1], u \in \Omega_1,$$

with the discontinuous initial condition

$$x(t) = \varphi(t), t < t_0, x(t_0) = x_0. \tag{4.2}$$

Definition 4.1. Let $\nu = (t_0, t_1, \tau_1, ..., \tau_s,\ x_0, \varphi, u) \in A_1$. A function $x(t) = x(t; \nu) \in O, t \in [\hat{\tau}, t_1]$ is called a solution of equation (4.1) with the initial condition (4.2) or the solution corresponding to ν and defined on the interval $[\hat{\tau}, t_1]$ if it satisfies condition (4.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (4.1) a.e. on $[t_0, t_1]$.

By the step method and Gronwall inequality it can be proved that for every element $\mu \in \Lambda_1$ there exists the unique solution $x(t; \mu)$ defined on the interval $[\hat{\tau}, b]$ and it is continuous with respect to μ.

Let the scalar-valued functions $q^i(t_0, t_1, \tau_1, ..., \tau_s, x_0, x_1), i = \overline{0, l}$, be continuously differentiable on $I \times I \times [h_{11}, h_{21}] \times ... \times [h_{1s}, h_{2s}] \times O^2$.

Definition 4.2. An element $\nu = (t_0, t_1, \tau_1, \cdots, \tau_s,\ x_0, \varphi, u) \in A_1$ is said to be admissible if the corresponding solution $x(t) = x(t; \nu)$ satisfies the boundary conditions

$$q^i(t_0, t_1, \tau_1, ..., \tau_s, x_0, x(t_1)) = 0, i = \overline{1, l}. \tag{4.3}$$

Denote by A_{10} the set of admissible elements.

Definition 4.3. An element $\nu_0 = (t_{00}, t_{10}, \tau_{10}, \cdots, \tau_{s0},\ x_{00}, \varphi_0, u_0) \in A_{10}$ is said to be optimal if for arbitrary element $\nu \in A_{10}$ the inequality

$$q^0(t_{00}, t_{10}, \tau_{10}, ..., \tau_{s0}, x_{00}, x(t_{10})) \leq q^0(t_0, t_1, \tau_1, ..., \tau_s, x_0, x(t_1)) \tag{4.4}$$

holds. Here $x_0(t)=x(t;\nu_0)$ and $x(t)=x(t;\nu)$.

The problem (4.1) – (4.4) is called an optimal control problem with the discontinuous initial condition.

Theorem 4.1. *Let $\nu_0\in A_{10}$ be an optimal element with $t_{00},t_{10}\in(a,b)$ and $A_{10}\cap A_1^-\neq\emptyset$, where $A_1^-=(a,t_{00}]\times(t_{00},t_{10}]\times[h_{11},h_{21}]\times...\times[h_{1s},h_{2s}]\times X_0\times\Phi_1\times\Omega_1$. Moreover, let the following conditions hold:*

4.1) $\tau_{s0}>\dots>\tau_{10}$ and $t_{00}+\tau_{s0}<t_{10}$ with $\tau_{i0}\in(h_{i1},h_{i2}), i=\overline{1,s}$;

4.2) *the function $\varphi_0(t)$ is absolutely continuous and $\dot\varphi_0(t),t\in I_1$ is bounded;*

4.3) *there exists the finite limit*

$$\lim_{w\to w_0} f(w,u_0(t),u_0(t-\theta_1),...,u_0(t-\theta_k))=f^-, w\in(a,t_{00}]\times O^{1+s},$$

where $w_0=(t_{00},x_{00},\varphi_0(t_{00}-\tau_{10}),...,\varphi_0(t_{00}-\tau_{s0}))$;

4.4) *there exist the finite limits*

$$\lim_{(w_{1i},w_{2i})\to(w_{1i}^0,w_{2i}^0)}[f(w_{1i},u_0(t),u_0(t-\theta_1),...,u_0(t-\theta_k))$$

$$-f(w_{2i},u_0(t),u_0(t-\theta_1)...,u_0(t-\theta_k))]=f_i,$$

where $w_{1i},w_{2i}\in(a,b)\times O^{1+s}, i=\overline{1,s}$,

$$w_{1i}^0=(t_{00}+\tau_{i0},x_0(t_{00}+\tau_{i0}),x_0(t_{00}+\tau_{i0}-\tau_{10}),...,x_0(t_{00}+\tau_{i0}-\tau_{i-10}),$$

$$x_{00},x_0(t_{00}+\tau_{i0}-\tau_{i+10}),...,x_0(t_{00}+\tau_{i0}-\tau_{s0})),$$

$$w_{2i}^0=(t_{00}+\tau_{i0},x_0(t_{00}+\tau_{i0}),x_0(t_{00}+\tau_{i0}-\tau_{10}),...,x_0(t_{00}+\tau_{i0}-\tau_{i-10}),$$

$$\varphi_0(t_{00}),x_0(t_{00}+\tau_{i0}-\tau_{i+10}),...,x_0(t_{00}+\tau_{i0}-\tau_{s0})),$$

4.5) *there exists the finite limit*

$$\lim_{w\to w_{s+1}} f(w,u_0(t),u_0(t-\theta_1),...,u_0(t-\theta_k))=f_{s+1}^-, w\in(t_{00},t_{10})\times O^{1+s},$$

$$w_{s+1}=(t_{10},x_0(t_{10}),x_0(t_{10}-\tau_{10}),...,x_0(t_{10}-\tau_{s0})).$$

Then there exist a vector $\pi=(\pi_0,...,\pi_l)\neq0$, with $\pi_0\le0$ and a solution $\psi(t)=(\psi_1(t),...,\psi_n(t))$ of the equation

$$\dot{\psi(t)}=-\psi(t)f_x[t]-\sum_{i=1}^{s}\psi(t+\tau_{i0})f_{x_i}[t+\tau_{i0}], t\in[t_{00},t_{10}], \tag{4.5}$$

$$\psi(t)=0, t>t_{10},$$

such that the following conditions hold:

4.6) *the conditions for the moments t_{00} and t_{10} :*

$$\pi Q_{0t_0}\ge\psi(t_{00})f^-+\sum_{i=1}^{s}\psi(t_{00}+\tau_{i0})f_i,$$

$$\pi Q_{0t_1}\ge-\psi(t_{10})f_{s+1}^-,$$

where

$$Q=(q^0,\cdots,q^l)^T,\ Q_0=Q(t_{00},t_{10},\tau_{10},...,\tau_{s0},x_{00},x_0(t_{10})),$$

$$Q_{0t_0} = \frac{\partial}{\partial t_0} Q_0;$$

4.7) *the conditions for the delays* $\tau_{i0}, i = \overline{1,s}$:

$$\pi Q_{0\tau_{i0}} = \psi(t_{00}+\tau_{i0})f_i + \int_{t_{00}}^{t_{00}+\tau_{i0}} \psi(t) f_{x_i}[t]\dot{\varphi}_0(t-\tau_{i0})dt$$

$$+ \int_{t_{00}+\tau_{i0}}^{t_{00}} \psi(t) f_{x_i}[t]\dot{x}_0(t-\tau_{i0})dt;$$

4.8) *the conditions for the vector* x_{00}:

$$\Big(\pi Q_{0x_0} + \psi(t_{00})\Big)x_{00} = \max_{x_0 \in X_0} \Big(\pi Q_{0x_0} + \psi(t_{00})\Big)x_0;$$

4.9) *the integral maximum principle for the initial function* $\varphi_0(t)$:

$$\int_{t_{00}-\tau_{i0}}^{t_{00}} \psi(t+\tau_{i0}) f_{x_i}[t+\tau_{i0}]\varphi_0(t)dt = \max_{\varphi(t)\in\Phi_1} \int_{t_{00}}^{t_{10}} \psi(t+\tau_{i0}) f_{x_i}[t+\tau_{i0}]\varphi(t)dt;$$

4.10) *the integral maximum principle for the control function* $u_0(t)$:

$$\int_{t_{00}}^{t_{10}} \psi(t)\Big[f_u[t]u_0(t) + \sum_{i=1}^{k} f_{u_i}[t]u_0(t-\theta_i)\Big]dt$$

$$= \max_{u(t)\in\Omega_1} \int_{t_{00}}^{t_{10}} \psi(t)\Big[f_u[t]u(t) + \sum_{i=1}^{k} f_{u_i}[t]u(t-\theta_i)\Big]dt;$$

4.11) *the condition for the function* $\psi(t)$,

$$\psi(t_{10}) = \pi Q_{0x_1}.$$

Theorem 4.2. *Let* ν_0 *be an optimal element with* $t_{00}, t_{10} \in (a,b)$ *and* $A_{10} \cap A_1^+ \neq \emptyset$, where $A_1^+ = [t_{00}, t_{10}) \times [t_{10}, b) \times [h_{11}, h_{21}] \times ... \times [h_{1s}, h_{2s}] \times X_0 \times \Phi_1 \times \Omega_1$. *Moreover, the conditions 4.1), 4.2), and 4.4) of Theorem 4.1 hold and there exist the finite limits*

$$\lim_{w\to w_0} f(w, u_0(t), u_0(t-\theta_1), ..., u_0(t-\theta_k)) = f^+, w \in [t_{00}, t_{10}) \times O^{1+s},$$

$$\lim_{w\to w_{s+1}} f(w, u_0(t), u_0(t-\theta_1), ..., u_0(t-\theta_k)) = f_{1+s}^+, w \in [t_{10}, b) \times O^{s+1}.$$

Then there exist a vector $\pi = (\pi_0, ..., \pi_l) \neq 0$, *with* $\pi_0 \le 0$ *and a solution* $\psi(t) = (\psi_1(t), \cdots, \psi_n(t))$ *of equation* (4.5), *such that conditions 4.7)-4.11) hold. Moreover,*

$$\pi Q_{0t_0} \le \psi(t_{00})f^+ + \sum_{i=1}^{s} \psi(t_{00}+\tau_{i0})f_i,$$

$$\pi Q_{0t_1} \le -\psi(t_{10})f_1^-.$$

Theorem 4.3. *Let* ν_0 *be an optimal element with* $t_{00}, t_{10} \in (a,b)$ *and the conditions of Theorems 4.1 and 4.2 hold. Moreover,*

$$f^- = f^+ := f, \; f_{s+1}^- = f_{s+1}^+ := f_{s+1}.$$

Then there exist a vector $\pi=(\pi_0,...,\pi_l)\neq 0$, *with* $\pi_0\leq 0$ *and a solution* $\psi(t)=(\psi_1(t),\cdots,\psi_n(t))$ *of equation* (4.5), *such that the conditions 4.7)-4.11) hold. Moreover,*

$$\pi Q_{0t_0}=\psi(t_{00})f+\sum_{i=1}^{s}\psi(t_{00}+\tau_{i0})f_i,$$

$$\pi Q_{0t_1}=-\psi(t_{10})f_{s+1}.$$

Let the function $u_0(t)$ be continuous at the points

$$t_{00},t_{00}-\tau_{i0},i=\overline{1,s};t_{00}+\tau_{i0},i=\overline{1,s};t_{10},t_{10}-\tau_{i0},i=\overline{1,s}.$$

Then we have

$$f=f\Big(t_{00},x_{00},\varphi_0(t_{00}-\tau_{00}),...,\varphi_0(t_{00}-\tau_{s0}),u_0(t_{00}),u_0(t_{00}-\theta_1),$$
$$...,u_0(t_{00}-\theta_k)\Big),$$
$$f_{s+1}=f\Big(t_{10},x_0(t_{10}),x_0(t_{10}-\tau_{10}),...,x_0(t_{10}-\tau_{s0}),u_0(t_{10}),u_0(t_{10}-\theta_1),$$
$$...,u_0(t_{10}-\theta_k)\Big);$$
$$f_i=f\Big(t_{00}+\tau_{i0},x_0(t_{00}+\tau_{i0}),x_0(t_{00}+\tau_{i0}-\tau_{10}),...,x_0(t_{00}+\tau_{i0}-\tau_{i-10}),x_{00},$$
$$x_0(t_{00}+\tau_{i0}-\tau_{i+10}),...,x_0(t_{00}+\tau_{i0}-\tau_{s0}),u_0(t_{00}+\tau_{i0}),u_0(t_{00}+\tau_{i0}-\theta_1),$$
$$...,u_0(t_{00}+\tau_{i0}-\theta_k)\Big)-f\Big(t_{00}+\tau_{i0},x_0(t_{00}+\tau_{i0}),x_0(t_{00}+\tau_{i0}-\tau_{10}),...,$$
$$x_0(t_{00}+\tau_{i0}-\tau_{i-10}),\varphi_0(t_{00}),x_0(t_{00}+\tau_{i0}-\tau_{i+10}),...,x_0(t_{00}+\tau_{i0}-\tau_{s0})$$
$$u_0(t_{00}+\tau_{i0}),u_0(t_{00}+\tau_{i0}-\theta_1),...,u_0(t_{00}+\tau_{i0}-\theta_k)\Big).$$

It is clear that, if $\varphi_0(t_{00})=x_{00}$ then $f_i=0,i=\overline{1,s}$.

Theorem 4.3 is a corollary to Theorems 4.1 and 4.2.

Proof of Theorem 4.1. On the basis of the variation formula of solution (see Theorem 2.1) Theorem 4.1 will be proved by the scheme given in [8, 16].

On the convex set $Z=\mathbb{R}_+\times A_1^-$, where $\mathbb{R}_+=[0,\infty)$, let us define the mapping

$$P:Z\to\mathbb{R}^{1+l} \tag{4.6}$$

by the formula

$$P(z)=(p^0(z),...,p^l(z))^T,z=(\xi,\nu)\in Z,$$

where

$$p^0(z)=q^0(t_0,t_1,\tau_1,...,\tau_s,x_0,x(t_1;\nu))+\xi,\ p^i(z)=q^i(t_0,t_1,\tau_1,$$
$$...,\tau_s,x_0,x(t_1;\nu)),i=\overline{1,l}.$$

Consequently,

$$P(z)=Q(t_0,t_1,\tau_1,...,\tau_s,x_0,x(t_1;\nu))+(\xi,0...,0)^T.$$

It is clear that

$$p^0(z_0)\leq p^0(z),p^i(z)=0,i=\overline{1,l},\forall z\in\mathbb{R}_+\times(A_{10}\cap A_1^-)\subset Z,$$

where $z_0 = (0, \nu_0) = (0, t_{00}, t_{10}, \tau_{10}, ..., \tau_{s0}, x_{00}, \varphi_0, u_0)$.

Thus, the point $z_0 = (0, \nu_0) \in Z$ is critical (see [8, 16]), since $P(z_0) \in \partial P(Z)$. Moreover, mapping (4.6) is continuous.

There exists a small $\varepsilon_1 > 0$ such that for arbitrary $\varepsilon \in (0, \varepsilon_1)$ and

$$\delta z = (\delta\xi, \delta\nu) = (\delta\xi, \delta t_0, \delta t_1, \delta\tau_1, ..., \delta\tau_k, \delta x_0, \delta\varphi, \delta u),$$

where

$$\delta\xi \in [0, \alpha), \delta t_1 \in (-\alpha, 0] \text{ and } \delta\mu = (\delta t_0, \delta\tau_1, ..., \delta\tau_k, \delta x_0, \delta\varphi, \delta u) \in V_1^-$$

we get

$$z_0 + \varepsilon\delta z = (\varepsilon\delta\xi, t_{00} + \varepsilon\delta t_0, t_{10} + \varepsilon\delta t_1, \tau_{10} + \varepsilon\delta\tau_1, ..., \tau_{s0} + \varepsilon\delta\tau_s, x_{00}$$
$$+\varepsilon\delta x_0, \varphi_0 + \varepsilon\delta\varphi, u_0 + \varepsilon\delta u)) \in [0, \alpha) \times (a, t_{00}] \times (t_{10} - \delta_2, t_{10}] \times$$
$$(h_{11}, h_{12}) \times ... \times (h_{s1}, h_{s2}) \times X_0 \times \Phi_1 \times \Omega_1 \subset Z.$$

It easy to see that on the interval $[\hat{\tau}, t_{10}]$

$$x(t; \nu_0 + \varepsilon\delta\nu) = x(t; \mu_0 + \varepsilon\delta\mu) \text{ and } x_0(t) = x(t; \nu_0) = x(t; \mu_0),$$

therefore

$$\Delta x(t; \varepsilon\delta\nu) = x(t; \nu_0 + \varepsilon\delta\nu) - x_0(t) = \Delta x(t; \varepsilon\delta\mu)$$

On the basis of Theorem 2.1 we have

$$\Delta x(t; \varepsilon\delta\nu) = \varepsilon\delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu), \forall (t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10}] \times (0, \varepsilon_2) \times V_1^-,$$

where $\delta x(t; \delta\mu)$ has the form (2.4).

Now we calculate a differential of mapping (4.6) at the point z_0. We have,

$$P(z_0 + \varepsilon\delta z) - P(z_0) = Q(t_{00} + \varepsilon\delta t_0, t_{10} + \varepsilon\delta t_1, \tau_{10} + \varepsilon\delta\tau_1, ..., \tau_{s0} + \varepsilon\delta\tau_s, x_{00} + \varepsilon\delta x_0,$$
$$, ..., 0)^T.$$

where

$$Q_0 = Q(t_{00}, t_{10}, \tau_{10}, ..., \tau_{s0}, x_{00}, x_0(t_{10})).$$

We introduce the notation:

$$\vartheta(\varepsilon\delta t_1; \varepsilon\delta\mu) = x(t_{10} + \varepsilon\delta t_1; \varepsilon\delta\mu) - x_0(t_{10})$$

and

$$Q[\varepsilon; s] = Q(t_{00} + \varepsilon s\delta t_0, t_{10} + \varepsilon s\delta t_1, \tau_{10} + \varepsilon s\delta\tau_1, ..., \tau_{s0} + \varepsilon s\delta\tau_s, x_{00} + \varepsilon s\delta x_0,$$
$$x_0(t_{10}) + s\vartheta(\varepsilon\delta t_1; \varepsilon\delta\mu).$$

For $\vartheta(\varepsilon\delta t_1; \varepsilon\delta\mu)$ we have

$$\vartheta(\varepsilon\delta t_1; \varepsilon\delta\mu) = x(t_{10} + \varepsilon\delta t_1; \varepsilon\delta\mu) - x_0(t_{10} + \varepsilon\delta t_1) + x_0(t_{10} + \varepsilon\delta t_1) - x_0(t_{10})$$
$$= \varepsilon\delta x(t_{10} + \varepsilon\delta t_0; \varepsilon\delta\mu) + o(t_{10} + \varepsilon\delta\mu; \varepsilon\delta\mu) + \int_{t_{10}+\varepsilon\delta t_1}^{t_{10}} \dot{x}_0(t)dt$$
$$= \varepsilon[\delta x(t_{10}; \varepsilon\delta\mu) + f_{s+1}^- \delta t_1] + o(\varepsilon\delta\mu).$$

Let us transform the difference

$$Q(t_{00}+\varepsilon\delta t_0, t_{10}+\varepsilon\delta t_1, \tau_{10}+\varepsilon\delta\tau_1, ..., \tau_{s0}+\varepsilon\delta\tau_s, x_{00}+\varepsilon\delta x_0, x(t_{10}+\varepsilon\delta t_1;\varepsilon\delta\mu))$$

$$-Q_0=\int_0^1 \frac{d}{ds}\, Q[\varepsilon;s]ds=\int_0^1\Big[\varepsilon\Big(Q_{t_0}[\varepsilon;s]\delta t_0+Q_{t_1}[\varepsilon;s]\delta t_1+\sum_{i=1}^{s}Q_{\tau_i}[\varepsilon;s]\delta\tau_i$$

$$+Q_{x_0}[\varepsilon;s]\delta x_0\Big)+Q_{x_1}[\varepsilon;s]\vartheta(\varepsilon\delta t_1;\varepsilon\delta\mu)\Big]ds$$

$$=\varepsilon\Big[Q_{0t_0}\delta t_0+Q_{0t_1}\delta t_1+\sum_{i=1}^{s}Q_{0\tau_i}\delta\tau_i+Q_{0x_0}\delta x_0+Q_{0x_1}[\delta x(t_{10};\delta\mu)+f_{s+1}^{-}\delta t_1]\Big]+\gamma(\varepsilon\delta\mu),$$

where

$$\gamma(\varepsilon\delta\mu)=\varepsilon\int_0^1\Big\{[Q_{t_0}[\varepsilon;s]-Q_{0t_0}]\delta t_0+[Q_{t_1}[\varepsilon;s]-Q_{0t_1}]\delta t_1$$

$$+\sum_{i=1}^{s}[Q_{\tau_i}[\varepsilon;s]-Q_{0\tau_i}]\delta\tau_i+[Q_{x_0}[\varepsilon;s]-Q_{0x_0}]\delta x_0+[Q_{x_1}[\varepsilon;s]-Q_{0x_1}][\delta x(t_{10};\delta\mu)+f_{s+1}^{-}\delta t_1]$$

$$+Q_{0x_1}\frac{o(\varepsilon\delta w)}{\varepsilon}\Big\}ds.$$

It is easy to see that

$$\lim_{\varepsilon\to 0}[Q_{t_0}[\varepsilon;s]-Q_{0t_0}]=0,\ \lim_{\varepsilon\to 0}[Q_{t_1}[\varepsilon;s]-Q_{0t_1}]=0, \lim_{\varepsilon\to 0}[Q_{\tau_i}[\varepsilon;s]-Q_{0\tau_i}]=0,$$

$$\lim_{\varepsilon\to 0}[Q_{x_0}[\varepsilon;s]-Q_{0x_0}]=0,\ \lim_{\varepsilon\to 0}[Q_{x_1}[\varepsilon;s]-Q_{0x_1}]=0.$$

Therefore, $\gamma(\varepsilon\delta\mu)=o(\varepsilon\delta\mu)$. Thus,

$$P(z_0+\varepsilon\delta z)-P(z_0)=\varepsilon dP_{z_0}(\delta z)+o(\varepsilon\delta z),$$

where $o(\varepsilon\delta z):=o(\varepsilon\delta\mu)$ and differential $dP_{z_0}(\delta z)$ of the mapping (4.6) has the form

$$dP_{z_0}(\delta z)=Q_{0t_0}\delta t_0+Q_{0t_1}\delta t_1+\sum_{i=1}^{s}Q_{0\tau_i}\delta\tau_i+Q_{0x_0}\delta x_0+Q_{0x_1}[\delta x(t_{10};\delta\mu)+f_{s+1}^{-}\delta t_1]$$

Due to relation (2.4) we get

$$dP_{z_0}(\delta z)=\Big[Q_{0t_0}-Q_{0x_1}Y(t_{00};t_{10})f^{-}-\sum_{i=1}^{s}Q_{0x_1}Y(t_{00}+\tau_{i0})f_i\Big]\delta t_0$$

$$+\Big[Q_{0t_1}+Q_{0x_1}Y(t_{00};t_{10})f_{s+1}^{-}\Big]\delta t_1+\sum_{i=1}^{s}\Big[Q_{0\tau_i}-Q_{0x_1}Y(t_{00}+\tau_{i0};t_{10})f_i$$

$$+\int_{t_{00}}^{t_{10}}Q_{0x_1}Y(t;t_{10})f_{x_i}[t]\dot{x}(t-\tau_{i0})dt\Big]\delta\tau_i+\Big[Q_{0x_0}+Q_{0x_1}Y(t_{00};t_{10})\Big]\delta x_0$$

$$\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}}Q_{0x_1}Y(t+\tau_{i0};t_{10})\delta\varphi(t)dt+\int_{t_{00}}^{t_{10}}Q_{0x_1}Y(t;t_{10})\Big[f_u[t]\delta u(t)$$

$$+\sum_{i=1}^{k} f_{u_i}[t]\delta u(t-\theta_i)\Big]dt. \tag{4.7}$$

From the necessary condition of criticality [8, 16] it follows that: there exists a vector $\pi = (\pi_0, ..., \pi_l) \neq 0$ such that

$$\pi dP_{z_0}(\delta z) \leq 0, \forall\, \delta z \in \mathbb{R}_+ \times \mathbb{R}^2_- \times \mathbb{R}^s \times [X_0 - x_{00}]$$
$$\times[\Phi_1 - \varphi_0] \times [\Omega_1 - u_0]. \tag{4.8}$$

Introduce the function

$$\psi(t) = \pi Q_{0x_1} Y(t; t_{10}) \tag{4.9}$$

as is easily seen, it satisfies equation (4.5) and the initial condition.
Taking into account (4.7) and (4.9) from inequality (4.8) we obtain

$$\Big[\pi Q_{0t_0} - \psi(t_{00})f^- - \sum_{i=1}^{s}\psi(t_{00}+\tau_{i0})f_i\Big]\delta t_0$$
$$+\Big[\pi Q_{0t_1} + \psi(t_{00})f^-_{s+1}\Big]\delta t_1 + \sum_{i=1}^{s}\Big[\pi Q_{0\tau_i} - \psi(t_{00}+\tau_{i0})f_i$$
$$+\int_{t_{00}}^{t_{10}}\psi(t)f_{x_i}[t]\dot{x}(t-\tau_{i0})dt\Big]\delta\tau_i + \Big[\pi Q_{0x_0} + \psi(t_{00})\Big]\delta x_0$$
$$\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}}\psi(t+\tau_{i0})\delta\varphi(t)dt + \int_{t_{00}}^{t_{10}}\psi(t)\Big[f_u[t]\delta u(t)$$
$$+\sum_{i=1}^{k} f_{u_i}[t]\delta u(t-\theta_i)\Big]dt \leq 0. \tag{4.10}$$

Let $\delta t_0 = \delta t_1 = 0, \delta\tau_i = 0, i = \overline{1,s}, \delta x_0 = 0, \delta\varphi = 0, \delta u = 0$ in (4.10), then we obtain

$$\pi_0\delta\xi \leq 0, \forall\delta\xi \in \mathbb{R}_+.$$

This implies $\pi_0 \leq 0$.

Let $\delta\xi = 0, \delta\tau_i = 0, i = \overline{1,s}, \delta x_0 = 0, \delta\varphi = 0, \delta u = 0$ then taking into account that $\delta t_i \in \mathbb{R}_-, i = 0,1$ from (4.10) we obtain condition *4.6)*.

Let $\delta\xi = \delta t_0 = \delta t_1 = 0, \delta x_0 = 0, \delta\varphi = 0, \delta u = 0$ then, taking into account that $\delta x_0 \in X_0 - x_{00}$ from (4.10) we obtain condition *4.7)*.

Let $\delta\xi = \delta t_0 = \delta t_1 = 0, \delta\tau_i = 0, i = \overline{1,s}, \delta\varphi = 0, \delta u = 0$ then, taking into account that $\delta x_0 \in X_0 - x_{00}$ from (4.10) we obtain condition *4.8)*.

Let $\delta\xi = \delta t_0 = \delta t_1 = 0, \delta\tau_i = 0, i = \overline{1,s}, \delta x_0 = 0, \delta u = 0$ then, taking into account that $\delta\varphi \in \Phi_1 - \varphi_0$ from (14) we obtain condition *4.9)*.

Let $\delta\xi = \delta t_0 = \delta t_1 = 0, \delta\tau_i = 0, i = \overline{1,s}, \delta x_0 = 0, \delta\varphi = 0$ then, taking into account that $\delta u \in \Omega_1 - u_0$ from (14) we obtain condition *4.10)*.

Finally we note that Theorem 4.2 can be proved analogously to Theorem 4.1.

5. Optimal control problem with the continuous initial condition

Let us consider the optimal control problem

$$\dot{x}(t) = f(t, x(t), x(t-\tau_1), ..., x(t-\tau_s), u(t), u(t-\theta_1),$$

$$..., u(t-\theta_k)), t \in [t_0, t_1], u \in \Omega \tag{5.1}$$

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0], \tag{5.2}$$

$$q^i(t_0, t_1, \tau_1, ..., \tau_s, \varphi(t_0), x(t_1)) = 0, i = \overline{1, l}, \tag{5.3}$$

$$q^0(t_0, t_1, \tau_1, ..., \tau_s, \varphi(t_0), x(t_1)) \to \min . \tag{5.4}$$

Problem (5.1)-(5.4) is called an optimal control problem with the continuous initial condition.

Definition 5.1. Let $\rho = (t_0, t_1, \tau_1, ..., \tau_s, \varphi, u) \in A_2 = (a, b) \times (h_{11}, h_{12}) \times (h_{s1}, h_{s2}) \times \Phi_1 \times \Omega_1$. A function $x(t) = x(t; \rho) \in O, t \in [\hat{\tau}, t_1]$ is called a solution of equation (5.1) with the continuous initial condition (5.2) or the solution corresponding to ρ and defined on the interval $[\hat{\tau}, t_1]$ if it satisfies condition (5.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (5.1) a. e. on $[t_0, t_1]$.

Definition 5.2. An element $\rho = (t_0, t_1, \tau_1, ..., \tau_s, \varphi, u) \in A_2$ is said to be admissible if the corresponding solution $x(t) = x(t; \rho)$ satisfies the boundary conditions (5.3).

Denote by A_{20} the set of admissible elements.

Definition 5.3. An element $\rho_0 = (t_{00}, t_{10}, \tau_{10}, ..., \tau_{s0}, \varphi_0, u_0) \in A_{20}$ is said to be optimal if for an arbitrary element $\rho \in A_{20}$ the inequality

$$q^0(t_{00}, t_{10}, \tau_{10}, ..., \tau_{s0}, \varphi_0(t_{00}), x_0(t_{10})) \le q^0(t_0, t_1, \tau_1, ..., \tau_s, \varphi(t_0), x(t_1))$$

holds.

Theorem 5.1. *Let ρ_0 be an optimal element with $t_{00}, t_{10} \in (a, b)$ and $A_{20} \cap A_2^- \neq \emptyset$, where $A_2^- = (a, t_{00}] \times (t_{00}, t_{10}] \times [h_{11}, h_{21}] \times ... \times [h_{1s}, h_{2s}] \times \Phi_1 \times \Omega_1$. Moreover, the following conditions hold:*

5.1) *the function $\varphi_0(t)$ is absolutely continuous and $\dot{\varphi}_0(t), t \in I_1$ is bounded;*

5.2) *there exist the finite limits*

$$\lim_{t \to t_{00}-} \dot{\varphi}_0(t) = \dot{\varphi}_0^-,$$

$$\lim_{w \to w_0} f(w, u_0(t), u_0(t-\theta_1), ..., u_0(t-\theta_k)) = f^-, w \in (a, t_{00}] \times O^{1+s},$$

where $w_0 = (t_{00}, \varphi_0(t_{00} - \tau_{10}), ..., \varphi_0(t_{00} - \tau_{s0}))$;

5.3)*there exists the finite limit*

$$\lim_{w \to w_{s+1}} f(w, u_0(t), u_0(t-\theta_1), ..., u_0(t-\theta_k)) = f_{s+1}^-, w \in (t_{00}, t_{10}] \times O^{1+s},$$

$$w_{s+1} = (t_{10}, x_0(t_{10}), x_0(t_{10} - \tau_{10}), ..., x_0(t_{10} - \tau_{s0}));$$

Then there exists a vector $\pi=(\pi_0,...,\pi_l)\neq 0$, *with* $\pi_0\le 0$ *and a solution* $\psi(t)=(\psi_1(t),...,\psi_n(t))$ *of the equation*

$$\dot{\psi(t)}=-\psi(t)f_x[t]-\sum_{i=1}^{s}\psi(t+\tau_{i0})f_{x_i}[t+\tau_{i0}],t\in[t_{00},t_{10}], \tag{5.5}$$

$$\psi(t)=0,t>t_{10},$$

such that the following conditions hold:
5.4) *the conditions for the moments* t_{00} *and* t_{10} *:*

$$\pi Q_{0t_0}+\Big(\pi Q_{0x_0}+\psi(t_{00})\Big)\dot\varphi_0^-\ge\psi(t_{00})f^-,$$

$$\pi Q_{0t_1}\ge-\psi(t_{10})f_{s+1}^-,$$

5.5) *the conditions for the delays* $\tau_{i0},i=\overline{1,s}$:

$$\pi Q_{0\tau_{i0}}=\int_{t_{00}}^{t_{10}}\psi(t)f_{x_i}[t]\dot x_0(t-\tau_{i0})dt;$$

5.6) *the integral maximum principle for the initial function* $\varphi_0(t)$:

$$\Big[\pi Q_{0x_0}+\psi(t_{00})\Big]\varphi_0(t_{00})+\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}}\psi(t+\tau_{i0})f_{x_i}[t+\tau_{i0}]\varphi_0(t)dt$$

$$=\max_{\varphi(t)\in\Phi_1}\Big[\Big[\pi Q_{0x_0}+\psi(t_{00})\Big]\varphi(t_{00})+\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}}\psi(t+\tau_{i0})f_{x_i}[t+\tau_{i0}]\varphi(t)dt\Big];$$

5.7) *the integral maximum principle for the control function* $u_0(t)$:

$$\int_{t_{00}}^{t_{10}}\psi(t)\Big[f_u[t]u_0(t)+\sum_{i=1}^{k}f_{u_i}[t]u_0(t-\theta_i)\Big]dt$$

$$=\max_{u(t)\in\Omega_1}\int_{t_{00}}^{t_{10}}\psi(t)\Big[f_u[t]u(t)+\sum_{i=1}^{k}f_{u_i}[t]u(t-\theta_i)\Big]dt;$$

5.8) *the condition for the function* $\psi(t)$:

$$\psi(t_{10})=\pi Q_{0x_1}.$$

Theorem 5.2. Let ρ_0 be an optimal element with $t_{00},t_{10}\in(a,b)$ and $A_{20}\cap A_2^+\neq\emptyset$, where $A_2^+=[t_{00},t_{10})\times[t_{10},b)\times[h_{11},h_{21}]\times...\times[h_{1s},h_{2s}]\times\Phi_1\times\Omega_1$. Moreover, the condition 5.1) of Theorem 5.1 holds and there exist the finite limits

$$\lim_{t\to t_{00}+}\dot\varphi_0(t)=\dot\varphi_0^+,$$

$$\lim_{w\to w_0}f(w,u_0(t),u_0(t-\theta_1),...,u_0(t-\theta_k))=f^+,w\in[t_{00},b)\times O^{1+s},$$

$$\lim_{w\to w_{s+1}}f(w,u_0(t),u_0(t-\theta_1),...,u_0(t-\theta_k))=f_{1+s}^+,w\in[t_{10},b)\times O^{1+s}.$$

Then there exist a vector $\pi = (\pi_0, ..., \pi_l) \neq 0$, *with* $\pi_0 \leq 0$ *and a solution* $\psi(t) = (\psi_1(t), \cdots, \psi_n(t))$ *of equation* (5.5), *such that conditions 5.5)-5.8) hold. Moreover,*

$$\pi Q_{0t_0} + \big(\pi Q_{0x_0} + \psi(t_{00})\big)\dot{\varphi}_0^+ \leq \psi(t_{00})f^+,$$

$$\pi Q_{0t_1} \leq -\psi(t_{10})f_{s+1}^+.$$

Theorem 5.3. *Let* ν_0 *be an optimal element with* $t_{00}, t_{10} \in (a, b)$ *and the conditions of Theorems 5.1 and 5.2 hold. Moreover,*

$$\dot{\varphi}_0^- = \dot{\varphi}_0^+ := \dot{\varphi}_0, f^- = f^+ := f,\ f_{s+1}^- = f_{s+1}^+ := f_{s+1}.$$

Then there exists a vector $\pi = (\pi_0, ..., \pi_l) \neq 0$, *with* $\pi_0 \leq 0$ *and a solution* $\psi(t) = (\psi_1(t), \cdots, \psi_n(t))$ *of equation* (3.5), *such that conditions 5.5)-5.8) hold. Moreover,*

$$\pi Q_{0t_0} + \big(\pi Q_{0x_0} + \psi(t_{00})\big)\dot{\varphi}_0 = \psi(t_{00})f,\ \pi Q_{0t_1} = -\psi(t_{10})f_{s+1}.$$

Theorem 5.3 is a corollary to Theorems 5.1 and 5.2. Finally we note that, Theorem 5.1 can be proved analogously to Theorem 4.1 on the basis of variation formula (see Theorem 3.1).

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Authors' addresses:

T. Tadumadze
Department of Mathematics &
I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
2 University St.,Tbilisi 0186
Georgia
E-mail: tamaz.tadumadze@tsu.ge

T. Shavadze
I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
2 University St.,Tbilisi 0186
Georgia
E-mail: tea.shavadze@gmail.com