Seminar of I. Vekua Institute of Applied Mathematics REPORTS, Vol. 45, 2019

ON THE SOLUTION OF THE FIRST PLANE INTERIOR BOUNDARY VALUE PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURE BY VARIATION METHOD

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Abstract. It is shown that a solution of the boundary value problem of statics of the linear theory of elastic mixture, in the case of a finite simply connected plane domain, is the minimizing vector-function of the functional whose integrand represents the double potential energy of elastic mixture.

Keywords and phrases: Elastic mixture, variation method, first interior boundary value problem, functional, minimizing vector-function.

AMS subject classification (2010): 74E35.

1. Introduction

The basic two-dimensional boundary value problems of statics of the linear theory of elastic mixtures are studied in [1], [2], [5] [6] and also by many other authors. Two-dimensional boundary value problems of statics are investigated by potential method and the theory of singular integral equations in [1]. Using potentials with complex densities the solutions of basic plane boundary value problems of statics are reduced to the solution of Fredholm's linear integral equation of second kind in [2]. In the paper, the first boundary value problem of statics is investigated by variation method in the case of the plane theory of elastic mixture for a simply connected finite domain, when on the boundary a displacement vector is given. To solve the problem we use the method described in [3] and [4].

2. Some auxiliary formulas and operators

The homogeneous equation of statics of the linear theory of elastic mixture for the two-dimensional case can be written in the matrix form as [1]

$$\mathbf{A}(\partial \mathbf{x})\mathbf{U}(\mathbf{x}) = 0 \tag{2.1}$$

where

$$A(\partial x) = \begin{bmatrix} A^{(1)}(\partial x) & A^{(2)}(\partial x) \\ A^{(2)}(\partial x) & A^{(3)}(\partial x) \end{bmatrix}, \qquad A^{(p)}(\partial x) = [A^{(p)}_{kj}(\partial x)]_{2\times 2}, \quad p = 1, 2, 3,$$

$$A^{(2q-1)}(\partial x) = a_q \delta_{kj} \Delta + b_q \frac{\partial^2}{\partial x_k \partial x_j} \quad q = 1, 2; k, j = 1, 2,$$

$$A^{(2)}(\partial x) = c\delta_{kj}\Delta + d\frac{\partial^2}{\partial x_k \partial x_j} \quad k, j = 1, 2;$$

δ_{kj} is Kroneker's symbol, and Δ is the Laplace operator, $\mathbf{u} = (u', u'')^T$, $\mathbf{u}' = (u_1, u_2)^T$ and $\mathbf{u}'' = (u_3, u_4)^T$ are partial displacements, $\mathbf{x} = (x_1, x_2)$,

$$a_1 = \mu_1 - \lambda_5, \quad a_2 = \mu_2 - \lambda_5, \quad c = \mu_3 + \lambda_5, \quad b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2\rho_2/\rho,$$

$$b_2=\mu_2+\lambda_2+\lambda_5-\alpha_2\rho_1/\rho,\quad \alpha_2=\lambda_3-\lambda_4,\quad \rho=\rho_1+\rho_2,$$
$$d=\mu_3+\lambda_3-\lambda_5-\alpha_2\rho_1/\rho\equiv\mu_3+\lambda_4-\lambda_5+\alpha_2\rho_2/\rho.$$

Here $\mu_1,\mu_2,\mu_3;\lambda_p,\ p=\overline{1,5}$ are elastic constants, ρ_1 and ρ_2 are partial densities (positive constants).
In the sequel it is assumed that

$$\begin{cases}\mu_1>0,\lambda_5<0,\quad \mu_1\mu_2-\mu_3^2>0,\quad b_1-\lambda_5>0,\\ (b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_5)^2>0.\end{cases}\tag{2.2}$$

Let D^+ be a finite two-dimensional region bounded by the contour $S\in C^{2,\alpha},\ 0<\alpha<1$. A vector-function $\mathbf{U}=(u',u'')^T=(u_1,u_2,u_3,u_4)^T$ is said to be regular in D^+ if $\mathbf{U}\in C^2(D^+)\bigcap C^{1,\alpha}(D^+\bigcup S)$.

Note that for a regular $U=(u_1,u_2,u_3,u_4)^T$ and $V=(v_1,v_2,v_3,v_4)^T$ vector-functions we have the Green formula [1],

$$\int_{D^+}[\mathbf{VA}(\partial\mathbf{x})\mathbf{U}+\mathbf{E}(\mathbf{u},\mathbf{v})]\mathbf{dx}=\int_{\mathbf{S}}[\mathbf{V}(\mathbf{y})]^+[\mathbf{Tu}(\mathbf{y})]^+\mathbf{dyS},\tag{2.3}$$

where $y=y_1+iy_2,y\in S,\quad \mathbf{TU}\ =[(Tu)_1,(Tu)_2,(Tu)_3,(Tu)_4]^T$ is the stress vector [1]

$$E(u,v)=E(v,u)=(b_1-\lambda_5)\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}\right)$$
$$+(d+\lambda_5)\left[\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial v_3}{\partial x_1}+\frac{\partial v_4}{\partial x_2}\right)+\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_4}{\partial x_2}\right)\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}\right)\right]$$
$$+(b_2-\lambda_5)\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_4}{\partial x_2}\right)\left(\frac{\partial v_3}{\partial x_1}+\frac{\partial v_4}{\partial x_2}\right)$$
$$+\mu_1\left[\left(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial v_1}{\partial x_1}-\frac{\partial v_2}{\partial x_2}\right)+\left(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\right)\left(\frac{\partial v_2}{\partial x_1}+\frac{\partial v_1}{\partial x_2}\right)\right]$$
$$+\mu_3\left[\left(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial v_3}{\partial x_1}-\frac{\partial v_4}{\partial x_2}\right)+\left(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_4}{\partial x_2}\right)\left(\frac{\partial v_1}{\partial x_1}-\frac{\partial v_2}{\partial x_2}\right)\right]$$
$$+\left[\left(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\right)\left(\frac{\partial v_4}{\partial x_1}+\frac{\partial v_3}{\partial x_2}\right)\left(\frac{\partial u_4}{\partial x_1}+\frac{\partial u_3}{\partial x_2}\right)\left(\frac{\partial v_2}{\partial x_1}+\frac{\partial v_1}{\partial x_2}\right)\right]$$
$$+\mu_2\left[\left(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_4}{\partial x_2}\right)\left(\frac{\partial v_3}{\partial x_1}-\frac{\partial v_4}{\partial x_2}\right)+\left(\frac{\partial u_4}{\partial x_1}+\frac{\partial u_3}{\partial x_2}\right)\left(\frac{\partial v_4}{\partial x_1}+\frac{\partial v_3}{\partial x_2}\right)\right]$$
$$-\lambda_5\ [\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)\left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right)-\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)\left(\frac{\partial v_4}{\partial x_1}-\frac{\partial v_3}{\partial x_2}\right)$$
$$-\left(\frac{\partial u_4}{\partial x_1}-\frac{\partial u_3}{\partial x_2}\right)\left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right)+\left(\frac{\partial u_4}{\partial x_1}-\frac{\partial u_3}{\partial x_2}\right)\left(\frac{\partial v_4}{\partial x_1}-\frac{\partial v_3}{\partial x_2}\right)].\tag{2.4}$$

From (2.4) when **V=U** we obtain

$$\int_{D^+}[\mathbf{U}\ A(\partial x)\mathbf{U}+E(\mathbf{u,u})]dx=\int_S[\mathbf{U}(y)]^+[\mathbf{TU}(y)]^+dyS$$

where $E(\mathbf{u},\mathbf{u})$ is the double potential energy of the form

$$E(\mathbf{u},\mathbf{u}) = (b_1-\lambda_5)\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)^2 + 2(d+\lambda_5)\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_4}{\partial x_2}\right)$$

$$+(b_2-\lambda_5)\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_4}{\partial x_2}\right)^2 + \mu_1\left[\left(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\right)^2+\left(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\right)^2\right]$$

$$+2\mu_3\left[\left(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_4}{\partial x_2}\right)+\left(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\right)\left(\frac{\partial u_4}{\partial x_1}+\frac{\partial u_3}{\partial x_2}\right)\right]$$

$$+\mu_2\left[\left(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_4}{\partial x_2}\right)^2+\left(\frac{\partial u_4}{\partial x_1}+\frac{\partial u_3}{\partial x_2}\right)^2\right]$$

$$-\lambda_5\left[\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)-\left(\frac{\partial u_4}{\partial x_1}-\frac{\partial u_3}{\partial x_2}\right)\right]^2. \tag{2.5}$$

Owing to (2.2) it follows that $E(\mathbf{u},\mathbf{u})$ is the positively defined quadratic form, also note that the equation $E(\mathbf{u},\mathbf{u})=0$ admits a solution

$$\mathbf{U}=(\mathbf{u}',\mathbf{u}'')^T \quad \mathbf{u}'=\begin{pmatrix}\alpha_1-\varepsilon x_2\\ \alpha_2+\varepsilon x_1\end{pmatrix}, \qquad \mathbf{u}''=\begin{pmatrix}\alpha_3-\varepsilon x_2\\ \alpha_4+\varepsilon x_1\end{pmatrix}$$

where $\alpha_q,\ \ q=\overline{1,4}$ and ε are arbitrary real constants.

Let us consider the functional

$$\Pi(\mathbf{u})=\int_{D^+} E(\mathbf{u},\mathbf{u})dx, \tag{2.6}$$

where $E(\mathbf{u},\mathbf{u})$ is defined by (2.5).

On the basis of the above results we have that (2.6) functional is a positively defined quadratic form.

3. Solution of the first interior boundary value problem

The first interior boundary value problem of statics is formulated as follows [1]; find a regular solution of equation (1.1) in D^+ satisfies the boundary condition

$$\mathbf{U}^+(y)=\mathbf{f}(y), \quad y\in S, \tag{3.1}$$

where $\mathbf{f}\in C^{1,\beta}(S),\quad 0<\beta<\alpha<1$ is a given vector-function.

Let us denote by $(I)_f^+$ the (1.1) (3.1) problem.

The following assertion is true [1].

Theorem 3.1. *The $(I)_f^+$ problem is uniquely solvable.*

For the solution of the problem by the variation method we have used the way, developed in [3] and [4].

Let us now prove the following

Theorem 3.2. T*he vector-function $U(x)$ minimizes the functional (2.6) is a solution of problem $(I)_f^+$ if and only if the condition (3.1) is fulfilled*

Proof. At first let us prove sufficiency of equality (3.1), Let $\mathbf{f}(y)$ be such that minimization of the vector-function $\mathbf{U}(x)$ of the functional (2.6) satisfies condition (3.1). Let us show that the vector-function $\mathbf{U}(x)$ is the solution of problem $(I)_f^+$.

To this end let us consider the vector-function $\mathbf{u}(x) + \varepsilon\mathbf{h}(x)$ where ε is an arbitrary real scalar constant, and $\mathbf{h} = (h_1, h_2, h_3, h_4)^T \neq 0$ is an arbitrary vector=function in D^+ and satisfies the condition

$$\mathbf{h}^+(y) = 0, \quad y \in S. \tag{3.2}$$

Elementary calculations yield (see (2.4), (2.5) and (2.6))

$$\Pi(\mathbf{u} + \varepsilon\mathbf{h}) \equiv \Pi(\mathbf{u}) + 2\varepsilon\Pi(\mathbf{u},\mathbf{h}) + \varepsilon^2\Pi(\mathbf{h}) > 0, \tag{3.3}$$

where

$$\Pi(\mathbf{u},\mathbf{h}) = \int_{D^+} E(\mathbf{u},\mathbf{h})dx \tag{3.4}$$

From (2.3) if $\mathbf{V} = \mathbf{h}$ by virtue of (3.2) and (3.4) we obtain

$$\int_{D^+} \mathbf{h}(x)\mathbf{A}\ (\partial x)U(x)dx = -\Pi(\mathbf{u},\mathbf{h}). \tag{3.5}$$

Let us note that since in (3.3) ε is an arbitrary real scalar constant and the $\Pi(u)$ functional at $\mathbf{U}(x)$ attains minimum therefore we have

$$\Pi(\mathbf{u},\mathbf{h}) = 0. \tag{3.6}$$

By virtue of the fact $\mathbf{h}(x) \neq 0$ is an arbitrary regular vector-function in D^+ therefore owing to (3.6) from (3.5) it follows that $\mathbf{U}(x)$ is a solution of equation (1.1) in the domain D^+.

Finally, from the above arguments and owing to Theorem 3.1 we conclude that if (3.1) condition is fulfilled then the minimization vector-function $\mathbf{U}(x)$ of the functional (2.6) is a uniquely solution of the problem $(I)_\mathbf{f}^+$.

Now let us show necessity of condition (3.1). Let $\mathbf{f}(y)$ be such that minimization vector-function $\mathbf{U}(x)$ of the functional (2.6) is the solution of the problem $(I)_f^+$. We shall show that condition (3.1) is fulfilled. Since the minimization vector-function $\mathbf{U}(x)$ of the functional (2.6) is the solution of the problem $(I)_f^+$, owing to uniqueness Theorem 3.1 we can conclude that (3.1) is fulfilled.

Finally, from Theorem 3.1 and Theorem 3.2 we conclude that minimization vector-function $\mathbf{U}(x)$ of the functional (2.6) is the uniquely solution of the problem $(I)_f^+$.

R E F E R E N C E S

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Received 12.05 2019; revised 11.06.2019; accepted 09.07.2019

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