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OSCILLATORY PROPERTIES OF SOLUTIONS OF HIGHER ORDER ESSENTIAL NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. The differential equation

$$u^{(n)}(t) + F(u)(t) = 0$$

is considered, where $F : \mathbb{C}(\mathbb{R}_+;\mathbb{R}) \to \mathbb{L}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$ is a continuous mapping. In the case operator F has essential nonlinear minorant, sufficient conditions are established for equation to have properties **A** and **B**.

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1. Introduction

Let $\tau \in \mathbb{C}(\mathbb{R}_+;\mathbb{R})$ and $\lim\limits_{t\to+\infty} \tau(t) = +\infty$. Denote by $V(\tau)$ the set of continuous mappings $F : \mathbb{C}(\mathbb{R}_+;\mathbb{R}) \to \mathbb{L}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$ satisfying the conditions $F(x)(t) = F(y)(t)$ holds for any $t \in \mathbb{R}_+$ and $x; y \in \mathbb{C}(\mathbb{R}_+;\mathbb{R})$ provided that $x(s) = y(s)$ for $s \geq \tau(t)$.

This work is dedicated to the study of oscillatory properties of solutions of a functional differential equation of the form

$$u^{(n)}(t) + F(u)(t) = 0, \tag{1.1}$$

where $n \geq 2$ and $F \in V(\tau)$.

For any $t \in \mathbb{R}_+$, we denote by $H_{t_0,\tau}$ the set of all functions $u \in \mathbb{C}(\mathbb{R}_+;\mathbb{R})$ satisfying $u(t) \neq 0$ for $t \geq t_*$, where $t_* = \min\{t_0, \tau_*(t_0)\}$, $\tau_*(t) = \inf\{\tau(s) : s \geq t\}$.

Throughout the work whenever the notation $V(\tau)$ and $H_{t_0,\tau}$ occurs, it will be understood unless otherwise specified that the function τ satisfies the conditions stated above.

It will always be assumed that either the condition

$$F(u)(t)\,u(t) \geq 0 \quad \text{for} \quad t \geq t_0, \quad u \in H_{t_0,\tau} \tag{1.2}$$

or the condition

$$F(u)(t)\,u(t) \leq 0 \quad \text{for} \quad t \geq t_0, \quad u \in H_{t_0,\tau} \tag{1.3}$$

is fulfilled.

Let $t_0 \in \mathbb{R}_+$. A function $u : [t_0, +\infty) \to \mathbb{R}$ is said to be a proper solution of equation (1.1) if it is locally absolutely continuous together with its derivatives up to order $n-1$ inclusive $\sup\{|u(s)| : s \in [t, +\infty)\} > 0$ for $t \geq t_0$ and there exist a function $\overline{u} \in \mathbb{C}(\mathbb{R}_+;\mathbb{R})$ such that $\overline{u}(t) = u(t)$ on $[t_0, +\infty)$ and the equality $\overline{u}^{(n)}(t) + F(\overline{u})(t) = 0$ holds for $t \in [t_0, +\infty)$. A proper solution $u : [t_0, +\infty) \to \mathbb{R}$ of equation (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution u is said to be nonoscillatory.

Definition 1.1 [1]. We say that equation (1.1) has Property **A** if any of its proper solutions is oscillatory when n is even and either is oscillatory or satisfies

$$\left|u^{(i)}(t)\right| \downarrow 0 \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.4}$$

when n is odd.

Definition 1.2. We say that equation (1.1) has Property **B** if any of its proper solutions either is oscillatory or satisfies (1.4) or

$$\big|u^{(i)}(t)\big| \uparrow +\infty \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.5}$$

when n is even and either is oscillatory or satisfies (1.5) when n is odd.

Study of oscillatory properties of differential equation of type (1.1) began in 1990. Namely in [3] for the first time a new approach was used for establishing oscillatory properties. Investigation of "almost linear" and essential nonlinear differential equation, in our opinion for the first time, was carried out in [5–10].

In the present paper we study both cases of Properties **A** and **B**, when the operator F has an essential nonlinear minorant.

Some results analogous to those of the paper, for Emden-Fowler type differential equations are given in [3].

2. Necessary conditions of the existence of monotone solutions

Let $t_0 \in \mathbb{R}_+$, $\ell \in \{1, \dots, n-1\}$. By U_{ℓ,t_0} we denote the set of proper solutions of equation (1.1) satisfying the condition

$$\begin{gathered} u^{(i)}(t) \geq 0 \quad \text{for} \quad t \geq t_0 \quad (i = 0, \dots, \ell), \\ (-1)^{i+\ell} u^{(i)}(t) > 0 \quad \text{for} \quad t \geq t_0 \quad (i = \ell, \dots, n-1). \end{gathered} \tag{2.1}$$

Everywhere below we assume that the inequality

$$\big|F(u)(t)\big| \geq p(t)\, \omega\big(|u(\sigma(t))|\big) \quad t \geq t_0, \quad u \in H_{t_0,\tau} \tag{2.2}$$

holds, where

$$p \in L_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}_+), \quad \omega(0) = 0, \quad \omega(s) > 0 \quad \text{for} \quad s > 0, \tag{2.3}$$

ω is a nondecreasing function and there exist $c > 0$, such that

$$\omega(x \cdot y) \geq c\, \omega(x)\, \omega(y) \quad \text{for} \quad x; y \in [1, +\infty) \tag{2.4}$$

and

$$\int_0^1 \frac{ds}{\omega(s)} < +\infty. \tag{2.5}$$

Theorem 2.1. *Let $F \in V(\tau)$, let conditions (1.2), (1.3), (2.2)–(2.5) be fulfilled, $\ell \in \{1, \dots, n-1\}$, $\ell + n$ odd ($\ell + n$ even) and*

$$\int_0^{+\infty} t^{n-\ell} \omega\big(\sigma^{\ell-1}(t)\big)\, p(t)\, dt = +\infty. \tag{2.6}$$

Moreover, let $U_{\ell,t_0} \neq \varnothing$ for some $t_0 \in \mathbb{R}_+$, then

$$\int_0^{+\infty} t^{n-\ell-1} p(t)\, \omega(t)\, \omega\big(\sigma^{\ell-1}(t)\big) dt < +\infty$$

and for any $k \in \mathbb{N}$

$$\int_0^{+\infty} t^{n-\ell-1} p(t)\, \omega\Big(\frac{1}{\ell!} \sigma^{\ell-1}(t)\, \rho_{\ell,k}(\sigma(t))\Big) dt < +\infty,$$

where

$$\rho_{\ell,1}(t) = \Phi^{-1}\Big(\frac{1}{\ell!(n-\ell)!}\int_0^t\int_s^{+\infty}\xi^{n-\ell-1}p(\xi)\,\omega\big(\sigma^{\ell-1}(\xi)\big)d\xi\,ds\Big), \tag{2.7}$$

$$\rho_{\ell,k}(t) = \frac{1}{(n-\ell)!}\int_0^t\int_s^{+\infty}\xi^{n-\ell-1}p(\xi)\,\omega(\sigma(\xi))\,\omega\big(\rho_{\ell,k-1}(\sigma(\xi))\big)d\xi\,ds, \tag{2.8}$$

$$(k=2,3,\dots),$$

$$\Phi(t) = \int_1^t\frac{ds}{\omega(s)}\,. \tag{2.9}$$

3. Sufficient conditions of nonexistence of monotone solutions

Theorem 3.1. *Let* $F\in V(\tau)$, *conditions (1.2), (1.3), (2.2)–(2.6) be fulfilled,* $\ell\in\{1,\dots,n-1\}$, $\ell+n$ *odd* ($\ell+n$ even). *Moreover, if one of two conditions*

$$\int_0^{+\infty}t^{n-\ell-1}p(t)\,\omega(t)\,\omega\big(\sigma^{\ell-1}(t)\big)dt=+\infty, \tag{3.1}$$

or for some $k\in\mathbb{N}$

$$\int_0^{+\infty}t^{n-\ell-1}p(t)\,\omega\big(\sigma^{\ell-1}(t)\,\rho_{\ell,k}(\sigma(t))\big)dt=+\infty \tag{3.2}$$

holds, then for any $t_0\in\mathbb{R}_+$, $U_{\ell,k_0}=\varnothing$, *where* $\rho_{\ell,k}$ *functions are defined by (2.7)–(2.9).*

Corollary 3.1. Let $F\in V(\tau)$, let conditions (1.2), ((1.3)), (2.2)–(2.6) be fulfilled, $\ell\in\{1,\dots,n-1\}$, $\ell+n$ odd ($\ell+n$ even) and for some $\alpha\in(0,1)$

$$\liminf_{t\to+\infty}t^{\alpha}\int_0^{+\infty}s^{n-\ell-1}\omega(\sigma^{\ell-1}(s)\,p(s)\,ds>0. \tag{3.3}$$

Moreover, if for any $c>0$

$$\int_0^{+\infty}t^{n-\ell-1}p(t)\,\omega\big(\sigma^{\ell-1}(t)\,\Phi^{-1}(c\,\sigma^{1-\alpha}(t)\big)dt=+\infty, \tag{3.4}$$

then for any $t_0\in\mathbb{R}_+$ we have $U_{\ell,t_0}=\varnothing$.

Corollary 3.2. Let $F\in V(\tau)$, let conditions (1.2), ((1.3)), (2.2) be fulfilled, $\ell+n$ odd ($\ell+n$ even) and

$$\omega(x)=|x|^{\lambda},\quad 0<\lambda<1. \tag{3.5}$$

There exist $\alpha\in(0,1)$ and $\delta>1$, such that

$$\liminf_{t\to+\infty}t^{\alpha}\int_t^{+\infty}s^{n-\ell-1}p(s)\,\sigma^{\lambda(\ell-1)}(s)ds>0 \tag{3.6}$$

and

$$\liminf_{t\to+\infty}\frac{\sigma(t)}{t^{\delta}}>0. \tag{3.7}$$

Moreover, if at last one of the conditions

$$\delta\,\lambda \geq 1 \tag{3.8}$$

or if $\delta\lambda < 1$, for some $\varepsilon > 0$

$$\int_0^{+\infty} t^{n-\ell-1+\frac{\delta\lambda(1-\alpha)}{1-\alpha\lambda}-\varepsilon}\big(\sigma(t)\big)^{\lambda(\ell-1)}p(t)\,dt = +\infty \tag{3.9}$$

holds, then for any $t_0 \in \mathbb{R}_+$ we have $U_{\ell,t_0} = \varnothing$.

4. Differential equations with property A

Theorem 4.1. *Let $F \in V(\tau)$, let conditions (1.2), (2.2)–(2.5) be fulfilled and for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd, condition (3.1) holds. Let, moreover,*

$$\int_0^{+\infty} t^{n-1}p(t)\,dt = +\infty \tag{4.1}$$

when n is odd, then equation (1.1) has Property A.

Theorem 4.2. *Let $F \in V(\tau)$, let conditions (1.2), (2.2)–(2.5) be fulfilled and for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd, for some $k \in \mathbb{N}$ the condition (3.2) and when n is odd the condition (4.1) holds. Then equation (1.1) has Property A.*

Corollary 4.1. Let $F \in V(\tau)$, conditions (1.2), (2.2)–(2.5) be fulfilled and for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd (3.3), (3.4) and for odd n condition (4.1) holds. Then equation (1.1) has Property **A**.

Corollary 4.2. Let $F \in V(\tau)$, let conditions (1.2), (2.2), (3.5) be fulfilled and for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd (3.6), (3.8) or if $\alpha\lambda < 1$ (3.6), (3.7) and (3.9) holds, then equation (1.1) has Property **A**.

Theorem 4.3. *Let $F \in V(\tau)$, conditions (1.2), (2.2)–(2.6) and for odd n (4.1) be fulfilled and*

$$\liminf_{t\to+\infty}\frac{\omega(\sigma(t))}{t} > 0. \tag{4.2}$$

Moreover, if

$$\int_0^{+\infty} t^{n-2}\omega(t)\,p(t)\,dt = +\infty, \tag{4.3}$$

then equation (1.1) has Property A.

Theorem 4.4. *Let $F \in V(\tau)$, let conditions (1.2), (2.2)–(2.6), (4.2) and for odd n (4.1) be fulfilled and for some $k \in \mathbb{N}$*

$$\int_0^{+\infty} t^{n-2}p(t)\,\omega\big(\rho_{1,k}(\sigma(t)\big)\,dt = +\infty,$$

then equation (1.1) has Property A.

Theorem 4.5. *Let $F \in V(\tau)$, conditions (1.2), (2.2)–(2.6) and for odd n (4.1) be fulfilled. Moreover, there exist $M > 1$, such that*

$$\omega(x\cdot y) \leq M\,\omega(x)\,\omega(y) \quad \text{for} \quad x;y \in [1,+\infty) \tag{4.4}$$

and

$$\limsup_{t\to+\infty}\frac{\omega(\sigma(t))}{t} < +\infty. \tag{4.5}$$

Then the condition

$$\int_0^{+\infty} \omega(t)\,\omega\big(\sigma^{n-2}(t)\big)p(t)\,dt = +\infty$$

is sufficient for equation (1.1) to have Property A.

5. Differential equation with property B

Theorem 5.1. *Let $F \in V(\tau)$, let conditions (1.3), (2.2)–(2.5) be fulfilled and for any $\ell \in \{1,\dots,n-2\}$ with $\ell+n$ even, equation (3.1) holds. Let moreover (4.1) hold when n is even and*

$$\int_0^{+\infty} \omega\big(\sigma^{n-1}(t)\big)p(t)\,dt = +\infty. \tag{5.1}$$

Then equation (1.1) has Property B.

Theorem 5.2. *Let $F \in V(\tau)$, conditions (1.3), (2.2)–(2.5), (5.1) be fulfilled and for any $\ell \in \{1,\dots,n-2\}$ with $\ell+n$ even for some $k \in \mathbb{N}$ condition (3.2) holds. Let moreover, when n is even the condition (4.1) be fulfilled, then equation (1.1) has Property B.*

Corollary 5.1. Let $F \in V(\tau)$, let conditions (1.3), (2.2)–(2.5), (5.1) be fulfilled and for any $\ell \in \{1,\dots,n-2\}$ with $\ell+n$ even (3.3) and (3.4) holds. If, moreover, condition (4.1) hold for even n, then equation (1.1) has Property **B**.

Corollary 5.2. Let $F \in V(\tau)$, let conditions (1.3), (2.2), (3.5) be fulfilled and for any $\ell \in \{1,\dots,n-2\}$ with $\ell+n$ even conditions (3.6)–(3.8) or if $\alpha\lambda < 1$, (3.6), (3.7) and (3.9) holds. Then equation (1.1) has Property **B**.

Theorem 5.3. *Let $F \in V(\tau)$, let conditions (1.3), (2.2)–(2.6), (4.3) and (5.1) be fulfilled and the condition (4.1) holds for even n. Then equation (1.1) has Property B.*

Theorem 5.4. *Let $F \in V(\tau)$, let conditions (1.3), (2.2)–(2.6), (4.2) and (5.1) be fulfilled. Moreover, if the condition (4.1) holds for even n and for some $k \in \mathbb{N}$*

$$\int_0^{+\infty} t^{n-1}p(t)\,\omega\big(\rho_{1,k}(\sigma(t))\big)\,dt = +\infty,$$

then equation (1.1) has Property B.

Theorem 5.5. *Let $F \in V(\tau)$, let conditions (1.3), (2.2)–(2.6), (4.4) and (4.5) be fulfilled. Moreover, if the condition (4.1) holds for even n and for some $k \in \mathbb{N}$*

$$\int_0^{+\infty} t\,p(t)\,\omega\big(\sigma^{n-3}(t)\,p_{n-2,k}(\sigma(t))\big)\,dt = +\infty,$$

then equation (1.1) has Property B.

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