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BOUNDARY VALUE PROBLEMS OF THE THEORY OF THERMOELASTICITY FOR THE SPHERE WITH VOIDS

Bitsadze L.

Abstract. In the present paper the 3D equations of thermoelasticity for materials with voids is considered. The representation of general solution of the system of equations is constructed by means of elementary (harmonic, meta-harmonic and bi-harmonic) functions, which makes it possible to solve the BVPs for a sphere. The Dirichlet type BVPs for the sphere with voids and for the space with spherical cavity are solved explicitly. The obtained solutions are represented in the form of absolutely and uniformly convergent series.

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1. Introduction

The linear theory of thermoelasticity for materials with voids or empty pores is the generalization of the classical theory of elasticity. This theory is used for investigated various types of geological and biological materials for which the classical theory of elasticity is not adequate. Porous materials with voids have applications in many fields of engineering, such as the petroleum industry, material science and biology. This theory enable us to analyze the behaviour of elastic porous materials which can be found in engineering, such as rock and soil, bone, the manufactured porous materials. The voids are assumed to contain nothing of mechanical or energetic significance.

The non-linear version of elastic materials with voids was proposed by Nunziato and Cowin [1] and the linear version was developed by Cowin and Nunziato [2] to study mathematically the mechanical behavior of porous solids. Ieşan in [3] established a variational theory for thermoelastic materials with voids. In [4,5] Ciarletta and Scalia studied a linear thermoelastic theory of materials with voids, and established uniqueness and reciprocal theorems. In [6] Ieşan and Quintanilla have developed the theory of Nunziato and Cowin for thermoelastic deformable materials with double porosity structure.

In the last years many authors have investigated the BVPs for elastic materials with voids, using the theory developed by Cowin and his co-workers, also the BVPs for elastic materials with double porosity structure. Below is mentioned a few works(see [7-22]), where also the bibliographical information can be found.

Along with the development of the linear theory of elasticity for materials with voids, a great deal of attention is attached to the construction of explicit solutions of boundary value problems for concrete domains, useful for engineering practice.

In the present paper the 3D linear theory of thermoelasticity for materials with voids is considered. The representation of general solution of the system of equations in the considered theory is constructed by means of elementary (harmonic, meta-harmonic and bi-harmonic) functions, which makes it possible to solve the BVPs for a sphere. The Dirichlet type BVPs for the sphere with voids and for the space with spherical cavity are solved explicitly. The obtained solutions are represented in the form of absolutely and uniformly convergent series.

2. Basic equations. Boundary value problems

Let us assume that D is a ball of radius R centered at origin $O(0,0,0)$ in the Euclidean 3D space E_3 and S is a spherical surface of radius R. Let $\mathbf{x} = (x_1, x_2, x_3) \in E_3$. Let D^- be the whole space with spherical cavity, with boundary S. Let us assume that the domain $D(D^-)$ is filled with an isotropic material consisting of empty pores.

The basic system of equations in the linear theory of thermoelasticity for isotropic materials with voids, can be written as [3]:

$$\begin{cases} \mu\Delta\mathbf{u} + (\lambda+\mu)\mathrm{graddiv}\mathbf{u} + b\mathrm{grad}\varphi - \beta\mathrm{grad}\theta = 0, \\ (\alpha\Delta + b_0)\varphi - b\mathrm{div}\mathbf{u} + m\theta = 0, \\ (k\Delta + b_1)\theta + b_2\mathrm{div}\mathbf{u} + b_3\varphi = 0. \end{cases} \tag{1}$$

where $\mathbf{u}$ is the displacement vector in a solid, φ is the change of volume fraction, θ is the temperature, $b_0 = -\xi$, $b_1 = aT_0 i\omega$, $b_2 = \beta T_0 i\omega$, $b_3 = mT_0 i\omega$, λ, μ, β, α, ξ, m, a, k are constitutive coefficients, $T_0 = const > 0$ is the absolute temperature in the reference state, Δ is the Laplacian.

Let us introduce the definition of a regular vector-function.

Definition. A vector-function $\mathbf{U} = (\mathbf{u}, \varphi, \theta)^\top$ defined in the domain D is called regular if

$$\mathbf{U} \in C^2(D) \cap C^1(\overline{D})$$

and for the infinite domain D^- the vector $\mathbf{U}$ additionally should satisfy the following conditions at the infinity:

$$\mathbf{U}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad \frac{\partial \mathbf{U}}{\partial x_j} = O(|\mathbf{x}|^{-2}), \quad |\mathbf{x}|^2 = x_1^2 + x_2^2 + x_3^2 >> 1, \quad j = 1,2,3.$$

Let us now formulate the Dirichlet type boundary value problems(BVPs):

Problem 1. Find a regular solution $\mathbf{U}$ of system (1) in the domain D, satisfying the following boundary conditions on S :

$$\mathbf{u}^+(\mathbf{z}) = \mathbf{F}^+(\mathbf{z}), \quad \varphi^+(\mathbf{z}) = f_4^+(\mathbf{z}), \quad \theta^+ = f_5^+(\mathbf{z}), \quad \mathbf{z} \in S.$$

Problem 2. Find a regular solution $\mathbf{U}$ of system (1) in the domain D^-, satisfying the following boundary conditions on S :

$$\mathbf{u}^-(\mathbf{z}) = \mathbf{F}^-(\mathbf{z}), \quad \varphi^-(\mathbf{z}) = f_4^-(\mathbf{z}), \quad \theta^- = f_5^-(\mathbf{z}), \quad \mathbf{z} \in S,$$

where the vector-function $\mathbf{F}^\pm(\mathbf{z}) = (f_1, f_2, f_3)$, and the functions $f_4^\pm(\mathbf{z})$, $f_5^\pm(\mathbf{z}, \ )$ are prescribed on S, at $\mathbf{z}$. Under $\mathbf{U}^\pm(\mathbf{z})$ we mean limits of $\mathbf{U}(\mathbf{x})$ at $\mathbf{z} \in S$ from $D(D^-)$

$$[\mathbf{U}(\mathbf{z})]^+ = \lim_{D \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}), \quad [\mathbf{U}(\mathbf{z})]^- = \lim_{D^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}).$$

Throughout this paper we assume that

$$\mu > 0, \quad \alpha > 0, \quad \xi > 0, \quad 3\lambda + 2\mu > 0, \quad (3\lambda + 2\mu)\xi > 3\beta^2, \quad k > 0.$$

Theorem 1. *The Dirichlet type boundary value problem has at most one regular solution in domain $D(D^-)$.*

Theorem 1 can be proved similarly to the corresponding theorem in the classical theory of thermoelasticity (for details see [23]).

3. Some auxiliary formulas

Let us introduce the spherical coordinates and the following notations:

$$\begin{cases} x_1 = \rho \sin\xi \cos\eta, \quad x_2 = \rho \sin\xi \sin\eta, \quad x_3 = \rho\cos\xi, \\ y_1 = R\sin\xi_0 \cos\eta_0, \quad y_2 = R\sin\xi_0\sin\eta_0, \quad y_3 = R\cos\xi_0, \quad y \in S, \\ |\mathbf{x}| = \rho = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad 0 \le \xi \le \pi, \quad 0 \le \eta \le 2\pi, \quad 0 \le \rho \le R. \end{cases} \tag{2}$$

$(\mathbf{x}\cdot\mathbf{w}) = \sum\limits_{k=1}^{3} x_k w_k$ denotes the usual scalar product of two vectors $\mathbf{x}$ and $\mathbf{w}$, $[\mathbf{x}\cdot\mathbf{w}]$-denotes the vector product of the two vectors.
The operator $\dfrac{\partial}{\partial S_k(x)}$ is defined as follows

$$[\mathbf{x}\cdot\nabla]_k = \frac{\partial}{\partial S_k(x)}, \qquad k = 1,2,3, \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right).$$

Below we use the following identities [24]

$$\begin{cases} (\mathbf{x}\cdot \mathrm{grad} g) = \rho\dfrac{\partial g}{\partial \rho}, \qquad \dfrac{\partial}{\partial S_k(x)}\dfrac{\partial}{\partial x_k} = \dfrac{\partial}{\partial x_k}\dfrac{\partial}{\partial S_k(x)}, \\ \sum\limits_{k=1}^{3} \dfrac{\partial^2}{\partial S_k^2(x)} = \dfrac{\partial^2}{\partial \xi^2} + ctg\xi\dfrac{\partial}{\partial\xi} + \dfrac{1}{sin^2\xi}\dfrac{\partial^2}{\partial\eta^2}, \\ \sum\limits_{k=1}^{3} \dfrac{\partial}{\partial S_k(x)}\dfrac{\partial}{\partial x_k} = 0, \qquad \dfrac{\partial g(\rho)Y(\xi,\eta)}{\partial S_k(x)} = g(\rho)\dfrac{\partial Y(\xi,\eta)}{\partial S_k(x)}, \\ \sum\limits_{k=1}^{3} \dfrac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{g}]_k = \rho^2 \mathrm{div}\mathbf{g}(\mathbf{x}) - \left[1 + \rho\dfrac{\partial}{\partial\rho}\right](\mathbf{x}\cdot\mathbf{g}). \end{cases} \tag{3}$$

If g_m is the spherical harmonic, then

$$\sum_{k=0}^{3} \frac{\partial^2 g_m(\mathbf{x})}{\partial S_k^2(\mathbf{x})} = -m(m+1)g_m(\mathbf{x}).$$

We introduce the following functions:

$$\begin{cases} (\mathbf{z}\cdot\mathbf{F})^{\pm} =: h_1^{\pm}(\mathbf{z}), \quad \left(\sum\limits_{k=1}^{3} \dfrac{\partial}{\partial S_k(\mathbf{z})}[\mathbf{z}\cdot\mathbf{f}]_k\right)^{\pm} =: h_2^{\pm}(\mathbf{z}), \\ \sum\limits_{k=1}^{3}\left(\dfrac{\partial f_k}{\partial S_k(\mathbf{z})}\right)^{\pm} =: h_3^{\pm}(\mathbf{z}), \quad \varphi^{\pm} = h_4^{\pm}(\mathbf{y}), \quad \vartheta^{\pm} =: h_5^{\pm}(\mathbf{y}), \quad \mathbf{y}\in S. \end{cases} \tag{4}$$

In what follows we assume that the functions $h_k^{\pm}$, $k = 1, .., 5$, can be expanded in the form of the series

$$h_k^{\pm}(\mathbf{y}) = \sum_{n=0}^{\infty} h_{kn}^{\pm}(\xi_0, \eta_0),$$

where $h_{kn}^{\pm}$ $k = 1, .., 5$ are the spherical harmonics of order n

$$h_{kn}^{\pm} = \frac{2n+1}{4\pi R^2}\int\limits_S P_n(\cos\gamma) h_k^{\pm}(\mathbf{y}) dS_y,$$

P_n Legendre polynomial of the n th order,γ is an angle formed by the radius-vectors Ox and Oy,

$$\cos\gamma = \frac{1}{|\mathbf{x}||\mathbf{y}|}\sum_{k=1}^{3} x_k y_k.$$

4. A representation of regular solutions

In this section we present the general solution of system (1) by means of elementary (harmonic, meta-harmonic and bi-harmonic) functions.

Theorem 2. *If $\boldsymbol{U} := (\boldsymbol{u}, \varphi, \vartheta)$ is a regular solution of the homogeneous system (1) then $\boldsymbol{u}$, $\mathrm{div}\boldsymbol{u}$, φ and θ satisfy the following equations*

$$\begin{cases} \Delta\Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)\mathbf{u} = 0, \\ \Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)\Psi = 0, \end{cases} \tag{5}$$

where $\Psi = (\mathrm{div}\mathbf{u}, \varphi, \theta)$.

Proof. Let $\mathbf{U} = (\mathbf{u}, \varphi, \theta)$ be a regular solution of equation (1). Upon taking the divergence operation to equation $(1)_1$, from (1) we obtain

$$\begin{cases} \mu_0\Delta\mathrm{div}\mathbf{u} + b\Delta\varphi - \beta\Delta\theta = 0, \\ (\alpha\Delta + b_0)\varphi - b\mathrm{div}\mathbf{u} + m\theta = 0, \\ (k\Delta + b_1)\theta + b_2\mathrm{div}\mathbf{u} + b_3\varphi = 0. \end{cases} \tag{6}$$

Rewrite the latter system as follows

$$D(\Delta)\Psi := \begin{pmatrix} \mu_0\Delta & b\Delta & -\beta\Delta \\ -b & \alpha\Delta + b_0 & m \\ b_2 & b_3 & k\Delta + b_1 \end{pmatrix}\Psi = 0,$$

the determinant of this system is equal to

$$detD = k\mu_0\alpha\Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2),$$

where λ_j^2, $\quad j=1,2$, are roots of the equation

$$\alpha k\mu_0\xi^2 - a_1\xi + a_2 = 0,$$

$$a_1 = \mu_0(\alpha b_1 + kb_0) + \alpha\beta b_2 + kb^2, \quad \mu_0 = \lambda + 2\mu,$$

$$a_2 = \mu_0(b_0b_1 - mb_3) + b_1b^2 + mbb_2 + \beta(bb_3 + b_0b_2).$$

We assume that the values λ_j^2 are distinct and different from zero.

Clearly, from the system (6) it follows that

$$\begin{cases} \Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)\mathrm{div}\mathbf{u} = 0, \\ \Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)\varphi = 0, \\ \Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)\theta = 0. \end{cases} \tag{7}$$

Further, applying the operator $\Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2))$ to equation $(1)_1$, and using the last relations we obtain

$$\Delta\Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)\mathbf{u} = 0. \tag{8}$$

The last formulas prove the theorem.

Theorem 3. *The regular solution $\boldsymbol{U}$ of system (1) admits in the domain of regularity a representation*

$$\begin{cases} \mathbf{u} = \mathbf{\Phi} - \mathrm{grad}\left[A_0\vartheta_0 + \sum\limits_{j=1}^{2}\dfrac{A_j\vartheta_j}{\lambda_j^2}\right], \quad \varphi = B\vartheta + \sum\limits_{k=1}^{2}B_k\vartheta_k, \\ \theta = \vartheta + \sum\limits_{k=1}^{2}\vartheta_k, \qquad \mathrm{div}\mathbf{u} = A\vartheta + \sum\limits_{k=1}^{2}A_k\vartheta_k, \end{cases} \tag{9}$$

where

$$\begin{cases} A = \dfrac{mb_3 - b_0b_1}{b_0b_2 + bb_3}, \quad B = -\dfrac{mb_2 + bb_1}{b_0b_2 + bb_3}, \\ A_j = \dfrac{mb_3 - (b_0 - \alpha\lambda_j^2)(b_1 - k\lambda_j^2)}{b_2(b_0 - \alpha\lambda_j^2) + bb_3}, \\ B_j = -\dfrac{mb_2 + b(b_1 - k\lambda_j^2)}{b_2(b_0 - \alpha\lambda_j^2) + bb_3}, \quad \mu_0A_j + bB_j - \beta = 0, \\ A_0 = \dfrac{(\lambda+\mu)A + bB - \beta}{\mu} = -\dfrac{\mu(mb_3 - b_0b_1) + a_2}{\mu(b_0b_2 + bb_3)}, \end{cases} \tag{10}$$

the functions ϑ_0 *and* $\mathbf{\Phi}$ *are chosen such that*

$$\Delta\vartheta_0 = \vartheta, \quad \Delta\mathbf{\Phi} = 0, \quad \operatorname{div}\mathbf{\Phi} = \frac{-a_2}{\mu(b_0 b_2 + b b_3)}\vartheta,$$

ϑ *and* ϑ_j, $j = 1, 2$, *are solutions of the following equations*

$$\Delta\vartheta = 0, \quad (\Delta + \lambda_j^2)\vartheta_j = 0, \quad j = 1, 2.$$

Proof. By an immediate verification we make sure that the functions φ, ϑ and $\operatorname{div}\mathbf{u}$ satisfy equations $(1)_2$, and $(1)_3$.

If supposing that φ, ϑ and $\operatorname{div}\mathbf{u}$, are known values, we can rewrite Eq. $(1)_1$ in the following form

$$\Delta\mathbf{u} = -\operatorname{grad}\left[A_0\vartheta - \sum_{j=1}^{2} A_j\vartheta_j\right]. \tag{11}$$

The general solution of equation (11) has the following form

$$\mathbf{u} = \mathbf{\Phi} + \mathbf{u}_0, \tag{12}$$

where the vector-function $\mathbf{\Phi}$ is a harmonic function, satisfying the conditions

$$\Delta\mathbf{\Phi} = 0, \quad \operatorname{div}\mathbf{\Phi} = \frac{-a_2}{\mu(b_0 b_2 + b b_3)}\vartheta, \quad \Delta\operatorname{div}\mathbf{\Phi} = \mathbf{0}$$

and $\mathbf{u}_0$ is one of the particular solutions of the nonhomogeneous equation (11)

$$\mathbf{u}_0 = -\operatorname{grad}\left[A_0\vartheta_0 + \sum_{j=1}^{2} \frac{A_j\vartheta_j}{\lambda_j^2}\right], \tag{13}$$

the function ϑ_0 is chosen such that $\Delta\vartheta_0 = \vartheta$. It is obvious that ϑ_0 is a bi-harmonic function $\Delta\Delta\vartheta_0 = 0$.

Thus the solution of system (1), is represented by formulas (9).

From (9) we conclude that the representation of a solution of equation $(1)_1$ contains a harmonic, bi-harmonic and a metaharmonic functions, while the representation of φ and θ contains a harmonic and a metaharmonic functions.

5. The solution of problem 1

From the point of view of applications, it is interesting to investigate and construct explicit solutions of boundary-value problems of thermoelasticity theory for concrete domains (circle, plane with circular hole, sphere, the space with spherical cavity, ellipse and ect.).

In this section, we will construct, an explicit solution of Problem 1 in details. Quite similarly, we can construct the solution of problem 2.

We look for the solution to system (1) in the form (9) and taking into account the identities (2) and (3), from (9) we arrive at the following relations

$$\begin{cases} (\mathbf{x}\cdot\mathbf{u}) = (\mathbf{x}\cdot\mathbf{\Phi}) - \rho\dfrac{\partial}{\partial\rho}\left[A_0\vartheta_0 + \displaystyle\sum_{j=1}^{2}\frac{A_j\vartheta_j}{\lambda_j^2}\right], \\ \displaystyle\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{u}]_k = \sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{\Phi}]_{\mathbf{k}} \\ -\displaystyle\sum_{k=1}^{3}\frac{\partial^2}{\partial S_k^2(x)}\left[A_0\vartheta_0 + \sum_{j=1}^{2}\frac{A_j\vartheta_j}{\lambda_j^2}\right], \\ \displaystyle\sum_{k=1}^{3}\frac{\partial u_k}{\partial S_k(x)} = \sum_{k=1}^{3}\frac{\partial\Phi_k}{\partial S_k(x)}, \\ \varphi = B\vartheta + \displaystyle\sum_{k=1}^{2}B_k\vartheta_k, \quad \theta = \vartheta + \sum_{k=1}^{2}\vartheta_k. \end{cases} \tag{14}$$

Let us replace functions $(\mathbf{x}\cdot\mathbf{\Phi})$ and $\sum\limits_{k=1}^{3}\dfrac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{\Phi}]_k$ with function ϑ. To this end, using the following identities

$$\Delta(\mathbf{x}\cdot\mathbf{\Phi}) = 2\mathrm{div}\,\mathbf{\Phi} = -2\frac{a_2}{\mu(b_0b_2+bb_3)}\vartheta,$$

$$\sum_{j=1}^{3}\frac{\partial}{\partial S_k}[\mathbf{x}\cdot\mathbf{g}]_j = \rho^2\mathrm{div}\,\mathbf{g} - \left(\rho\frac{\partial}{\partial\rho}+1\right)(\mathbf{x}\cdot\mathbf{g}),$$

we obtain

$$\begin{cases} (\mathbf{x}\cdot\mathbf{\Phi}) = \Omega - 2\dfrac{a_2}{\mu(b_0b_2+bb_3)}\vartheta_0, \\ \displaystyle\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{\Phi}]_k = -\frac{a_2}{\mu(b_0b_2+bb_3)}\rho^2\vartheta - \left(\rho\frac{\partial}{\partial\rho}+1\right)(\mathbf{x}\cdot\mathbf{\Phi}), \end{cases} \tag{15}$$

where Ω is an arbitrary harmonic function $\Delta\Omega = 0$.

Let us assume that the functions ϑ, ϑ_j, $j=1,2$, Ω and $\sum\limits_{k=1}^{3}\dfrac{\partial\Phi_k}{\partial S_k(x)}$ in (14) are sought in the form [25]

$$\begin{cases} \vartheta(\mathbf{x}) = \sum\limits_{n=0}^{\infty} \dfrac{\rho^n}{R^n} Y_n(\xi,\eta), \\ \\ \vartheta_j(\mathbf{x}) = \sum\limits_{n=0}^{\infty} \phi_n(\lambda_j\rho) Y_{jn}(\xi,\eta), \\ \\ \Omega(\mathbf{x}) = \sum\limits_{n=0}^{\infty} \dfrac{\rho^n}{R^n} Z_n(\xi,\eta), \\ \\ \sum\limits_{k=1}^{3} \dfrac{\partial\Phi_k}{\partial S_k(x)} = \sum\limits_{n=0}^{\infty} \left(\dfrac{\rho}{R}\right)^n Z_{4n}(\xi,\eta), \quad \rho < R, \end{cases} \tag{16}$$

where Y_n, Y_{kn}, Z_n and Z_{4n}, $k = 1,2$ are the unknown spherical harmonics of order n,

$$\phi_n(\lambda_k\rho) = \frac{\sqrt{R} J_{n+\frac{1}{2}}(\lambda_k\rho)}{\sqrt{\rho} J_{n+\frac{1}{2}}(\lambda_k R)} \qquad k = 1,2,$$

$J_{n+\frac{1}{2}}(\lambda_k\rho)$ is the Bessel function. Taking into account (16), we can write the particular solutions of equation $\Delta\vartheta_0 = \vartheta$ in the following form

$$\vartheta_0(\mathbf{x}) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\rho^2}{3+2n} \left(\frac{\rho}{R}\right)^n Y_n(\vartheta,\eta). \tag{17}$$

Substituting (16) and (17) into (14), passing to the limit as $\rho \to R$, and bearing in mind (4), for the determination of Y_n, Y_{kn}, Z_n and Z_{4n}, $k = 1,2$ we arrive at the following system of algebraic equations

$$\begin{cases} Z_n - \left[\dfrac{A_0(n+2)}{2} + \dfrac{a_2}{\mu(b_0 b_2 + b b_3)}\right] \dfrac{R^2}{3+2n} Y_n - \\ \\ R \sum\limits_{j=1}^{2} \dfrac{A_j}{\lambda_j^2} \left[\dfrac{\partial\phi_n(\lambda_j\rho)}{\partial\rho}\right]_{\rho=R} Y_{jn} = h_{1n}^+, \\ \\ -(n+1)Z_n + \left[\dfrac{(n+1)A_0}{2} - \dfrac{a_2}{\mu(b_0 b_2 + b b_3)}\right] \dfrac{nR^2 Y_n}{(3+2n)} \\ \\ + \sum\limits_{j=1}^{2} \dfrac{A_j Y_{jn}}{\lambda_j^2} = h_{2n}^+, \quad Z_{4n} = h_{3n}^+, \\ \\ BY_n + \sum\limits_{j=1}^{2} B_j Y_{jn} = h_{4n}^+, \quad Y_n + \sum\limits_{j=1}^{2} Y_{jn} = h_{5n}^+. \end{cases} \tag{18}$$

By applying Theorem 1 we conclude that the determinant of system (18) for $n \geq 0$ is different from zero. Therefore, system (18) is uniquely solvable.

This finally leads to the conclusion suggesting that the solution of Problem 1 is given by formula (9), where functions $\vartheta, \vartheta_j,\ \Omega, \vartheta_0$ are defined from (16),(17) and (18).

6. The solution of problem 2

In quite the same way, as above, we can construct the solution to the BVP for the space with spherical cavity.

Let us assume that functions $\vartheta,\ \ \vartheta_j,\ \ j=1,2,\ \ \ \Omega$ and $\sum\limits_{k=1}^{3}\dfrac{\partial\Phi_k}{\partial S_k(x)}$ in (14) are sought in the form [25]

$$\begin{cases} \vartheta(\mathbf{x}) = \sum\limits_{n=0}^{\infty}\left(\dfrac{R}{\rho}\right)^{n+1} Y_n, \\ \vartheta_j(\mathbf{x}) = \sum\limits_{n=0}^{\infty}\Psi_n(\lambda_j\rho)Y_{jn}(\xi,\eta). \\ \Omega(\mathbf{x}) = \sum\limits_{m=0}^{\infty}\left(\dfrac{R}{\rho}\right)^{n+1} Z_n(\xi,\eta), \\ \sum\limits_{k=1}^{3}\dfrac{\partial\Phi_k}{\partial S_k(x)} = \sum\limits_{n=0}^{\infty}\left(\dfrac{R}{\rho}\right)^{n+1} Z_{4n}(\xi,\eta), \qquad \rho > R \end{cases} \tag{19}$$

respectively, where $Y_n,\ \ Y_{kn},\ \ Z_n$ and $Z_{4n},\ \ k=1,2$ are the sought functions,

$$\Psi_n(\lambda_k\rho) = \frac{\sqrt{R}H^{(1)}_{n+\frac{1}{2}}(\lambda_k\rho)}{\sqrt{\rho}H^{(1)}_{n+\frac{1}{2}}(\lambda_k R)} \qquad k=1,2,$$

$H^{(1)}_{n+\frac{1}{2}}(\lambda\rho)$ is Hankel's function. The solution ϑ_0 of equation $\Delta\theta_0 = \vartheta$ is defined as

$$\vartheta_0(\mathbf{x}) = \frac{\rho^2}{2}\sum_{n=0}^{\infty}\frac{Y_n(\vartheta,\eta)}{(1-2n)}\left(\frac{R}{\rho}\right)^{n+1}, \quad \rho > R. \tag{20}$$

Substituting (19) into (14), passing to the limit as $\rho\to R$, and bearing in mind (4), for the determination of $Y_j, Y_{jn},\ Z_{3n}\ Z_{4n}$, we arrive at the following system of algebraic equations

$$
\begin{cases}
Z_n - \left[\dfrac{a_2}{\mu(b_0b_2+bb_3)} + \dfrac{A_0(1-n)}{2}\right] \dfrac{R^2 Y_n}{1-2n} \\[2ex]
- \left[\rho \dfrac{\partial}{\partial \rho} \displaystyle\sum_{j=1}^{2} \dfrac{A_j}{\lambda_j^2} \Psi_n(\lambda_j \rho)\right]_{\rho=R} Y_{jn} = h_{1n}^{-}, \\[2ex]
nZ_n + \left[\dfrac{nA_0}{2} + \dfrac{a_2}{\mu(b_0b_2+bb_3)}\right] \dfrac{(n+1)R^2 Y_n}{1-2n} \\[2ex]
+n(n+1) \sum\limits_{j=1}^{2} \dfrac{A_j}{\lambda_j^2} Y_{jn} = h_{2n}^{-}, \quad Z_{4n} = h_{3n}^{-}, \\[2ex]
BY_n + \sum\limits_{j=1}^{2} B_j Y_{jn} = h_{4n}^{-}, \quad Y_n + \sum\limits_{j=1}^{2} Y_{jn} = h_{5n}^{-}.
\end{cases} \tag{21}
$$

By applying the uniqueness theorem we conclude that the determinant of system (21) for $n \geq 0$ is different from zero. Therefore, system (21) is uniquely solvable.

For the absolute and uniform convergence of series in (16),(17),(19) and (20), together with their first derivatives, it is sufficient to assume that

$$h_j \in C^5(S), \quad j = 1, 2, .., 5.$$

Under these conditions the resulting series are absolutely and uniformly convergent.

We note that in the elasticity theory of isotropic bodies, the basic BVPs for the sphere in the classical setting for potential methods are thoroughly investigated in [23] (see also references therein).

7. Conclusions

In this paper, system of equations of the thermoelasticity for isotropic materials with voids is considered and the following results are obtained:

1. The regular solution system of equations is constructed explicitly by means of elementary functions.

2. Solutions of the Problems 1 and 2 are obtained in the form of the absolutely and uniformly convergent series.

3. On the basis of the method offered in this paper, it is possible to construct the solutions of BVPs for the sphere and for the space with spherical cavity.

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Author's address:

L. Bitsadze
I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi, 0186
Georgia
E-mail: lamarabitsadze@gmail.com