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# SOLUTION OF THE BOUNDARY VALUE PROBLEMS OF ELASTOSTATICS FOR A PLANE WITH CIRCULAR HOLE WITH VOIDS

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**Abstract**. In the present paper, the special representations of a general solution of a system of differential equations of the theory of elastic materials with voids is constructed by using harmonic, biharmonic and metaharmonic functions which make it possible to reduce the initial system of equations to equations of simple structure and facilitate the solution of initial problems. These representations are used to solve problems for an elastic plane with circular hole and with voids. The solutions are written explicitly in the form of absolutely and uniformly converging series. The uniqueness of regular solutions of the considered problems is also investigated.

**Keywords and phrases**: Porous media with voids, boundary value problems, explicit solutions.

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Dedicated to my teacher professor Tengiz Gegelia on the occasion of his $90^{th}$ birthday anniversary

## 1. Introduction

The history of construction and research of models of the mechanics of porous materials has been around for about two centuries. The first main results were of an experimental nature. From the middle of the previous century the works of M. Bio [1] begin a new period in the mechanics of porous materials. Both linear and nonlinear theories are constructed. Intensively there is a study of the problems of the theoretical and practical nature of these theories. For a history of developments and a review of main results in the theory of porous media, see the books of de Boer [2], Straughan [3],[4].

Two types of porous materials are considered. 1) porous materials saturated (or unsaturated) with liquid, and 2) porous materials with voids. In this paper we study boundary problems for elastic materials with empty pores. The foundations of the linear theory of elastic materials with voids were first proposed by Cowin and Nunziato [5]. Such materials include, in particular, rocks and soils, granulated and some other manufactured porous materials.

Problems of elasticity for materials with voids were investigated by many authors. Below we mention only a few works where the bibliographical information can also be found. In the framework of the theory of linear elastic materials with voids in [6] some basic theorems on the existence and uniqueness of a solution are given, reciprocity relations and the variation characteristic of the solution. In [7], solutions are obtained for equations of the classical theory of elastodynamics of homogeneous and isotropic elastic materials with voids. In the work [8] boundary value problems of stationary oscillations in the linear theory of homogeneous and isotropic materials with voids are investigated. In [9] the behavior of plane harmonic waves in an elastic material with pores is studied and the main properties of these waves are established. Also established are the existence and uniqueness of solutions for external problems. Existence theorems in equilibrium theory are proved in [10].

The linear theory of thermoelastic materials with voids was for the first time considered in the work of Iesan [11]. Solutions of Galerkin type and uniqueness theorems in the theory of thermo-elasticity for materials with voids are proved in [12, 13]. Problems of steady vibrations of elastic and thermoelastic materials with voids are investigated in [15]. The linear theory of micropolar thermoelasticity is considered for materials with voids in the papers [14] and [15].The nonlinear theory of elastic porous materials with voids was proposed in the work of Nunziato and Cowin [16].

From the point of view of applications, it is actual to construct explicit solutions of problems, which makes it possible to effectively analyze the problem under study. Some of these results are presented in [17-25] and in references to them.

In the present paper, the special representations of a general solution of a system of differential equations of the theory of elastic materials with voids is constructed by using harmonic, biharmonic and metaharmonic functions which make it possible to reduce the initial system of equations to equations of simple structure and facilitate the solution of initial problems. These representations are used to solve problems for an elastic plane with circular hole and with voids. The solutions are written explicitly in the form of absolutely and uniformly converging series. The uniqueness of regular solutions of the considered problems is also investigated.

## 2. Basic equations and boundary value problems

Let $D$ be a plane with a circular hole $S$, consisting of empty pores. A system of equations of the linear theory of elastic materials with voids has the form [5, 8]:

$$\begin{cases} \mu\Delta\mathbf{u} + (\lambda+\mu)\mathrm{graddiv}\mathbf{u} + \beta\mathrm{grad}\varphi = 0, \\ \\ \alpha\Delta\varphi - \xi\varphi - \beta\mathrm{div}\mathbf{u} = 0, \end{cases} \tag{1}$$

where $\mathbf{u} = \mathbf{u}(u_1, u_2)$ is the displacement vector in a solid, $\varphi$ is a change with respect to the pore area; $\lambda$ and $\mu$ are the Lamé constants; $\alpha, \beta$ and $\xi$ are the constants characterizing the body porosity.

Let us now formulate the boundary value problems.

Find, a regular vector $\mathbf{U(x)} = (\mathbf{u(x)}, \varphi)$ in $D$ that satisfies the system of equations (1) and, on the boundary $S$ , one of the following conditions :

$$\begin{aligned} &I.\mathbf{u}(\mathbf{z}) = \mathbf{f}(\mathbf{z}), \quad \varphi(\mathbf{z}) = f_3(\mathbf{z}); \\ &II.\mathbf{P}_1(\partial_{\mathbf{z}}, \mathbf{n})\mathbf{U}(\mathbf{z}) = \mathbf{f}(\mathbf{z}), \quad \partial_n\varphi(\mathbf{z}) = f_3(\mathbf{z}), \quad \mathbf{z} \in S, \end{aligned} \tag{2}$$

where $\mathbf{f}(\mathbf{z}) = (f_1(\mathbf{z}), f_2(\mathbf{z})), f_3(\mathbf{z})$ are the given functions on the circumference $S$.

Vector $\mathbf{U(x)}$ satisfies the following conditions at infinity:

$$\mathbf{U}(\mathbf{x}) = o(1), \quad r^2\partial_{x_k}\mathbf{U}(\mathbf{x}) = O(1) \tag{3}$$

-in the problem I, and

$$\mathbf{U}(\mathbf{x}) = O(1), \quad r^2\partial_{x_k}\mathbf{U}(\mathbf{x}) = O(1) \tag{4}$$

-in the problem II, where $r^2 = x_1^2 + x_2^2, \quad k = 1, 2.$

The stress vector can be written as [5]

$$\mathbf{P}\left(\partial_x, \mathbf{n}\right)\mathbf{U}(\mathbf{x}) = \begin{pmatrix} \mathbf{P}_1\left(\partial_x, \mathbf{n}\right)\mathbf{U}(\mathbf{x}) \\ \alpha\partial_n\varphi(\mathbf{x}) \end{pmatrix},$$

where

$$\mathbf{P}_1\left(\partial_x, \mathbf{n}\right)\mathbf{U}(\mathbf{x}) = \mathbf{T}\left(\partial_x, \mathbf{n}\right)\mathbf{u}(\mathbf{x}) + \beta\mathbf{n}\varphi(\mathbf{x}),$$

$$\mathbf{T}\left(\partial_x, \mathbf{n}\right)\mathbf{u}(\mathbf{x}) = \mu\partial_n\mathbf{u}(\mathbf{x}) + \lambda\mathbf{n}\mathrm{div}\mathbf{u}(\mathbf{x}) + \mu\sum_{i=1}^{2} n_i(\mathbf{x})\mathrm{grad}u_i(\mathbf{x})$$

-is the stress vector in the classical theory of elasticity [27].

## 3. A representation of the general solution

The solution of system (1) is written in the form

$$\left[\begin{array}{l} \mathbf{u}(x) = c_0\hat{\mathbf{u}}(x) + c_1\check{\mathbf{u}}(x), \\ \\ \varphi(x) = \varphi_1(x) + \varphi_2(x), \end{array}\right. \tag{5}$$

where $\varphi_1$ is a harmonic function, $\Delta\varphi_1 = 0$, and $\varphi_2$ is a metaharmonic function with the parameter $s_1^2$, $(\Delta + s_1^2)\varphi_2 = 0$; $s_1 = i\sqrt{\dfrac{\mu_0\xi - \beta^2}{\mu_0\alpha}} = is_0$, $i = \sqrt{-1}$,

$$\lambda > 0, \quad \mu > 0, \quad \alpha > 0, \quad \mu_0\xi - \beta^2 > 0; \tag{6}$$

$c_0$ and $c_1$ are the unknown constants for the time being. A general solution $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2)$ of the homogeneous equation, corresponding to the nonhomogeneous equation $(1)_1$ with respect to $\hat{\mathbf{u}}$, is represented as follows [24]

$$\hat{\mathbf{u}}(\mathbf{z}) = \mathrm{grad}[\Phi_1(\mathbf{x}) + \Phi_2(\mathbf{x})] + \mathrm{rot}\Phi_3(\mathbf{x}) + c_2\mathbf{\Gamma}(\mathbf{x}), \tag{7}$$

where the functions $\Phi_2(\mathbf{x})$ and $\Phi_3(\mathbf{x})$ are related to each other by the equality

$$\mu_0\mathrm{grad}\Delta\Phi_2(\mathbf{x}) + \mu\mathrm{rot}\Delta\Phi_3(\mathbf{x}) = 0, \tag{8}$$

$$\Delta\Phi_1(\mathbf{x}) = 0, \quad \Delta\Delta\Phi_2(\mathbf{x}) = 0, \quad \Delta\Delta\Phi_2(\mathbf{x}) = 0,$$

$\Phi_1(\mathbf{x})$, $\Phi_2(\mathbf{x})$, $\Phi_3(\mathbf{x})$- are scalar functions, $\mathbf{\Gamma}(\mathbf{x}) = \mathbf{\Gamma}(x_2, -x_1)$, $\mathrm{div}\mathbf{\Gamma} = 0$, $c_2$ is the sought coefficient, $\mathrm{rot} = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$, $\check{\mathbf{u}} = (\check{u_1}, \check{u_2})$ is one of the particular solutions of equation $(1)_1$:

$$\check{\mathbf{u}}(\mathbf{z}) = -\frac{\beta}{\mu_0}\mathrm{grad}\left(-\frac{1}{s_1^2}\varphi_2 + \varphi_0\right), \tag{9}$$

where $\varphi_0$ is chosen such that $\Delta\varphi_0 = \varphi_1$ . It is obvious that $\varphi_0$ is a biharmonic function: $\Delta\Delta\varphi_0 = \Delta\varphi_1 = 0$. For simplicity, the function $\varphi_1$ is chosen such that $\varphi_1 = \mathrm{div}\hat{\mathbf{u}} \equiv \Delta\Phi_2$. Then we can take $\varphi_0 = \Phi_2$. Let us calculate the values of the coefficients $c_0$ and$c_1$ in representation (7). We apply the operator *div* to the first equality in (7) and compare the obtained expression with $\mathrm{div}\mathbf{u}$ defined by equation $(1_2)$. We obtain

$$c_0 = \frac{\mu_0\xi - \beta^2}{\mu_0\beta}, \quad c_1 = 1.$$

By an immediate verification we make sure that representations (7) satisfy equations $(1)_1$ and $(1)_2$.

**4. Uniqueness theorems**

For a regular solution $\mathbf{U}$ of equation (1) under conditions (3) and (4) Green's formulas [27]

$$\left[\begin{array}{l} \int\limits_D [E(\mathbf{u},\mathbf{u}) + \beta\varphi \mathrm{div}\mathbf{u}]dx = -\int\limits_S \mathbf{u}P_1\left(\partial\mathbf{y},\mathbf{n}\right)\mathbf{U}d_yS, \\ \\ \int\limits_D \left(\alpha|\mathrm{grad}\varphi|^2 + \xi\varphi^2 + \beta\varphi div\mathbf{u})\right\} d\mathbf{x} = -\int\limits_S \alpha\varphi\partial_n d_yS \end{array}\right. \tag{10}$$

are valid, where

$$E(\mathbf{u},\mathbf{u}) = (\lambda+\mu)(\mathrm{div}\mathbf{u})^2 + \mu\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right)^2 + \mu\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)^2$$

under conditions (6) , is a non negative quadratic form [25].
Let $\mathbf{U}'$ and $\mathbf{U}''$ be two arbitrary solutions of any of the problems I and II. For their difference $\mathbf{U} = (\mathbf{u},\varphi) = \mathbf{U}' - \mathbf{U}''$, the right-hand sides of formulas (10) are zero on K. From these equalities, taking into account conditions (6) and the equality

$$\xi\varphi^2 + \beta\varphi\mathrm{div}\mathbf{u} = \frac{\mu_0\xi - \beta^2}{\mu_0}\varphi_2(\varphi_1+\varphi_2),$$

we obtain:

$$\varphi_1(\mathbf{x}) = k, \quad \varphi_2(\mathbf{x}) = 0, \quad E(\mathbf{u},\mathbf{u}) + \beta k\mathrm{div}\mathbf{u} = 0, \tag{11}$$

where $k$ is an arbitrary constant.

In the case of the first homogeneous problem

$$\varphi(\mathbf{x}) = 0, \quad \mathbf{x}\in D.$$

Consequently, $k = 0$, and $E(\mathbf{u},\mathbf{u}) = 0$. The solution of the above equation has the form [25]:

$$u_1 = q_1 - px_2, \quad u_2 = q_2 + px_1,$$

where $p, q_1, q_2$ are arbitrary constants. In addition, in Problem I , by virtue of the homogeneous boundary condition we get $p = q_1 = q_2 = 0$, and hence,

$$u_1(\mathbf{x}) = u_2(\mathbf{x}) = \varphi(\mathbf{x}) = 0, \quad \mathbf{x}\in D.$$

Thus, we have

**Theorem 1**. *Problem I has a unique solution.*
The solution of the homogeneous Problem II, which also satisfies equations (14), has the form:

$$\mathbf{u}(\mathbf{x}) = l\mathbf{x} + \boldsymbol{\alpha}, \tag{12}$$

where $l = -\dfrac{k\beta}{2(\lambda+\mu)}$, $\boldsymbol{\alpha} = (\alpha_1,\alpha_2)$ -is an arbitrary constant vector. Taking into account conditions (4), we can write from (12): $l = 0$. So, we get $\mathbf{u}(\mathbf{x}) = 0$. Thus, we have

**Theorem 2**.*Two arbitrary solutions of Problem II can differ from each other only by a constant vector.*

### 5. Solution of the problems

Let us rewrite representations (7) in terms of polar coordinates $r$ and $\psi$ as normal and tangent components

$$\begin{cases} u_n = \partial_r(c_0\Phi_1 + c_3\Phi_2 + c_4\varphi_2) - c_0\dfrac{1}{r}\partial_\psi\Phi_3, \\ u_s = \dfrac{1}{r}\partial_\psi(c_0\Phi_1 + c_3\Phi_2 + c_4\varphi_2) + c_0\partial_r\Phi_3 - c_2 r, \\ \varphi = \varphi_1 + \varphi_2, \end{cases} \tag{13}$$

where $c_3 = -\dfrac{\xi}{\beta}$, $\quad c_4 = \dfrac{\beta}{\mu_0} s_1^2$, $\quad r^2 = x_1^2 + x_2^2$.

Using formula (8) and the equality $\varphi_1 = \Delta\Phi_2$ , the harmonic and biharmonic functions and also the metaharmonic function contained in (13) are represented in the domain $D$ as series [28, 29]:

$$\begin{cases} \Phi_1 = \sum\limits_{m=0}^{\infty}\left(\dfrac{R}{r}\right)^m (\mathbf{X}_{m3}\cdot\boldsymbol{\nu}_m(\psi)), \\ \Phi_2 = \dfrac{R^2}{4}\sum\limits_{m=2}^{\infty}\dfrac{1}{1-m}\left(\dfrac{R}{r}\right)^{m-2} (\mathbf{X}_{m1}\cdot\boldsymbol{\nu}_m(\psi)), \\ \Phi_3 = \dfrac{R^2\mu_0}{4\mu}\sum\limits_{m=2}^{\infty}\dfrac{1}{1-m}\left(\dfrac{R}{r}\right)^{m-2} (\mathbf{X}_{m1}\cdot\mathbf{s}_m(\psi)), \\ \varphi_1 = \sum\limits_{m=0}^{\infty}\left(\dfrac{R}{r}\right)^m (\mathbf{X}_{m1}\cdot\boldsymbol{\nu}_m(\psi)), \quad \varphi_2 = \sum\limits_{m=0}^{\infty} K_m(s_0 r)(\mathbf{X}_{m2}\cdot\boldsymbol{\nu}_m(\psi)), \end{cases} \tag{14}$$

where $\mathbf{X}_{mk}$ is the sought two-component vector, $k = 1,2,3,$ $\quad \mathbf{x} = (r,\psi);$ $\quad \boldsymbol{\nu}_m = (\cos m\psi, \sin m\psi),$ $\quad \mathbf{s}_m = (-\sin m\psi, \cos m\psi),$ $\quad K_m(s_0 r)$ - is a modified Hankel function of the imaginary argument; $K_m(s_0 r) \to 0, \quad r \to \infty.$

Let us write the boundary conditions of Problem I in the form of normal and tangent components

$$u_n(\mathbf{z}) = f_n(\mathbf{z}), \quad u_s(\mathbf{z}) = f_s(\mathbf{z}), \quad \varphi(\mathbf{z}) = f_3(\mathbf{z}). \tag{15}$$

Let the functions $f_n, f_s$ and $f_3$ be expanded in the Fourier series, whose coefficients are $\alpha_m, \beta_m$ and $\gamma_m$, respectively. $m$ is the summation index.

Let us substitute expressions (14) into (13) and pass to the limit as $r \to R$. Then obtained results substitute into (15). We obtain the linear algebraic systems of equations:

For $m = 0$

$$\begin{cases} c_3 R X_{01} + 2s_0 c_4 K_0'(s_0 R) X_{02} = \alpha_0, \\ c_0 R\mu_0 X_{01} - 2\dfrac{1}{R} X_{03} = \beta_0, \\ 2X_{01} + 2K_0(s_0 R) X_{02} = \gamma_0, \end{cases} \tag{16}$$

where, for convenience, we have introduced the notation $X_{03} \equiv c_2$.

For $m = 1$, we have

$$\begin{cases} c_4 s_0 K_1'(s_0 R) X_{12} - \dfrac{c_0}{R} X_{13} = \alpha_1, \\ c_4 K_1(s_0 R) X_{12} + c_0 X_{13} = R\beta_1, \\ X_{11} + K_1(s_0 R) X_{12} = \gamma_1. \end{cases} \tag{17}$$

For $m = 2, 3, \ldots$ we obtain

$$\begin{cases} \dfrac{R}{4\mu(m-1)}[c_3(m-2)\mu - c_0\mu_0 m]X_{m1} + c_4 s_0 K_m'(s_0 R) X_{m2} - \dfrac{c_0 m}{R} X_{m3} = \alpha_m, \\ \dfrac{R}{4\mu(m-1)}[-c_3\mu + c_0\mu_0(m-2)m]X_{m1} + \dfrac{c_4 m}{R} K_m(s_0 R) X_{m2} + \dfrac{c_0 m}{R} X_{m3} = \beta_m, \\ X_{m1} + K_m(s_0 R) X_{m2}. \end{cases} \tag{18}$$

We solve systems (16), (17) and (18). The obtained values of vectors $\boldsymbol{X_{mk}}$ are substituted into (14). Then, using formulas (7), (9) and (5), we obtain the solution of the considered problem I.

The problem II is solved similarly. Rewrite boundary conditions $(2)_2$ in the form

$$\left[ \begin{array}{l} \mathbf{P}_1(\partial_z, \mathbf{n})\, \mathbf{U}(\mathbf{z})_n = f_n(z), \quad \mathbf{P}_1(\partial_z, \mathbf{n})\, \mathbf{U}(\mathbf{z})_s = f_s(z), \\ \alpha \partial_r \varphi(\mathbf{z}) = f_3(\mathbf{z}), \end{array} \right. \tag{19}$$

where

$$\mathbf{P}_1(\partial_x, \mathbf{n})\, \mathbf{U}(\mathbf{x})_n = \mu_0 \partial_r u_n(\mathbf{x}) + \frac{\lambda}{r} \partial_\psi u_s(\mathbf{x}) + \beta\varphi(\mathbf{x}),$$

$$\mathbf{P}_1(\partial_x, \mathbf{n})\, \mathbf{U}(\mathbf{x})_s = \mu[\partial_r u_s(\mathbf{x}) + \frac{1}{r} \partial_\psi u_n(\mathbf{x})],$$

$$\partial_n \varphi(\mathbf{x}) = \partial_r \varphi(\mathbf{x}).$$

Substitute (13) and (14) into (19). Passing to the limit as $r \to R$, we obtain the algebraic systems of equations:

For $m = 0$ we have

$$\begin{cases} \left(c_3 \dfrac{\mu_0}{2} + \beta\right) X_{01} + [c_4 s_0^2 K_0''(s_0 R) + \beta K_0(s_0 R)] X_{02} = \dfrac{\alpha_0}{2}, \\ c_0 \dfrac{\mu_0}{2} X_{01} + \dfrac{\mu}{R^2} X_{03} = \dfrac{\beta_0}{2}, \\ s_0 K_0'(s_0 R) X_{02} = \dfrac{\gamma_0}{2}, \end{cases}$$

where $X_{03} \equiv c_2$.

For $m = 1$, we have

$$\begin{cases} \dfrac{\mu_0}{4}\left(\dfrac{c_0}{\mu} + c_3\right) X_{11} + \left[c_4\mu_0 s_0^2 K_1''(s_0R) - \dfrac{c_4\lambda}{R} K_1(s_0R) + \beta K_1(s_0R)\right] X_{12} \\ + \dfrac{c_0}{R}\dfrac{2\mu_0}{R} X_{13} = \alpha_1, \\ \dfrac{1}{4}\left(c_3R + \dfrac{c_0\mu_0}{\mu}\right) X_{11} + \dfrac{c_4}{R}\left[-\dfrac{1}{R}K_1(s_0R) + 2s_0K_1'(s_0R)\right] X_{12} - \dfrac{3}{R^2}c_0X_{13} = \dfrac{\beta_1}{\mu}, \\ -\dfrac{1}{R} + s_0K_1'(s_0R)X_{12} = \gamma_1. \end{cases}$$

For $m = 2, 3, \ldots$ we get

$$\begin{cases} \dfrac{1}{4}\left[\dfrac{c_0\mu_0 m}{\mu} - c_3(m-2)\mu_0 + \dfrac{c_3mR\lambda}{m-1}\beta - \dfrac{c_0\lambda R\mu_0 m(m-2)}{\mu(m-1)}\right] X_{m1} \\ + \left[c_4\mu_0 s_0^2 K_m''(s_0R) - \dfrac{c_4\lambda m^2}{R} K_m(s_0R) + \beta K_m(s_0R)\right] X_{m2} \\ + \left[\dfrac{c_0\mu_0 m(m+1)}{R^2} - \dfrac{c_0\lambda m^2}{R}\right] X_{m3} = \alpha_m, \\ \dfrac{1}{4}\left[c_3mR - \dfrac{c_0\mu_0(m-2)}{\mu} - \dfrac{c_3m(m-2)}{m-1} - \dfrac{c_0\mu_0 m^2}{\mu}(m-1)\right] X_{m1} \\ + \dfrac{1}{R}\left[c_4m[-\dfrac{1}{R}K_m(s_0R) + s_0K_m'(s_0R)] + c_4s_0mK_m'(s_0R)\right] X_{m2} \\ - \dfrac{(2m+1)c_0m}{R^2} X_{m3} = \dfrac{\beta_m}{\mu}, \\ -\dfrac{m}{R}X_{m1} + s_0K_m'(s_0R)X_{m2} = \gamma_m. \end{cases}$$

Solving these systems and substituting the obtained values of the vectors into (14), by formulas (7), (9) and (5) we obtain the solution of Problem II.

For the obtained series to converge absolutely and uniformly it suffices to require that

$$\mathbf{f}, f_3 \in C^3(K)$$

In Problem I:

$$\mathbf{f}, f_3 \in C^2(K)$$

In Problem II:

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