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ON THE EFFECTIVE SOLUTION OF THE FIRST BOUNDARY VALUE PROBLEM OF STATICS OF THE THEORY OF ELASTIC MIXTURE IN A CIRCULAR RING

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Abstract. The paper deals with the construction of explicit solution of the first boundary value problem (when on the boundary is given a displacement vector) of the homogeneous equation of statics of the elastic mixture in the case of a circular ring. The solution is represented by absolutely and uniformly convergent series.

Keywords and phrases: Elastic mixture first boundary, value problem, Kolosov -Muskhelishvuli type formula, Laurent series.

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1. Introduction

The basic two-dimensional boundary value problems of statics of the linear theory of elastic mixtures are studied in [1], [2], [4], [5] and also by many other authors. The paper deals with the construction of explicit solution to the first boundary value problem of the linear theory of elastic mixture in the case of a circular ring. For the solution of the problem will be used the generalized Kolosov-Muskhelisvhiis formula [2] and the method developed in [3] and [5] The solution is obtained in the form of absolutely and uniformly convergent series.

2. Basic equation and boundary value problem

The homogeneous equation of statics of the linear theory of elastic mixture complex form is written as [2]

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + K \frac{\partial^2 \overline{U}}{\partial \overline{z}^2} = 0. \quad U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix}, \tag{1}$$

where $u_p, \quad p = \overline{1,4}$ are components of the displacement vector, $z = x_1 + ix_2$,
$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)$,

$$K = -\frac{1}{2}em^{-1} \qquad e = \begin{bmatrix} e_4 & e_5 \\ e_5 & e_6 \end{bmatrix}, \qquad m^{-1} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}^{-1}, \qquad \Delta_0 = det|m| > 0,$$

$m_k, \quad e_{3+k}, \quad k = 1,2,3$ are expressed in terms of elastic constants [2] or [5].

In [2] by M. Basheleishvili was obtained the representation (Kolosov-Muskhelis hvili type formula)

$$U = (u_1 + iu_2, u_3 + iu_4)^T = m\varphi(z) + \frac{1}{2}ez\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{2}$$

where $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions.

The following formulas are valid (see [1], pp. 2-6)

$$\frac{2(m_1 + X_j m_2)}{e_4 + X_j e_5} = \frac{2(m_2 + X_j m_3)}{e_5 + X_j e_6} = -\frac{1}{K_j}, \qquad K_j \neq 0, \quad |K_j| < 1, \quad j = 1,2 \tag{3}$$

where X_1 and X_2 are real constants, $X_1 - X_2 \neq 0$

Let $G = \{r < |z| < 1\}$, $\Gamma_0 = \{|z| = 1\}$, $\Gamma_1 = \{|z| = r\}$, $\Gamma = \{\Gamma_0 \bigcup \Gamma_1\}$. We consider the following problem. Find in the domain G, a vector $U = (u_1 + iu_2, u_3 + iu_4)^T$ which belongs to the class $C^2(G) \bigcap C^{1,\alpha}(G \bigcup \Gamma)$, is a solution of equation (1) and satisfies the following boundary conditions on Γ

$$U(t) = m\varphi(t) + \frac{1}{2} et\overline{\varphi'(t)} + \overline{\psi(t)} = \begin{cases} cf(\theta_0), t = e^{i\theta_0} \\ F(\theta_0), t = re^{i\theta_0} \end{cases}, 0 \leq \theta_0 \leq 2\pi, \tag{4}$$

where $f = (f_1, f_2)^T$ and $F = (F_1, F_2)^T$ are given complex vector-functions. Below we assume that f'' and F'' belongs to the Dirichlet class, $0 \leq \theta_0 \leq 2\pi$.
The following theorem is valid [4].

Theorem. *The problem (1), (4) is uniquely solvable.*
To solve the problem we use the method described in [3] (§ 59, also in [5]).
Let us expand the boundary vector-functions f and F into a Laurent series:

$$f(\theta_0) = \sum_{k=-\infty}^{\infty} f_k e^{ik\theta_0}, \qquad F(\theta_0) = \sum_{k=-\infty}^{\infty} F_k e^{ik\theta_0}, \qquad 0 \leq \theta_0 \leq 2\pi, \tag{5}$$

where $f_k = (f_{k1}, f_{k2})^T$ and $F_k = (F_{k1}, F_{k2})^T$; $(k = 0, \pm 1, \pm 2, \pm ...)$ are the Laurent coefficients

We look for the vector-functions φ and ψ in the following form

$$\varphi(t) = \gamma \ln t + \sum_{k=-\infty}^{\infty} \beta_k t^k, ; \qquad \psi(t) = m\overline{\gamma} \ln t + \sum_{k=-\infty}^{\infty} \gamma_k t^k; \quad (r < |t| < 1) \tag{6}$$

where $\gamma = (\gamma_1, \gamma_2)^T, \beta_k = (\beta_{k1}, \beta_{k2})^T$ and $\gamma_k = (\gamma_{k1}, \gamma_{k2})$, $(k = 0, \pm 1 \pm 2, \pm ...)$ are unknown coefficeints. Further note that $\gamma_0 = m\overline{\beta_0}$, $\gamma = -\frac{X+iY}{4\pi}$, where $X + iY$ is the principal vector of stresses applied on Γ_1 (see [4]).

Taking into account (5) and (6) in (4) we obtain the following systems:

$$2m\beta_0 = f_0, \qquad \gamma \ln r + 2m\beta_0 = F_0, \tag{7}$$

$$m\beta_2 + \frac{1}{2} e\overline{\gamma} + \overline{\gamma_{-2}} = f_2, \qquad r^4 m\beta_2 + \frac{1}{2} r^2 e\overline{\gamma} + \overline{\gamma_{-2}} = r^2 F_2, \tag{8}$$

$$m\beta_{k+1} + \frac{1}{2}(1-k)e\overline{\beta}_{-k+1} + \overline{\gamma}_{-k-1} = f_{k+1},$$

$$(9)$$

$$r^{2k+2} m\beta_{k+1} + \frac{1}{2}(1-k) r^2 e\overline{\beta}_{-k+1} + \overline{\gamma}_{-k-1} = r^{k+1} F_{k+1},$$

From (9) we get

$$\begin{cases}(r^{-k}-r^{k+2})m\beta_{k+1}+\frac{1}{2}(1-k)(1-r^2)r^{-k}e\overline{\beta}_{-k+1}=r^{-k}f_{k+1}-rF_{k+1},\\ \frac{1}{2}(1+k)(1-r^2)r^{k}e\beta_{k+1}+(r^{k}-r^{2-k})m\overline{\beta}_{-k+1}=r^{k}\overline{f}_{-k+1}-r\overline{F}_{-k+1}.\end{cases}\tag{10}$$

Owing to (9) we can rewrite the system (10) as follows:

$$\begin{pmatrix} m_1+X_jm_2\\ m_2+X_jm_3\end{pmatrix}\left[(r^{-k}-r^{k+2})\beta_{k+1}-K_j(1-k)(1-r^2)r^{-k}\overline{\beta}_{-k+1}\right]=$$

$$=\begin{pmatrix}1\\ X_j\end{pmatrix}(r^{-k}f_{k+1}-rF_{k+1}),\quad j=1,2 \qquad (11)_j$$

$$\begin{pmatrix} m_1+X_jm_2\\ m_2+X_jm_3\end{pmatrix}\left[K_j(1+k)(r^2-1)r^{k}\beta_{k+1}+(r^{k}-r^{2-k})\overline{\beta}_{-k+1}\right]=$$

$$=\begin{pmatrix}1\\ X_j\end{pmatrix}(r^{k}\overline{f}_{-k+1}-r\overline{F}_{-k+1}),\quad j=1,2, \qquad (12)_j$$

From $(11)_1-(12)_1$ and $(11)_2-(12)_2$ we have e.g. (due to the Kramers theorem)

$$\begin{pmatrix} m_1+X_1m_2\\ m_2+X_1m_3\end{pmatrix}\beta_{k+1}=\frac{g^{(1)}}{D^{(1)}_{k+1}},\qquad \begin{pmatrix} m_1+X_2m_2\\ m_2+X_2m_3\end{pmatrix}\beta_{k+1}=\frac{g^{(2)}}{D^{(2)}_{k+1}}, \qquad (13)^{(1)}$$

$$\begin{pmatrix} m_1+X_1m_2\\ m_2+X_1m_3\end{pmatrix}\overline{\beta_{-k+1}}=\frac{h^{(1)}}{D^{(1)}_{k+1}},\qquad \begin{pmatrix} m_1+X_2m_2\\ m_2+X_2m_3\end{pmatrix}\overline{\beta_{-k+1}}=\frac{h^{(2)}}{D^{(2)}_{k+1}} \qquad (13)^{(2)}$$

here $k=\pm2,\pm3,\pm...$, and

$$D^{(j)}_{k+1}=1+r^4-r^2(r^{2k}+r^{-2k})-K_j^2(1-k^2)(1-r^2)^2,\quad j=1,2.$$

It is easy to check that $D^{(j)}_1>0,\quad j=1,2,$ and $D^{(j)}_{k+1}<0\quad j=1,2\quad k=\pm2,\pm3,\pm...$ (see [3], p. 210, footnote 2) or [5]).

Remarking that the determinant of the system $(13)^{(j)}$ $j=1,2$ equal to (X_2-X_1) $det|m|$ and different from zero.

By the above given reasoning we can conclude that from the system (7), (8), $(13)^{(1)}$,$(13)^{(2)}$ and (9) it is possible to define uniquely coefficients γ,β_k and $\gamma_k, k=0,\pm1\pm2\pm...$ by means of Laurent coefficients of the f and F vector-functions (see(5)). further, note that f'' and F'' belongs to the Dirichlet class.

Having found the coefficients γ,β_k and $\gamma_k(k=0,\pm1\pm2\pm...)$ using formulas (6) we can find $\varphi(t)$ and $\psi(t)$ given by absolutely and uniformly convergent series.

Using the expressions of the above-mentioned vector-functions and substituting them into the expression for the displacement vector (see(2)) we obtain the solution of the posed problem given by absolutely and uniformly convergent series.

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