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NECESSARY OPTIMALITY CONDITIONS OF DELAYS PARAMETERS FOR ONE CLASS OF CONTROLLED FUNCTIONAL DIFFERENTIAL EQUATION WITH THE DISCONTINUOUS INITIAL CONDITION

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Abstract. The nonlinear optimal control problem with constant delays in the phase coordinates and controls is considered. The necessary conditions of optimality are obtained for the initial and final moments, for delays parameters and the initial vector, for the initial function and control.

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Let $O \subset \mathbb{R}^n$ be an open set and let $U \subset \mathbb{R}^r$ be a convex and compact set. Let $0 < \tau_1 < \tau_2$, $0 < \theta_1 < \theta_2$ be given numbers and the n-dimensional function $f(t, x, x_1, u, u_1), (t, x, x_1, u, u_1) \in I \times O^2 \times U^2$, where $I = [a, b]$, satisfies the following conditions:

a) for almost all fixed $t \in I$ the function $f(t, x, x_1, u, u_1)$ is continuously differentiable with respect to $(x, x_1, u, u_1) \in O^2 \times U^2$;

b) for each fixed $(x, x_1, u, u_1) \in O^2 \times U^2$ the functions

$$f(t, x, x_1, u, u_1), f_x(t, \cdot), f_{x_1}(t, \cdot), f_u(t, \cdot), f_{u_1}(t, \cdot)$$

are measurable on I;

c) for any compact set $K \subset O$ there exists a function $m_K(t) \in L_1(I, [0, \infty))$, such that for any $(x, x_1, u, u_1) \in K^2 \times U^2$ and for almost all $t \in I$ we have

$$|f(t, x, x_1, u, u_1)| + |f_x(t, \cdot)| + |f_{x_1}(t, \cdot)| + |f_u(t, \cdot)| + |f_{u_1}(t, \cdot)| \leq m_K(t).$$

Furthermore, let $N \subset O$ and let $X_0 \subset O$ be convex compact sets; let Φ and Ω be sets of continuously differentiable functions $\varphi : I_1 = [\hat{\tau}, b] \to N$ and $u : I_2 = [\hat{\theta}, b] \to U$, respectively, where $\hat{\tau} = a - \tau_2$ and $\hat{\theta} = a - \theta_2$.

To each element

$$\nu = (t_0, t_1, \tau, \theta, x_0, \varphi, u) \in A = [a, b) \times (a, b] \times [\tau_1, \tau_2] \times [\theta_1, \theta_2] \times X_0 \times \Phi \times \Omega$$

on the interval $[t_0, t_1]$ we assign the controlled delay functional differential equation

$$\dot{x}(t) = f(t, x(t), x(t - \tau), u(t), u(t - \theta)) \quad (1)$$

with the discontinuous initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0), x(t_0) = x_0. \quad (2)$$

The condition (2) is called discontinuous because, in general, $x(t_0) \neq \varphi(t_0)$.

Definition 1 . Let $\nu = (t_0, t_1, \tau, \theta, x_0, \varphi, u) \in A$. A function $x(t) = x(t; \nu) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b]$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element ν and defined on the interval $[t_0, t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let the scalar-valued functions $q^i(t_0, t_1, \tau, \theta, x_0, x_1),\ i = \overline{0, l}$, be continuously differentiable on $I^2 \times [\tau_1, \tau_2] \times [\theta_1, \theta_2] \times O^2$.

Definition 2. An element $\nu = (t_0, t_1, \tau, \theta, x_0, \varphi, u) \in A$ is said to be admissible if the corresponding solution $x(t) = x(t; \nu)$ satisfies the boundary conditions

$$q^i(t_0, t_1, \tau, \theta, x_0, x(t_1)) = 0,\ i = \overline{1, l}. \tag{3}$$

Denote by A_0 the set of admissible elements.

Definition 3. An element $\nu_0 = (t_{00}, t_{10}, \tau_0, \theta_0, x_{00}, \varphi_0, u_0) \in A_0$ is said to be optimal if for an arbitrary element $\nu \in A_0$ the inequality

$$q^0(t_{00}, t_{10}, \tau_0, \theta_0, x_{00}, x_0(t_{10})) \leq q^0(t_0, t_1, \tau, \theta_0, x_0, x(t_1)) \tag{4}$$

holds, where $x_0(t) = x(t; \nu_0),\ x(t) = x(t; \nu)$.

The problem (1)-(4) is called the optimal control problem with delays in the phase coordinates and controls, and with the discontinuous initial condition.

Theorem 1. *Let ν_0 be an optimal element with $t_{00}, t_{10} \in (a, b)$ and $t_{00} + \tau_0 < t_{10}$, and let the following conditions hold:*
1) *the function $f(w, u, u_1)$, where $w = (t, x, x_1)$, is bounded on $I \times O^2 \times U^2$;*
2) there exists the finite limit

$$\lim_{w \to w_0} f_0(w) = f_0^-, w \in (a, t_{00}] \times O^2,$$

where

$$f_0(w) = f(w, u_0(t), u_0(t - \theta_0)), w_0 = (t_{00}, x_{00}, \varphi_0(t_{00} - \tau_0));$$

3) there exists the finite limit

$$\lim_{(w_1, w_2) \to (w_{10}, w_{20})} \Big[f_0(w_1) - f_0(w_2)\Big] = f_1,\ w_1, w_2 \in I \times O^2,$$

where

$$w_{10} = (t_{00} + \tau_0, x_0(t_{00} + \tau_0), x_{00}), w_{20} = (t_{00} + \tau_0, x_0(t_{00} + \tau_0), \varphi_0(t_{00})).$$

4) there exists the finite limit

$$\lim_{w \to w_{30}} f_0(w) = f_2^-, w \in (t_{00}, t_{10}] \times O^2,$$

where $w_{30} = (t_{10}, x_0(t_{10}), x_0(t_{10} - \tau_0))$.
Then there exist a vector $\pi = (\pi_0, ..., \pi_l) \neq 0$, with $\pi_0 \leq 0$, *and a solution $\psi(t) = (\psi_1(t), ..., \psi_n(t))$ of the equation*

$$\dot{\psi}(t) = -\psi(t) f_{0x}[t] - \psi(t + \tau_0) f_{0x_1}[t + \tau_0], t \in [t_{00}, t_{10}], \tag{5}$$

with the initial condition

$$\psi(t) = 0, t > t_{10},$$

where $f_{0x}[t] = f_{0x}(t, x_0(t), x_0(t - \tau_0))$, such that the following conditions hold:
5) *the conditions for the moments t_{00} and t_{10} :*

$$\pi Q_{0t_0} \geq \psi(t_{00})f_0^- + \psi(t_{00}+\tau_0)f_1, \;\; \pi Q_{0t_1} \geq -\psi(t_{10})f_2^-,$$

where

$$Q_{0t_0} = \frac{\partial}{\partial t_0} Q(t_{00}, t_{10}, \tau_0, \theta_0, x_{00}, x_0(t_{10})), Q = (q^0, ..., q^l)^T;$$

6) *the conditions for the delay* τ_0

$$\pi Q_{0\tau_0} = \psi(t_{00}+\tau_0)f_1 + \int_{t_{00}}^{t_{00}+\tau_0} \psi(t) f_{0x_1}[t]\dot{\varphi}_0(t-\tau_0)dt$$

$$+ \int_{t_{00}+\tau_0}^{t_{10}} \psi(t) f_{0x_1}[t]\dot{x}_0(t-\tau_0)dt;$$

7) *the condition for the delay* θ_0

$$\pi Q_{0\theta} = \int_{t_{00}}^{t_{10}} \psi(t) f_{0u_1}[t]\dot{u}_0(t-\theta_0)dt;$$

8) *the condition for the vector* x_{00},

$$(\pi Q_{0x_0} + \psi(t_{00}))x_{00} = \max_{x_0 \in X_0} (\pi Q_{0x_0} + \psi(t_{00}))x_0;$$

9) *the integral maximum principle for the initial function* $\varphi_0(t)$,

$$\int_{t_{00}-\tau_0}^{t_{00}} \psi(t+\tau_0) f_{0x_1}[t+\tau_0]\varphi_0(t)dt = \max_{\varphi(t)\in\Phi} \int_{t_{00}-\tau_0}^{t_{00}} \psi(t+\tau_0) f_{0x_1}[t+\tau_0]\varphi(t)dt;$$

10) *the integral maximum principle for the control function* $u_0(t)$,

$$\int_{t_{00}}^{t_{10}} \psi(t)\Big[f_{0u}[t]u_0(t) + f_{0u_1}[t]u_0(t-\theta_0)\Big]dt = \max_{u(t)\in\Omega} \int_{t_{00}}^{t_{10}} \psi(t)\Big[f_{0u}[t]u(t)$$

$$+ f_{0u_1}[t]u(t-\theta_0)\Big]dt;$$

11) *the condition for the function* $\psi(t)$

$$\psi(t_{10}) = \pi Q_{0x_1}.$$

Theorem 2. *Let ν_0 be an optimal element with $t_{00}, t_{10} \in (a,b)$ and $t_{00}+\tau_0 < t_{10}$, and the conditions 1), 3) of Theorem 1 hold. Moreover, there exists the finite limits*

$$\lim_{w\to w_0} f_0(w) = f_0^+, w \in [t_{00}, t_{10}) \times O^2, \lim_{w\to w_{30}} f_0(w) = f_2^+, w \in [t_{10}, b) \times O^2,$$

Then there exist a vector $\pi = (\pi_0, ..., \pi_l) \neq 0$, with $\pi_0 \leq 0$, and a solution $\psi(t) = (\psi_1(t), ..., \psi_n(t))$ of equation (5) such that conditions 6)-11) hold. Moreover,

$$\pi Q_{0t_0} \leq \psi(t_{00})f_0^+ + \psi(t_{00}+\tau_0)f_1, \;\; \pi Q_{0t_1} \leq -\psi(t_{10})f_2^+.$$

Theorems 1,2 are proved by the scheme given in [1,2].

Theorem 3. *Let ν_0 be an optimal element with $t_{00}, t_{10} \in (a,b)$ and $t_{00}+\tau_0 < t_{10}$, and the conditions of Theorems 1 and 2 hold. Moreover,*

$$f_0^- = f_0^+ := f_0, \;\; f_2^- = f_2^+ := f_2.$$

Then there exist a vector $\pi = (\pi_0, ..., \pi_l) \neq 0$, with $\pi_0 \leq 0$, and a solution $\psi(t) = (\psi_1(t), ..., \psi_n(t))$ of equation (5) such that conditions 6)-11) hold. Moreover,

$$\pi Q_{0t_0} = \psi(t_{00})f_0 + \psi(t_{00}+\tau_0)f_1, \;\; \pi Q_{0t_1} = -\psi(t_{10})f_2.$$

Theorem 3 is a corollary to Theorems 1 and 2. All assumptions of Theorem 3 are satisfied if the function $f(t,x,x_1,u,u_1)$ is continuous and bounded. In this case we have

$$f_0 = f(t_{00}, x_0(t_{00}), x_0(t_{00}-\tau_0), u_0(t_{00}), u_0(t_{00}-\theta_0)), f_2 = f(t_{10}, x_0(t_{10}), x_0(t_{10}-\tau_0),$$

$$u_0(t_{10}), u_0(t_{10}-\theta_0)); f_1 = f(t_{00}+\tau_0, x_0(t_{00}+\tau_0), x_{00}, u_0(t_{00}+\tau_0), u_0(t_{00}+\tau_0-\theta_0))$$

$$-f(t_{00}+\tau_0, x_0(t_{00}+\tau_0), \varphi_0(t_{00}), u_0(t_{00}+\tau_0), u_0(t_{00}+\tau_0-\theta_0)).$$

Theorem 4. *Let $f(t,x,x_1,u,u_1) = A(t)x+B(t)x_1+C(t)u+D(t)u_1$, where $A(t), B(t), C(t)$ and $D(t)$ are continuous matrix functions. Let ν_0 be an optimal element with $t_{00}, t_{10} \in (a,b)$ and $t_{00}+\tau_0 < t_{10}$. Then there exists a vector $\pi = (\pi_0, ..., \pi_l) \neq 0$, with $\pi_0 \leq 0$, and a solution $\psi(t) = (\psi_1(t), ..., \psi_n(t))$ of the equation*

$$\dot{\psi}(t) = -\psi(t)A(t) - \psi(t+\tau_0)B[t+\tau_0], t \in [t_{00}, t_{10}], \;\; \psi(t) = 0, t > t_{10},$$

such that the following conditions hold:
the conditions for the moments t_{00} and t_{10} :

$$\pi Q_{0t_0} = \psi(t_{00})\Big(A(t_{00})x_{00} + B(t_{00})\varphi_0(t_{00}-\tau_0) + C(t_{00})u_0(t_{00}) + D(t_{00})u_0(t_{00}-\theta_0)\Big)$$

$$+\psi(t_{00}+\tau_0)B(t_{00}+\tau_0)\Big(x_{00} - \varphi_0(t_{00})\Big),$$

$$\pi Q_{0t_1} = -\psi(t_{10})\Big(A(t_{10})x_0(t_{10}) + B(t_{10})x_0(t_{10}-\tau_0) + C(t_{10})u_0(t_{10}) + D(t_{10})u_0(t_{10}-\theta_0)\Big);$$

the condition for the delay τ_0

$$\pi Q_{0\tau_0} = \psi(t_{00}+\tau_0)B(t_{00}+\tau_0)\Big(x_{00} - \varphi_0(t_{00})\Big) + \int_{t_{00}}^{t_{00}+\tau_0} \psi(t)B(t)\dot{\varphi}_0(t-\tau_0)dt$$

$$+\int_{t_{00}+\tau_0}^{t_{10}} \psi(t)B(t)\dot{x}_0(t-\tau_0)dt;$$

the condition for the delay θ_0

$$\pi Q_{0\theta} = \int_{t_{00}}^{t_{10}} \psi(t)D(t)\dot{u}_0(t-\theta_0)dt;$$

the condition 8) for the vector x_{00};
the integral maximum principle for the initial function $\varphi_0(t)$,

$$\int_{t_{00}-\tau_0}^{t_{00}} \psi(t+\tau_0)B(t+\tau_0)\varphi_0(t)dt = \max_{\varphi(t)\in\Phi} \int_{t_{00}-\tau_0}^{t_{00}} \psi(t+\tau_0)B(t+\tau_0)\varphi(t)dt;$$

the integral maximum principle for the control function $u_0(t)$,

$$\int_{t_{00}}^{t_{10}} \psi(t)\Big[C(t)u_0(t)+D(t)u_0(t-\theta_0)\Big]dt = \max_{u(t)\in\Omega} \int_{t_{00}}^{t_{10}} \psi(t)\Big[C(t)u(t)+D(t)u(t-\theta_0)\Big]dt;$$

the condition 11) for the function $\psi(t)$.

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