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WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR ONE CLASS OF NEUTRAL QUASI-LINEAR DIFFERENTIAL EQUATION WITH DISTRIBUTED DELAY

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Abstract. Theorems on the continuous dependence of a solution on perturbations of the initial data and the right-hand side are given for a neutral differential equation with distributed delay in the phase coordinates and whose right-hand side is linear with respect to phase velocity. The perturbations of the initial data (initial moment, initial vector, initial functions, initial matrix, distributed delay) are small in the standard norm, the perturbation of the right-hand side of equation is small in the integral sense.

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Let $I=[a,b]$ ba a finite interval and let $\mathbb{R}^n$ be the n-dimensional vector space of points $(x^1,\dots,x^n)^\top$, where $\top$ is the sign of transposition. Suppose that $O\subset\mathbb{R}^n$ is an open set and let E_f be the set of functions $f: I\times O^2\to\mathbb{R}^n$ which satisfy the following conditions: for each fixed $(x,y)\in O^2$ the function $f(\,\cdot\,,x,y): I\to\mathbb{R}^n$ is measurable; for each $f\in E_f$ and compact set $K\subset O$ there exist functions $M_{f,K}(t), L_{f,K}(t)\in L(I,R_+)$, where $\mathbb{R}_+=[0,+\infty)$ such that for almost all $t\in I$

$$|f(t,x,y)|\le M_{f,K}(t)\ \ \forall\,(x,y)\in K^2,$$

$$\Big|f(t,x_1,y_1)-f(t,x_2,y_2)\Big|\le L_{f,K}(t)\Big[\,|x_1-x_2|+|y_1-y_2|\,\Big]\ \ \forall\,(x_i,y_i)\in K^2,\ \ i=1,2.$$

We introduce the topology in E_f by the following basis of neighborhoods of zero:

$$\Big\{V_{K,\delta}:\ K\subset O \text{ is a compact set and } \delta>0 \text{ is an arbitrary number}\Big\},$$

where

$$V_{K,\delta}=\Big\{\delta f\in E_f:\ \Delta(\delta f;K)\le\delta\Big\},\Delta(\delta f;K)=\sup\Big\{\Big|\int\limits_{t'}^{t''}\delta f(t,x,y)\,dt\Big|$$

$$:\ t',t''\in I,\ x,y\in K\Big\}.$$

Let E_φ be the space of piecewise-continuous functions $\varphi(t)\in\mathbb{R}^n, t\in I_1=[\hat\tau,b]$, with finitely many first kind discontinuities equipped with the norm $\|\varphi\|_{I_1}=\sup\{\varphi(t):\ t\in I_1\}$. Let $\Phi_1=\{\varphi\in E_\varphi: cl\varphi(I_1)\subset O\}$ denote the set of initial functions of the trajectory, where $\varphi(I_1)=\{\varphi(t):\ t\in I_1\}$; we denote by Φ_2 the set of bounded measurable functions

$v : I_1 \to \mathbb{R}^n$ and $v(t)$ is called the initial function of the trajectory derivative. Let $\mathbb{R}^{n\times n}$ be the space of matrices $A = (a_{ij})_{i,j=1}^n$, $|A|^2 = \sum\limits_{i,j=1}^{n} |a_{ij}|^2$. Let Λ be the space of continuous matrix functions $A : I \to \mathbb{R}^{n\times n}$, $\|A\| = \sup\{|A(t)| :\ t \in I\}$. We denote by μ the collection of initial data $\mu = (t_0, \tau, x_0, A, \varphi, v, f) \in \mathfrak{M} = [a, b) \times D \times O \times \Lambda \times \Phi_1 \times \Phi_2 \times E_f$ we assign the neutral differential equation which is linear and nonlinear with respect to the phase velocity and distributed prehistory on the interval $[\tau(t), t]$, respectively:

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + \int\limits_{\tau(t)}^{t} f(t, x(t), x(s))\, ds \tag{1}$$

with the initial condition

$$x(t) = \varphi(t),\ \ \dot{x}(t) = v(t),\ \ t \in [\widehat{\tau}, t_0),\ \ x(t_0) = x_0. \tag{2}$$

Here $\sigma \in D$ is a fixed delay function in the phase velocity with $\sigma(t) < t$. The symbol $\dot{x}(t)$ on the interval $[\widehat{\tau}, t_0)$ is not connected with the derivative of the function $\varphi(t)$.

Definition 1. Let $\mu = (t_0, \tau, x_0, A, \varphi, v, f) \in \mathfrak{M}$. A function $x(t) = x(t; \mu) \in O$, $t \in [\hat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called the solution of equation (1) with the initial condition (2) or the solution corresponding to the element μ and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies condition (2), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

To formulate the main results, we introduce the following set:

$$W(K;\alpha) = \Big\{\delta f \in E_f :\ \exists\, M_{f,K}(t), L_{f,K}(t) \in L(I, R_+),\ \int\limits_I \big[M_{f,K}(t) + L_{f,K}(t)\big]\, dt \le \alpha\Big\},$$

where $K \subset O$ is a compact set and $\alpha > 0$ is a fixed number not depended on δf. The set $W(K;\alpha)$ is called the class of perturbations of the right-hand side of equation (1).

Furthermore,

$$B(t_{00};\delta) = \big\{t_0 \in I :\ |t_0 - t_{00}| < \delta\big\},\quad B_1(x_{00};\delta) = \big\{x_0 \in O :\ |x_0 - x_{00}| < \delta\big\},$$

$$V(\tau_0;\delta) = \big\{\tau \in D :\ \|\tau - \tau_0\|_{I_2} < \delta\big\},\quad V_1(A_0;\delta) = \big\{A \in \Lambda :\ \|A - A_0\|_I < \delta\big\},$$

$$V_2(\varphi_0;\delta) = \big\{\varphi \in \Phi_1 : \|\varphi - \varphi_0\|_{I_1} < \delta\big\},\quad V_3(v_0;\delta) = \big\{v \in \Phi_2 :\ \|v - v_0\|_{I_1} < \delta\big\},$$

where $t_{00} \in [a, b)$ and $x_{00} \in O$ are fixed points, $\tau_0 \in D$, $\varphi_0 \in \Phi_1$, $v_0 \in \Phi_2$ are fixed points, $\delta > 0$ is a fixed number, $I_2 = [a, \widehat{\gamma}]$, $\|\tau\|_I = \sup\{|\tau(t)| :\ t \in I\}$.

Theorem 1. *Let $x_0(t)$ be the solution corresponding to the element $\mu_0 = (t_{00}, \tau_0, x_{00}, A_0, \varphi_0, v_0, f_0) \in \mathfrak{M}$ and defined on $[\hat{\tau}, t_{10}], t_{10} < b$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = cl\varphi_0(I_1) \cup x_0([t_0, t_1])$. Then the following conditions hold:*

1) there exist numbers $\delta_i > 0$, $i = 0, 1$, such that, to each element

$$\mu = (t_0, \tau, x_0, A, \varphi, v, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha) = B(t_{00};\delta_0) \times V(\tau_0;\delta_0) \times B_1(x_{00};\delta_0)$$

$$\times V_1(A_0;\delta_0)\times V_2(\varphi_0;\delta_0)\times V_3(v_0;\delta_0)\times\big[f_0+(W(K_1;\alpha)\cap V_{K_1,\delta_0})\big],$$

corresponds solution $x(t;\mu)$ defined on the interval $[\widehat{\tau},t_{10}+\delta_1]\subset I_1$ and satisfying the condition $x(t;\mu)\in K_1$;

2) for an arbitrary $\varepsilon>0$ there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0)$ such that the following inequality holds for any $\mu\in V(\mu_0;K_1,\delta_2,\alpha)$:

$$\big|x(t;\mu)-x(t;\mu_0)\big|\le\varepsilon \ \text{ for all } \ t\in[\theta,t_{10}+\delta_1],\ \ \theta=\max\{t_{00},t_0\};$$

3) for an arbitrary $\varepsilon>0$ there exists a number $\delta_3=\delta_3(\varepsilon)\in(0,\delta_0)$ such that the following inequality holds for any $\mu\in V(\mu_0;K_1,\delta_3,\alpha)$:

$$\int\limits_{\widehat{\tau}}^{t_{10}+\delta_1}\big|x(t;\mu)-x(t;\mu_0)\big|\,dt\le\varepsilon.$$

The solution $x(t;\mu_0)$ is the continuation of the solution $x_0(t)$ and to the element $\mu=(t_0,\tau,x_0,A,\varphi,v,f_0+\delta f)\in V(\mu_0;K_1,\delta_0,\alpha)$ corresponds the perturbed differential equation

$$\dot{x}(t)=A(t)\dot{x}(\sigma(t))+\int\limits_{\tau(t)}^{t}\big[f_0(t,x(t),x(s))+\delta f(t,x(t),x(s))\big]\,ds$$

with the perturbed initial condition (2).

Now we introduce the set of variations

$$\mathfrak{I}=\Big\{\delta\mu=(\delta t_0,\delta\tau,\delta x_0,\delta A,\delta\varphi,\delta v,\delta f):\ |\delta t_0|\le\beta,\ \ \|\delta\tau\|_{I_2}\le\beta,\ \ |\delta x_0|\le\beta,$$

$$\|\delta A\|_I\le\beta,\ \ \|\delta\varphi\|_{I_1}\le\beta,\ \ \delta\varphi\in\Phi_1-\varphi_0,\ \ \|\delta v\|_{I_1}\le\beta,$$

$$\delta v\in\Phi_2-v_0,\ \ \delta f=\sum_{i=1}^{k}\lambda_i\delta f_i,\ \ |\lambda_i|\le\beta,\ \ i=1,\dots,k\Big\},$$

where $\beta>0$ is a fixed number and $\delta f_i\in E_f-f_0$, $i=1,\dots,k$, are fixed functions.

Theorem 2. *Let $x_0(t)$ be the solution corresponding to the element $\mu_0\in\mathfrak{M}$ and defined on $[\widehat{\tau},t_{10}]$, $t_{i0}\in(a,b)$, $i=0,1$. Let $K_1\subset O$ be a compact set containing a certain neighborhood of the set K_0. Then the following conditions hold:*

4) there exist numbers $\varepsilon_1>0$, $\delta_1>0$ such that for an arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_1)\times\mathfrak{I}$ the element $\mu_0+\varepsilon\delta\mu\in\mathfrak{M}$ and there corresponds the solution $x(t;\mu_0+\varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau},t_{10}+\delta_1]\subset I_1$. Moreover, $x(t;\mu_0+\varepsilon\delta\mu)\in K_1$;

5) the following relations are fulfilled:

$$\lim_{\varepsilon\to0}\Big[\sup\big\{\big|x(t;\mu_0+\varepsilon\delta\mu)-x(t;\mu_0)\big|:\ t\in[\theta,t_{10}+\delta_1]\big\}\Big]=0$$

and

$$\lim_{\varepsilon\to 0}\int\limits_{\widehat{\tau}}^{t_{10}+\delta_1}\big|x(t;\mu_0+\varepsilon\delta\mu)-x(t;\mu_0)\big|\,dt=0$$

uniformly for $\delta\mu\in I$, *where* $\theta=\max\{t_{00},t_{00}+\varepsilon\delta t_0\}$.

Theorem 2 is a simple corollary of Theorem 1 .

Let $U_0\subset\mathbb{R}^r$ be an open set and let Ω be the set of measurable functions $u(t)\in U_0$, $t\in I$, satisfying the conditions: $clu(I)$ is a compact set in $\mathbb{R}^r$ and $clu(I)\subset U_0$.

To each element $\mu=(t_0,\tau,x_0,A,\varphi,v,u)\in\mathfrak{M}_1=[a,b)\times D\times O\times\Lambda\times\Phi_1\times\Phi_2\times\Omega$ we assign the controlled neutral differential equation with distributed prehistory

$$\dot{x}(t)=A(t)\dot{x}(\sigma(t))+\int\limits_{\tau(t)}^{t}f(t,x(t),x(s),u(t))\,ds \tag{3}$$

Here the function $g(t,x,y,u)$ is defined on $I\times O^2\times U_0$ and satisfies the following conditions: for each fixed $(x,y,u)\in O^2\times U_0$ the function $g(\,\cdot\,,x,y,u):I\to\mathbb{R}^n$ is measurable; for each compact sets $K\subset O$ and $U\subset U_0$ there exist functions $M_{K,U}(t),L_{K,U}(t)\in L(I,\mathbb{R}_+)$ such that for almost all $t\in I$

$$|g(t,x,y,u)|\le M_{K,U}(t)\ \ \forall\,(x,y,u)\in K^2\times U,$$

$$\big|g(t,x_1,y_1,u_1)-g(t,x_2,y_2,u_2)\big|\le L_{f,K}(t)\big[\,|x_1-x_2|+|y_1-y_2|+|u_1-u_2|\,\big]$$

$$\forall\,(x_i,y_i)\in K^2,\ \ i=1,2,\ \ \forall\,(u_1,u_2)\in U^2.$$

Definition 2. Let $\rho=(t_0,\tau,x_0,A,\varphi,v,u)\in\mathfrak{M}_1$. A function $x(t)=x(t;\rho)\in O$, $t\in[\widehat{\tau},t_1]$, $t_1\in(t_0,b]$, is a called the solution of equation (3) with the initial condition (2) or the solution corresponding to the element ρ and defined on the interval $[\widehat{\tau},t_1]$, if it satisfies condition (2), is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (3) almost everywhere on $[t_0,t_1]$.

Theorem 3. *Let $x_0(t)$ be the solution corresponding to the element $\rho_0=(t_{00},\tau_0,x_{00},A_0,\varphi_0,v_0,u_0)\in\mathfrak{M}_1$ and defined on $[\widehat{\tau},t_{10}]$, $t_{10}<b$.* Let $K_1\subset O$ *be a compact set containing a certain neighborhood of the set $K_0=cl\varphi_0(I_1)\cup x_0([t_0,t_1])$. Then the following conditions hold:*

6) there exist numbers $\delta_i>0$, $i=0,1$ such that, to each element

$$\rho=(t_0,\tau,x_0,A,\varphi,v,u)\in\widehat{V}(\rho_0;\delta_0)=B(t_{00};\delta_0)\times V(\tau_0;\delta_0)$$

$$\times B_1(x_{00};\delta_0)\times V_1(A_0;\delta_0)\times V_2(\varphi_0;\delta_0)\times V_3(v_0;\delta_0)\times V_4(u_0;\delta_0)$$

corresponds solution $x(t;\rho)$ defined on the interval $[\widehat{\tau},t_{10}+\delta_1]\subset I_1$ and satisfying the condition $x(t;\rho)\in K_1$; here $V_4(u_0;\delta_0)=\{u\in\Omega:\ \|u-u_0\|_I<\delta_0\}$.

7) for an arbitrary $\varepsilon>0$ there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0)$ such that the following inequality holds for any $\rho\in\widehat{V}(\rho_0;\delta_2)$:

$$\big|x(t;\rho)-x(t;\rho_0)\big|\le\varepsilon\ \ \forall\, t\in[\theta,t_{10}+\delta_1],\ \ \theta=\max\{t_{00},t_0\};$$

8) for an arbitrary $\varepsilon>0$ there exists a number $\delta_3=\delta_3(\varepsilon)\in(0,\delta_0)$ such that the following inequality holds for any $\rho\in\hat V(\rho_0;\delta_3)$:

$$\int\limits_{\widehat{\tau}}^{t_{10}+\delta_1}\big|x(t;\rho)-x(t;\rho_0)\big|\,dt\le\varepsilon.$$

We introduce the set of variations

$$\Im_1=\Big\{\delta\rho=(\delta t_0,\delta\tau,\delta x_0,\delta A,\delta\varphi,\delta v,\delta u):\ |\delta t_0|\le\beta,\ \ \|\delta\tau\|_{I_2}\le\beta,\ \ |\delta x_0|\le\beta,$$

$$\|\delta A\|_I\le\beta,\ \ \|\delta\varphi\|_{I_1}\le\beta,\ \ \delta\varphi\in\Phi_1-\varphi_0,$$

$$\|\delta v\|_{I_1}\le\beta,\ \ \delta v\in\Phi_2-v_0,\ \ \|\delta u\|\le\beta,\ \ \delta u\in V_4-u_0\Big\},$$

Theorem 4. *Let $x_0(t)$ be the solution corresponding to $\rho\in\mathfrak{M}_1$ and defined on $[\widehat{\tau},t_{10}]$, $t_{i0}\in(a,b)$, $i=0,1$. Let $K_1\subset O$ be a compact set containing a certain neighborhood of the set K_0. Then the following conditions hold:*

9) there exist numbers $\varepsilon_1>0$, $\delta_1>0$ such that for an arbitrary $(\varepsilon,\delta\rho)\in(0,\varepsilon_1)\times\Im_1$ the element $\rho_0+\varepsilon\delta\rho\in\mathfrak{M}_1$ and there corresponds the solution $x(t;\rho_0+\varepsilon\delta\rho)$ defined on the interval $[\widehat{\tau},t_{10}+\delta_1]\subset I_1$. Moreover, $x(t;\rho_0+\varepsilon\delta\rho)\in K_1$;

10) the following relations are fulfilled:

$$\lim\Big[\sup\big\{\big|x(t;\rho_0+\varepsilon\delta\rho)-x(t;\rho_0)\big|:\ t\in[\theta,t_{10}+\delta_1]\big\}\Big]=0$$

and

$$\lim_{\varepsilon\to0}\int\limits_{\widehat{\tau}}^{t_{10}+\delta_1}\big|x(t;\rho_0+\varepsilon\delta\rho)-x(t;\rho_0)\big|\,dt=0$$

uniformly for $\delta\rho\in\Im_1$, where $\theta=\max\{t_{00},t_{00}+\varepsilon\delta t_0\}$.

Theorem 4 is a simple corollary of Theorem (3).

Finally, we note that theorems analogous to Theorem 1 for various classes of neutral equations are given in [1–5].

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