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ON THE ADJUSTMENT OF ARROW-HURWITZ MODEL IN THE FRAME OF WALRAS GENERAL EQUILIBRIUM THEORY

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Abstract. The Arrow-Hurwitz models adjustment possibilities, in the frame of Walras general equilibrium model, on the basis of modern optimization toolbox containing Matlab, is shown.

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It's well known (see [1-2]), that Arrow-Hurwitz model is the Walras general equilibrium model's realization for the case, when we have only one resource (labor), two firms using this resource and each of them producing only one product, one customer and “fair” auctioneer, who is defining prices of each product and labor resource.

Often described in well known textbooks, iterative algorithm of resolving of this problem, on the basis of sequential approximations, is based on non strict, heuristic approaches (for example, without considering demands on products, constrains on the amount of labor resource and so on). Although, it may be quite o.k. from the point of view of teaching, but on the basis of modern computer toolboxes, it's possible to adjust them using more strict approaches, what we will try to do below. In particular, for this aim we will use optimization toolbox on the basis of Matlab.

First of all, we will use this toolbox for solution of product volume optimal levels finding problem for each firm, subject to constraint on the amount of the labor resource (i.e. second item from Arrow-Hurwitz model [1-2]).

For this, for definiteness, we will consider that the product functions are having the form

$$Y_i^s(t) = F_i(L_i^d(t)) = c_i(L_i^d(t))^{a_i}\,, \quad (a_i < 1), \quad i = 1, 2; \tag{1}$$

where $Y_i^s(t)$ denotes volume of supply of i-the product (product level) for moment t, F_i - i-th moment function, $L_i^d(t)$ –i-th firms demand amount on the labor resource, c_i and a_i - constant coefficients.

In these conditions, the firms profit functions will have the forms:

$$\pi_i(t) = P_i(t)c_i(L_i^d(t))^{a_i} - W(t)L_i^d(t)\,, \quad i = 1, 2; \tag{2}$$

where $\pi(t)$ denotes i-th firms profit value for moment t, $P_i(t)$ - i-th product price for moment t and $W(t)$- labor resource cost for moment t.

From the above-said, the problem of definition of optimal levels of production the moment t, may be formulated (for given levels of prices on products and labor resource) as $\pi_1(t)+\pi_2(t)$ function maximization problem, within the constraint:

$$L_1^d(t) + L_2^d(t) \le L^s, \tag{3}$$

where L^s denotes labor resource supply volume (which, in the frame of the given model, can be considered as constant). Let now (by introducing additional variable) constraint (3) write as the following equity

$$L_1^d(t) + L_2^d(t) + \tau^2(t) = L^s \tag{4}$$

and consider $\pi_1(t) + \pi_2(t)$ function maximization problem, within this constraint. The Lagrange function of this problem will have the form:

$$L(t) = P_1(t)c_1(L_1^d(t))^{a_i} - W(t)(L_1^d(t) + L_2^d(t) + \lambda(L_1^d(t) + L_2^d(t) + \tau^2(t) - L^s) \tag{5}$$

where λ denotes the Lagrange multiplier.

Write now necessary conditions of extremum for the Lagrange function (5). We will have:

$$\begin{cases} \frac{\partial L(t)}{\partial L_1^d} = P_1(t)c_1a_1(L_1^d(t))^{a_1-1} - W(t) + \lambda = 0, \\ \frac{\partial L(t)}{\partial L_2^d} = P_2(t)c_2a_2(L_1^d(t))^{a_2-1} - W(t) + \lambda = 0, \\ \frac{\partial L(t)}{\partial \lambda} = L_1^d(t) + L_2^d(t) + \tau^2(t) - L^s = 0, \\ \frac{\partial L(t)}{\partial \tau} = 2\lambda\tau(t) = 0. \end{cases} \tag{6}$$

Then, from the first two equations of (6), we get:

$$P_1(t)c_1a_1(L_1^d(t))^{a_1-1} = P_2(t)c_2a_2(L_1^d(t))^{a_2-1}.$$

Taking it into account, system (6) can be written as follows:

$$\begin{cases} P_1(t)c_1a_1(L_1^d(t))^{a_1-1} - W(t) + \lambda = 0, \\ P_1(t)c_1a_1(L_1^d(t))^{a_1-1} = P_2(t)c_2a_2(L_1^d(t))^{a_2-1}, \\ L_1^d(t) + L_2^d(t) + \tau^2(t) - L^s = 0, \\ \lambda\tau(t) = 0. \end{cases} \tag{7}$$

Here we can consider two cases:

I. $\tau(\mathbf{t}) = \mathbf{0}$ (i.e. full employment) case. In this case, within $L_2^d(t) = L^s - L_1^d(t)$, (6) will give,

$$P_1(t)c_1a_1(L_1^d(t))^{a_1-1} = P_2(t)c_2a_2(L^s - L_1^d(t))^{a_2-1}.$$

or, equivalently,

$$P_2(t)c_2a_2(L^s - L_1^d(t))^{1-a_2} - P_1(t)c_1a_1(L_1^d(t))^{1-a_1} = 0, \quad (0 < L_1^d(t) < L^s. \tag{8}$$

Hence, in the case of full employment $L_1^d(t)$ (and, accordingly, $L_2^d(t) = L^s - L_1^d(t))$) can be found from equation (8). At the same time, it's clear, that this equation will always have solution in the interval $(0, L^s)$, because on the ends of this interval the left side of equation (8) (the continues function) has different signs.

II. $\tau(\mathbf{t}) \neq \mathbf{0}$ (i.e. **not full employment) case**. In this case, from (7) $\lambda = 0$ and from (6) we will have:

$$\begin{cases} (L_1^d(t))^{a_1-1} = \frac{W(t)}{P_1(t)c_1a_1}, \\ \\ (L_2^d(t))^{a_2-1} = \frac{W(t)}{P_2(t)c_2a_2}, \\ \\ L_1^d(t) + L_2^d(t) + \tau^2(t) - L^s = 0. \end{cases} \tag{9}$$

It's easy to find, that the first two equations of (9) give:

$$L_1^d(t) = \left(\frac{P_1(t)c_1a_1}{W(t)}\right)^{\frac{1}{1-a_1}} \quad \text{and} \quad L_2^d(t) = \left(\frac{P_2(t)c_2a_2}{W(t)}\right)^{\frac{1}{1-a_2}} \tag{10}$$

what is identical with results from [2].

However, it's clear, that these solutions may not satisfy $L_1^d(t) + L_2^d(t) < L^s$ constraint, i.e. defined from (4) $\tau^2(t)$ may not satisfy $\tau^2(t) > 0$ condition. (I.e. (10) are giving solutions not satisfying constraint on labor resource!).

In the last, proportional division of L^s may have sense with respect to $L_1^d(t)$ and $L_2^d(t)$, found from (10) i.e. the use of the following corrected formulas:

$$(L_1^d(t))_k = \frac{L_1^d(t)}{L_1^d(t) + L_2^d(t)} L^s \quad \text{and} \quad (L_2^d(t))_k = \frac{L_2^d(t)}{L_1^d(t) + L_2^d(t)} L^s.$$

From where, for appropriate (adjusted) values we will have:

$$\begin{cases} (Y_1^s(t))_k = F_1((L_1^d(t))_k) = c_i((L_1^d(t))_k)^{a_i} \\ \\ (Y_2^s(t))_k = F_1((L_2^d(t))_k) = c_i((L_2^d(t))_k)^{a_i}. \end{cases}$$

Now, as regards customers behavior (third item from Arrow-Hurwitz model (see [2])), instead of the model from this textbook

$$Y_1^s(t) = \max\{\beta \left(\frac{\partial U(t-1)}{\partial Y_1^d} - P_1(t)\right) + Y_1^d(t-1), 0\}, \quad i = 1,\ 2;$$

whose dignities are simplicity and iterativity (although, β parameters definition rule is not clear), we can use an approach more natural for this model, which assumes customers utilities $U(Y_1^d(t), Y_2^d(t))$ functions maximization, within budget constraint

$$P_1(t)Y_1^d(t) + P_2(t)Y_2^d(t) \leq B,$$

where B denotes the volume consumers expenditures (budget).

For this, for customers utilities function we can use the well known function

$$U = b_1 \ln Y_1^d(t) + b_2 \ln Y_2^d(t).$$

We must note, that the behavioure rule of the above customers, assuming constructing of demand function, can be simply realized on the basis of using modern computer packages (for example, Matlab). Even more, when using such a packages it's easy to foresee additional constraints, such as constraints on product supply, although following the Arrow-Hurwitz models logic, it is not needed.

At last we can leave unchanged product and labor resource prices defining mechanism using by auctioneer we can, i.e. take (see [2]):

$$P_i(t+1) = \max\{\alpha(Y_i^d(t) - Y_i^s(t)) + P_i(t-1), 0\}, \quad i = 1, 2;$$

and

$$W(t+1) = \max\{\gamma(L_1^d(t) + L_2^d(t) - L^s) + W(t-1), 0\},$$

where α and γ are constant parameters.

Show now all of this on the numerical example.

Numerical example. Consider fixed moment t and assume, that

$$c_1 = 3000; \quad a_1 = 0.15; \quad c_2 = 1800; \quad a_2 = 0.2;$$

$$P_1 = 4.4; P_2 = 5; \quad W = 1.5; \quad L^s = 100.$$

I. **Full employment case.** In this case, equation (8) will have the form

$$2(L_1^d)^{0.85} - 1.8(100 - L_1^d)^{0.8} = 0.$$

Let us solve this equation using Matlab.

syms x;
solve(2*x^0.85 – 1.8*(100 – x)^0.8)
$ans =$
41.003883955972537215124761534919
i.e. approximately $L_1^d = 41.004$ and $L_2^d = 58.996$.
Thus, In this case products issues volumes are, respectively:
$Y_1^s = 3000{*}41.004^{0.15} = 5236.5$,
$Y_2^s = 2000{*}58.996^{0.2} = 4520.6$.

II. Not full employment case (more exactly, without taking into account constraint on labor resource!). In this case,
$L_1^d = (1800/1.5)^{1/0,85} = 4193.4$
and
$L_2^d = (2000/1.5)^{1/0,8} = 8057.0$.
It's obvious, that these solutions don't satisfy constraint on labor supply. Therefore, defining our corrected amounts, we will have:
$(L_1^d)_k = (4193.4/(4193.4 + 8057.0)){*}100 = 34.2307$
and
$(L_1^d)_k = (8057.0/(4193.4 + 8057.0)){*}100 = 65.7693$.
Now, for corresponding (corrected) levels of production, we will have, respectively:
$(Y_1^s)k = 3000{*}34.2307^{0.15} = 5096.6$,
$(Y_2^s)k = 2000{*}65.7693^{0.2} = 4619.9$.

Now we can analyze which variant is better: foresighted from beginning full employment, or our correcting one?

For summary profits we will have, respectively:
$(\pi_1+\pi_2)_1 = 4{*}5236.5 + 5{*}4520.6 - 1.5{*}100 = 43399$,
$(\pi_1+\pi_2)_2 = 4{*}5096.6 + 5{*}4619.9 - 1.5{*}100 = 43336$.
I.e., as it was expected, first variant is better, although the difference is not so great!

III. Use of the numerical optimization model. Let us see what we will get the numerical solution of the problem, by using optimization toolbox on the basis of Matlab. For our example, objective function of conditional optimization problem (summary profit) will have the form:
$\pi = 4{*}3000{*}(L_1^d)^{0.15} + 5{*}2000{*}(L_2^d)^{0.2} - 1.5{*}(L_1^d + L_2^d)$
$= 12000{*}(L_1^d)^{0.15} + 10000{*}(L_2^d)^{0.2} - 1.5{*}(L_1^d + L_2^d)$.
Consider this problem as $-\pi$ minimization task and construct in Matlab so called anonym function of accounting $-\pi$
$miznf = @(x)(-12000{*}x(1)\hat{}0.15 - 10000{*}x(2)\hat{}0.2 + 1.5{*}(x(1)+x(2))$;
and find its minimum within the condition

$$x(1) + x(2) \leq 100.$$

In the frame of optimtool toolbox, fmincon function (which solves conditional minimization problems) gives $x(1) = 41.004, x(2) = 58.996$, what is identical to founded above solution for full employment case.

Of course best solution corresponds to full employment case, which can be found in various paths, as shown above or, directly, using optimization toolbox on the basis of Matlab. Supposed in [2] variant does not consider the constraint on labor resource and is not realistic. And what about adjustments our version, of course it can be used, but it's better initially solve problem with full employment! (Especially, accounting existence of such a solution, when using given class of product functions!)

Now, concerning consumers choice in the frame of our numerical example, assume that consumers budget value $B = 30000$ and b_1 and b_2 coefficients values are, correspondingly, 150 and 180.

In this case, consumers choice problem can be formulated as follows: find maximum of function,

$$U = 150 \ln Y_1^d(t) + 180 \ln Y_2^d(t)\,,$$

subject to budget constraint,

$$4 \ln Y_1^d(t) + 5 \ln Y_2^d(t) \leq 30000\,.$$

Solve now, as above, this problem using optimization toolbox on the basis of Matlab.
Consider minimization problem of function $-U$, within above mentioned budget constraint.
Built in Matlab anonym function for accounting $-U$
$miznf1 = @(x)(-150{*}\log(x(1)) - 180{*}\log(x(2)))$;
and find its minimum subject to

$$4{*}x(1) + 5{*}x(2) \leq 30000.$$

In the frame of optimtool, fmincon function gives $x(1) = 3409.091, x(2) = 3272.727$. At the same time, $4{*}3409.091 + 5{*}3272.727 = 30000$, i.e. in founded optimal variant all budget is used completely (budget constraint is fulfilled as equity).

Thus, in this case, customers demand on products is, accordingly, $Y_1^d = 3409.1$ and $Y_2^d = 3272.7$, what, is less than their supplies.

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