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PERSPECTIVE MAPS OF FREE MODULES

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Abstract. In the work we have studied perspective maps for free modules over left principal ideal domains and thus to give the first ring version of the problem. Free module can be non-commutative.

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One of fundamental theorems of Geometric Algebra is the theorem about the representation of perspective maps (perspectivities) by linear functions [1, 2]. However there has been intense research activity of Geometric Algebra over rings for recent 20-40 years [3, 4] and there exists no ring version of such kind of theorems. The reason is that the geometries over fields (classical version) from lattice-theoretical point of view are the lattices with complements. That is false in case of rings. Consequently, for the modules over rings it is necessary to change the definition of the perspectivity.

The aim of the paper is to study perspective maps for free modules over (possibly non-commutative) left principal ideal domains and thus to give the first ring version of this problem.

Let R be a (possibly non-commutative) left principal ideal domain; let Ω be a free R-module, $\dim \Omega = N \geq 3$, $N < \infty$. Suppose that M and M' are free submodules, $\dim M = \dim M' = n \geq 2$, $n < N$.

Definition 1. A ***submodule*** $T \subset \Omega$ will be called Δ-complement to the submodule M if

$$M \cap T = 0, \;\; T \oplus M \cong \Omega.$$

Proposition 1. For the submodules M and M' there exists a common Δ-complement T, i.e.

$$M \cap T = M' \cap T = 0, \;\; T \oplus M = T \oplus M' \cong \Omega.$$

Proposition 2.

(1)$(M \oplus T) \cap (M' \oplus T) = \Omega_1 \cong \Omega$;

(ii) if $M_1 = M \cap (M' \oplus T)$, $M'_1 = M' \cap (M \oplus T)$, then $M_1 \oplus T = M'_1 \oplus T = \Omega_1$;

(iii) if $U_1 \subset M_1$, $W \subset \Omega_1$, $T \subset W$, then $U_1 = M_1 \cap W$ if $W = U_1 \oplus T$.

Let $U_1 \subset M_1$ be any submodule and $W = U_1 \oplus T$, consider the intersection $W \cap M'_1 = U'_1$. We have

$$W = U'_1 \oplus T.$$

So between submodules M and M_1' we can construct a bijection $U_1 \to U_1'$ by the equation

$$W = U_1 \oplus T = U_1' \oplus T, \quad U_1 \subset M_1, \quad U_1' \subset M_1'.$$

Let $P(M_1)$ and $P(M_1')$ be projective spaces, i.e. the sets of all one-dimensional direct summands.

It is equivalent that

$$P(M_1) = \{Re,\ e \in M_1 \text{ is unimodular element in } M_1\},$$

$$P(M_1') = \{Re_1,\ e_1 \in M_1' \text{ is unimodular element in } M_1'\}.$$

Remark that an element $e \in M$ is unimodular if there exists a linear form $g : M \to R$, $g(e) = 1$ [3, 4, 5].

Definition 2. A ***bijection*** $p : P(M_1) \to P(M_1')$ is a perspectivity with the center of perspectivity $P(T)$ if it could be given by the equation

$$Rm \oplus T = Rm' \oplus T.$$

Theorem 1. *The perspectivity $p : P(M_1) \to P(M_1')$ is induced by a linear map $\alpha : M_1 \to M_1'$ which is invariant on the intersection $M \cap M_1'$.*

Definition 3. Bijection $\Delta p : P(M_1) \to P(M_1')$ is Δ-perspectivity with the center of Δ-perspectivity $P(T)$, if there exist submodule $M_1 \subset M$, $M_1' \subset M'$ such that between $P(M_1)$ and $P(M_1')$ perspectivity p with the center $P(T)$ could be found and the diagram will be commutative

$$\begin{array}{ccc} P(M_1) & \xrightarrow{p} & P(M_1') \\ i\downarrow & & \downarrow i' \\ P(M) & \xrightarrow{\Delta p} & P(M') \end{array}, \qquad i'op = \Delta poi,$$

$$i : Rm \to Re, \quad Rm = Re \cap M_1, \quad Rm \in P(M_1), \quad Re \in P(M),$$

$$i' : Rm' \to Re', \quad Rm' = Re' \cap M_1', \quad Rm' \in P(M_1'), \quad Re' \in P(M').$$

Theorem 2.*Every Δ-perspectivity $\Delta p : P(M) \to P(M')$ is induced by some linear mal $\alpha_1 : M_1 \to M_1'$ which is the identity on the intersection $M \cap M' = M_1 \cap M_1'$.*

Lemma. *The linear map $\alpha_1 : M_1 \to M_1'$ is invariant on the intersection $M_1 \cap M_1'$ iff the equations*

$$M_1 = (M_1 \cap M_1') \oplus M_0, \quad M_1' = (M_1 \cap M_1') \oplus M_0'$$

are true. Here M_0 and M_0' are complements of $M \cap M'$ up to M and M', respectively.

The map $\beta : P(M_1) \to P(M_1')$ is collineation if

$$Re_1 \subset Re_2 \oplus Re_3 \Longleftrightarrow \beta(Re_1) \subset \beta(Re_2) \oplus \beta(Re_3).$$

Theorem 3. *The collineation $\beta : P(M_1) \to P(M_1')$ will be perspectivity if it is induced by the linear map $\lambda : M_1 \to M_1'$ and λ is invariant on the intersection $M_1 \cap M_1'$.*

The collineation between projective spaces is called projection if it is a product of perspectivities.

Theorem 4. *Let $\beta : P(M_1) \to P(M_1')$ be a collineation which is invariant on the intersection $P(M_1) \cap P(M_1') = P(M_1 \cap M_1')$.*

(i) if $\dim(M_1 \cap M_1') \geq 2$, *then β is perspectivity;*

(ii) if $\dim(M_1 \cap M_1') = 0$ *and β is induced by linear map, then β is perspectivity;*

(iii) if $\dim(M_1 \cap M_1') = 1$, *β is induced by linear map and R is commutative, then β is perspectivity*

Let M be a R-free module, $\dim M = n < \infty$. The set of $n+1$ elements of projective space $P(M)$ no one n element of which generates free $(n-1)$-dimensional submodule will be called symplex.

Theorem 5. *Let $p : P(M) \to P(M)$ be projection which is invariant on every point of some symplex. If R is a commutative principal ideal domain, then p is an invariant map. The theorem is false if R is non-commutative.*

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