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THE NEUMANN BVP OF THE LINEAR THEORY OF THERMOELASTICITY FOR THE SPHERE WITH VOIDS AND MICROTEMPERATURES

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Abstract. In the present paper we investigate the elastic sphere with voids and microtemperatures. Special representations of a general solution of a system of equations for a homogeneous isotropic thermoelastic medium with voids and microtemperatures are constructed by means of the elementary (harmonic, bi-harmonic and meta-harmonic) functions. The Neumann type boundary value problems for the sphere are solved explicitly. The obtained solutions are represented by absolutely and uniformly convergent series.

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1. Introduction

The present paper considers the 3D linear theory of thermoelasticity for materials with voids and microtemperatures. This theory is the generalization of the classical theory of elasticity. The theory of porous materials with voids is used for investigated various types of geological and biological materials for which classical theory of elasticity is not adequate. Porous materials have applications in many fields of engineering, such as the petroleum industry, material science and biology. This theory study the behavior of elastic porous materials like the rock, the bone and the manufactured porous materials. The voids are assumed to contain nothing of mechanical or energetic significance.

Recently the linear theory of thermoelasticity for materials with voids and microtemperatures has been expanding and developing in different directions. For example, the non-linear version of elastic materials with voids was proposed by Nunziato and Cowin [1] and the linear version was developed by Cowin and Nunziato [2] to study mathematically the mechanical behaviour of porous solids. Ieşan in [3] established a variational theory for thermoelastic materials with voids. In [4,5] Ciarletta and Scalia studied a linear theory of thermoelasticity for materials with voids and established uniqueness and reciprocal theorems. In [6] Ieşan and Quintanilla have developed the theory of Nunziato and Cowin for thermoelastic deformable materials with double porosity structure by using the mechanics of materials with voids.

Many problem are investigated for elastic materials with microtemperatures by several researchers (some of those articles can be seen in [7-22] and references therein).

In the present work we consider the elastic sphere with voids and microtemperatures. The general solution of the equations for a homogeneous isotropic thermoelastic medium with voids and microtemperatures is constructed. The Neumann type boundary value problem for the sphere is explicitly solved. The obtained solution is represented as absolutely and uniformly convergent series.

2. Basic equations

Let us consider the isotropic elastic ball D consisting of voids and microtemperatures and bounded by the spherical surface S with center at the origin and radius R. Let D^- be the whole space with a spherical cavity and with boundary S. $D^- = E_3 \backslash \overline{D^+}$. Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point in the Euclidean 3D space E_3.

The homogeneous system of equations of the linear equilibrium theory of thermoelasticity with voids and with microtemperatures has the form [7,13,14]

$$\left(\mu - \frac{p}{2}\right) \Delta \mathbf{u} + \left(\lambda + \mu + \frac{p}{2}\right) \text{graddiv}\mathbf{u} + \lambda_0 \text{grad}\phi - \beta \text{grad} T = 0, \tag{1}$$

$$k_6 \Delta \mathbf{w} + (k_4 + k_5)\text{graddiv}\mathbf{w} - k_3 \text{grad} T - k_2 \mathbf{w} = 0, \tag{2}$$

$$k \Delta T + k_1 \text{div}\mathbf{w} = 0, \tag{3}$$

$$(\alpha \Delta - \varepsilon_1)\phi - \lambda_0 \text{div}\mathbf{u} - \mu_1 \text{div}\mathbf{w} + mT = 0, \tag{4}$$

where $\mathbf{u} := (u_1, u_2, u_3)^\top$ is the displacement vector, $\alpha, \lambda_0, \varepsilon_1, m$ are the material constants due to presence of voids, ϕ is the change in volume fraction field, k is the thermal conductivity, T is the absolute temperature, $\mu_1, k_i (i = 1, 2, ..., 6)$ are the constitutive coefficients, $\mathbf{w} := (w_1, w_2, w_3)^\top$ is the microtemperature vector, p is the initial pressure, λ, μ are the Lame's constants, $\beta = (3\lambda + 2\mu)\alpha_t$ such that α_t is the coefficient of thermal expansion, Δ is the 3D Laplace operator. The superscript $^\top$ denotes transposition.

Definition. A vector-function $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \phi, T)$ defined in the domain $D(D^-)$ is called regular if

$$\mathbf{U} \in C^2(D) \cap C^1(\overline{D})$$

and in the case of the domain D^-, the vector $\mathbf{U}$ should additionally satisfy the following conditions at the infinity:

$$\mathbf{U}(\mathbf{x}) = O(|x|^{-1}) \qquad \frac{\partial \mathbf{U}}{\partial x_j} = O(|\mathbf{x}|^{-2}) \qquad |x|^2 = x_1^2 + x_2^2 + x_3^2 >> 1, \qquad j = 1, 2, 3.$$

The Neumann type boundary value problems (BVPs) for Eqs.(1)-(4) are formulated as follows:

Problem 1. Find a regular solution $\mathbf{U}$ to Eqs. (1)-(4), in the domain D^+ satisfying the following boundary conditions on S:

$$\lim_{D^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{u} = \mathbf{G}^+(\mathbf{z}), \qquad \lim_{D^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{P}^{(2)}(\partial_\mathbf{x}, \mathbf{n})\, \mathbf{w}(\mathbf{x}) = \mathbf{f}^+(\mathbf{z}),$$

$$\lim_{D^+ \ni \mathbf{x} \to \mathbf{z} \in S} \phi(\mathbf{x}) = f_4^+(\mathbf{z}), \qquad \lim_{D^+ \ni \mathbf{x} \to \mathbf{z} \in S} T(\mathbf{x}) = f_5^+(\mathbf{z}), \quad \mathbf{z} \in S,$$

Problem 2. Find a regular solution $\mathbf{U}$ to Eqs. (1)-(4), in the domain D^- satisfying the following boundary conditions on S:

$$\lim_{D^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{u} = \mathbf{G}^-(\mathbf{z}), \qquad \lim_{D^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{P}^{(2)}(\partial_\mathbf{x}, \mathbf{n})\, \mathbf{w}(\mathbf{x}) = \mathbf{f}^-(\mathbf{z}),$$

$$\lim_{D^- \ni \mathbf{x} \to \mathbf{z} \in S} \phi(\mathbf{x}) = f_4^-(\mathbf{z}), \qquad \lim_{D^- \ni \mathbf{x} \to \mathbf{z} \in S} T(\mathbf{x}) = f_5^-(\mathbf{z}), \quad \mathbf{z} \in S,$$

where $\mathbf{n}(\mathbf{z})$ is the external unit normal vector on $\mathbf{z} \in S$, the vector-functions $\mathbf{G}(\mathbf{z}) = (G_1, G_2, G_3)$, $\mathbf{f}(\mathbf{z}) = (f_1, f_2, f_3)$, and the functions $f_4(\mathbf{z})$, $f_5(\mathbf{z})$, are prescribed on S, at $\mathbf{z}$, the vector $\mathbf{P}^{(2)}(\partial_\mathbf{x}, \mathbf{n})\mathbf{w}$ has the following form

$$\begin{aligned} &\mathbf{P}^{(2)}(\partial_\mathbf{x}, \mathbf{n})\mathbf{w} = (k_5 + k_6)\frac{\partial \mathbf{w}}{\partial \mathbf{n}} + k_4 \mathbf{n}\, \text{div}\mathbf{w} + k_5[\mathbf{n} \cdot \text{rot}\mathbf{w}], \\ &\frac{\partial}{\partial \mathbf{n}} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2} + n_3 \frac{\partial}{\partial x_3}. \end{aligned} \tag{5}$$

$[\mathbf{x}\cdot\mathbf{g}]$-denotes the vector product of the two vectors $\mathbf{x}$ and $\mathbf{g}$, The following assertion holds.

Theorem 1. *The Neumann type boundary value problem has at most one regular solution in the domain $D^+(D^-)$.*

3. Preliminaries

Let us introduce the spherical coordinates equalities

$$x_1=\rho\sin\xi\cos\eta,\quad x_2=\rho\sin\xi\sin\eta,\quad x_3=\rho\cos\xi,\quad x\in D^+,$$

$$y_1=R\sin\xi_0\cos\eta_0,\quad y_2=R\sin\xi_0\sin\eta_0,\quad y_3=R\cos\xi_0,\quad y\in S,$$

$$|x|^2=\rho^2=x_1^2+x_2^2+x_3^2,\quad 0\le\vartheta\le\pi,\quad 0\le\varphi\le2\pi\quad 0\le\rho\le R.$$

In the sequel we use the following notation: If $\mathbf{g}(\mathbf{x})=\mathbf{g}(g_1,g_2,g_3)$ and $\mathbf{q}(\mathbf{x})=\mathbf{q}(q_1,q_2,q_3)$ then by symbols $(\mathbf{g.q})$ and $[\mathbf{g.q}]$ will denote the scalar product and vector product respectively

$$(\mathbf{g.q})=\sum_{k=1}^{3}g_kq_k,\quad [\mathbf{g.q}]=(g_2q_3-g_3q_2,g_3q_1-g_1q_3,g_1q_2-g_2q_1),$$

We introduce the following notation:

$$[\mathbf{x}\cdot\nabla]_k=\frac{\partial}{\partial S_k(x)},\quad k=1,2,3,\quad \nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3,}\right),$$

The following identities are valid [24]:

$$(\mathbf{x}\cdot\mathrm{rot}\mathbf{g})=\sum_{k=1}^{3}\frac{\partial g_k(\mathbf{x})}{\partial S_k(\mathbf{x})},\quad \sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}(\mathrm{rot}[\mathbf{x}\cdot\nabla]h)_k=0,$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}(\mathrm{rot}\mathbf{g}(\mathbf{x}))_k=\rho\frac{\partial}{\partial\rho}\mathrm{div}\mathbf{g}(\mathbf{x})-\sum_{k=1}^{3}x_k\Delta\mathbf{g}_k(\mathbf{x}),$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{g}]_k=\rho^2\mathrm{div}\mathbf{g}(\mathbf{x})-\left(\rho\frac{\partial}{\partial\rho}+1\right)(\mathbf{x}\cdot\mathbf{g}(\mathbf{x})),$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(\mathbf{x})}[\mathbf{x}\cdot\mathrm{rot}\mathbf{g}(\mathbf{x})]_k=-\left(\rho\frac{\partial}{\partial\rho}+1\right)\sum_{k=1}^{3}\frac{\partial g_k(\mathbf{x})}{\partial S_k(x)},$$

$$\sum_{k=1}^{3}x_k\frac{\partial}{\partial S_k(x)}=0,\quad \frac{\partial}{\partial S_k(x)}\frac{\partial}{\partial x_k}=\frac{\partial}{\partial x_k}\frac{\partial}{\partial S_k(x)},$$

$$\sum_{k=1}^{3}\frac{\partial^2}{\partial S_k^2(x)}=\frac{\partial^2}{\partial\vartheta^2}+ctg\vartheta\frac{\partial}{\partial\vartheta}+\frac{1}{sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2},\quad \frac{\partial x_k}{\partial S_k}=0,$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}\frac{\partial}{\partial x_k}=0,\quad \frac{\partial g(\rho)Y(\vartheta,\varphi)}{\partial S_k(x)}=g(\rho)\frac{\partial Y(\vartheta,\varphi)}{\partial S_k(x)},$$

$$\frac{\partial}{\partial S_k(x)}\frac{\partial}{\partial\rho}=\frac{\partial}{\partial\rho}\frac{\partial}{\partial S_k(x)},\quad \Delta\frac{\partial g(x)}{\partial S_k(x)}=\frac{\partial}{\partial S_k}\Delta g(x),$$

Below we frequently used the formulas

$$[\mathbf{x}[\mathbf{x}\cdot\mathbf{g}]]=\mathbf{x}(\mathbf{x}\cdot\mathbf{g})-|\mathbf{x}|^2\mathbf{g}(x),$$

$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}[\mathbf{x}\cdot\mathbf{g}]]_k=-|\mathbf{x}|^2\sum_{k=1}^{3}\frac{\partial g_k(x)}{\partial S_k(x)}.$$

If g_m is the spherical harmonic, the operator $\dfrac{\partial}{\partial S_k}$, $\quad k=1,2,3$, does not affect the order of the spherical function:[23]

$$\sum_{k=0}^{3}\frac{\partial^2 g_m(\mathbf{x})}{\partial S_k^2(x)}=-m(m+1)g_m(\mathbf{x}).$$

Let us introduce the following notation

$$\begin{cases}(\mathbf{x}\cdot\mathbf{u})^{\pm}=h_1^{\pm},\quad \left(\sum\limits_{k=0}^{3}\dfrac{\partial}{\partial S_k(z)}[\mathbf{x}\cdot\mathbf{u}]_k\right)^{\pm}=h_2^{\pm},\\ \left(\sum\limits_{k=0}^{3}\dfrac{\partial}{\partial S_k(z)}[\mathbf{u}]_k\right)^{\pm}=h_3^{\pm},\quad \left(\mathbf{x}\cdot\mathbf{P}^{(2)}\mathbf{w}\right)^{\pm}=h_4^{\pm},\\ \left(\sum\limits_{k=0}^{3}\dfrac{\partial}{\partial S_k(z)}[\mathbf{x}\cdot\mathbf{P}^{(2)}\mathbf{w}]_k\right)^{\pm}=h_5^{\pm},\\ \left(\sum\limits_{k=0}^{3}\dfrac{\partial}{\partial S_k(z)}[\mathbf{P}^{(2)}\mathbf{w}]_k\right)^{\pm}=h_6^{\pm},\quad f_4^{\pm}=h_7^{\pm},\quad f_5^{\pm}=h_8^{\pm}.\end{cases}\tag{6}$$

We assume that the functions $h_k(\mathbf{y})$ are representable by the series form.

$$h_k^{\pm}(\mathbf{y})=\sum_{m=0}^{\infty}h_{km}^{\pm}(\xi_0,\eta_0),$$

where $h_{km}^{\pm}$ is the spherical harmonic of order m :

$$h_{km}^{\pm}=\frac{2m+1}{4\pi R^2}\int\limits_S P_m(\cos\gamma)h_m(y)dS_y,$$

P_m is Legender polynomial of the m-th order, γ is an angle formed by the radius-vectors Ox and Oy.

It is well known that the general solutions of the equations $(\Delta+\lambda_k^2)\psi=0$, $\quad k=1,2$, in the domain $D^+(D^-)$ have the form ([25])

$$\begin{cases}\psi(\mathbf{x})=\sum\limits_{m=0}^{\infty}\phi_m^{(1)}(\lambda_k\rho)Y_m(\vartheta,\varphi),\quad \phi_m^{(1)}(\lambda_k\rho)=\dfrac{\sqrt{R}J_{m+\frac12}(\lambda_k\rho)}{\sqrt{\rho}J_{m+\frac12}(\lambda_k R)},\quad \rho<R,\\ \psi(\mathbf{x})=\sum\limits_{m=0}^{\infty}\Psi_m^{(1)}(\lambda_k\rho)Y_m(\vartheta,\varphi),\quad \Psi_m^{(1)}(\lambda_k\rho)=\dfrac{\sqrt{R}H^{(1)}_{m+\frac12}(\lambda_k\rho)}{\sqrt{\rho}H^{(1)}_{m+\frac12}(\lambda_k R)},\quad \rho>R,\end{cases}\tag{7}$$

and the general solution of the equation $\Delta\phi = 0$ in the domain $D^+(D^-)$ has the form ([25])

$$\begin{cases} \phi(\mathbf{x}) = \sum\limits_{m=0}^{\infty} \dfrac{\rho^m}{R^m} Z_m(\xi,\eta), \quad \rho < R, \\ \\ \phi(\mathbf{x}) = \sum\limits_{m=0}^{\infty} \dfrac{R^{m+1}}{\rho^{m+1}} Z_m(\xi,\eta), \quad \rho > R, \end{cases} \tag{8}$$

respectively, where Y_m and Z_m are the spherical harmonics.

4. Expansion of regular solutions

In this section we obtain a general solution for system (1)-(4), which makes it possible to solve the BVPs for the sphere.

From (3)-(4) it follows that

$$\mathrm{div}\mathbf{w} = -\frac{k}{k_1}\Delta T, \tag{9}$$

$$\mathrm{div}\mathbf{u} = \frac{1}{\lambda_0}\left[\frac{\mu_1 k}{k_1}\Delta T + (\alpha\Delta - \varepsilon_1)\phi + mT\right], \tag{10}$$

Applying the operator div to the equation (2) and taking into account (9), we obtain

$$-\frac{(k_7\Delta - k_2)k}{k_1}\Delta T - k_3\Delta T = 0, \quad k_7 = k_4 + k_5 + k_6. \tag{11}$$

From here we find

$$-\frac{kk_7}{k_1}(\Delta - s_1^2)\Delta T = 0, \tag{12}$$

where

$$s_1^2 = \frac{kk_2 - k_1k_3}{kk_7}.$$

Applying the operator div to the equation (1), we get

$$\mu_0\Delta\mathrm{div}\mathbf{u} + \lambda_0\Delta\phi - \beta\Delta T = 0, \quad \mu_0 = \lambda + 2\mu. \tag{13}$$

In view of (10), the equation (13) reduces to the following equation

$$\Delta(\Delta - s_3^2)\phi = \frac{1}{\alpha\mu_0}\left[\beta\lambda_0 - \mu_0\left(\frac{\mu_1 k}{k_1}\Delta T + m\right)\right]\Delta T, \tag{14}$$

where

$$s_3^2 = \frac{\varepsilon_1\mu_0 - \lambda_0^2}{\alpha\mu_0}. \tag{15}$$

Since T is a solution of equation (12), we can write

$$T = \vartheta + \vartheta_1, \tag{16}$$

where

$$\Delta\vartheta = 0, \quad (\Delta - s_1^2)\vartheta_1 = 0.$$

The relations (9) and (14), in view of (16), can be written in the form

$$\begin{cases} \mathrm{div}\mathbf{w} = -\dfrac{k}{k_1} s_1^2 \vartheta_1, \\ \Delta(\Delta - s_3^2)\phi = \dfrac{s_1^2}{\alpha\mu_0}\left[\beta\lambda_0 - \mu_0\left(\dfrac{\mu_1 k}{k_1} s_1^2 + m\right)\right]\vartheta_1. \end{cases} \tag{17}$$

From (10) and (17) it follows that the functions ϕ and div$\mathbf{u}$ can be represented in the form

$$\phi = \psi + \psi_3 + \frac{A\vartheta_1}{s_1^2 - s_3^2},$$

$$\mathrm{div}\mathbf{u} = q\vartheta - \frac{\varepsilon_1}{\lambda_0}\psi - \frac{\lambda_0}{\mu_0}\psi_3 + q_1\vartheta_1,$$

where ψ is an arbitrary harmonic function $\Delta\psi = 0$,

$$q = \frac{m}{\lambda_0}, \quad A = \frac{1}{\alpha\mu_0}\left[\beta\lambda_0 - \mu_0\left(\frac{\mu_1 k}{k_1} s_1^2 + m\right)\right],$$

$$q_1 = \frac{1}{\mu_0\alpha(s_1^2 - s_3^2)}\left[m\lambda_0 + \beta(\alpha s_1^2 - \varepsilon_1) + \frac{\mu_1 k s_1^2}{k_1}\lambda_0\right].$$

Substituting the relations div$\mathbf{u}, \Phi$ and T into (1), we get the following nonhomogeneous equation with respect to $\mathbf{u}$

$$\Delta\mathbf{u} = \frac{2}{2\mu - p}\mathrm{grad}\left[q_0\psi + Q\vartheta - \frac{2\mu - p}{2\mu_0}\lambda_0\psi_3 + Q_1\vartheta_1\right], \tag{18}$$

where

$$\begin{cases} q_0 = \left(\lambda + \mu + \dfrac{p}{2}\right)\dfrac{\varepsilon_1}{\lambda_0} - \lambda_0, \\ Q = -\left(\lambda + \mu + \dfrac{p}{2}\right) q + \beta, \\ Q_1 = -\left(\lambda + \mu + \dfrac{p}{2}\right) q_1 + \beta - \dfrac{\lambda_0 A}{s_1^2 - s_3^2}. \end{cases} \tag{19}$$

The solution of equation (18) can be represented in the form

$$\mathbf{u} = \mathbf{\Psi} + \mathbf{u}_0, \tag{20}$$

where $\mathbf{u}_0$ denotes a particular solution of equation (18)

$$\mathbf{u}_0 = \frac{2}{2\mu - p}\mathrm{grad}\left[q_0\psi_0 + Q\vartheta_0 - \frac{2\mu - p}{2\mu_0}\frac{\lambda_0\psi_3}{s_3^2} + Q_1\frac{\vartheta_1}{s_1^2}\right], \tag{21}$$

the functions Ψ, ψ_0 and ϑ_0 are chosen such that

$$\begin{cases} \Delta\psi_0 = \psi, \quad \Delta\vartheta_0 = \vartheta, \quad \Delta\mathbf{\Psi} = 0, \\ \text{div}\mathbf{\Psi} = a_0\psi + a\vartheta, \\ a_0 = \dfrac{2}{\lambda_0(2\mu - p)}\left[\lambda_0^2 - \varepsilon_1\mu_0\right], \\ a = \dfrac{2}{2\mu - p}\left[q\mu_0 - \beta\right], \quad \dfrac{2}{2\mu - p}Q_1 = q_1. \end{cases} \tag{22}$$

Now let us prove the following theorem:

Theorem 2. *The regular solution* $\mathbf{w}$ *of equation (2) admits in the domain of regularity a representation*

$$\mathbf{w}(\mathbf{x}) = (\overset{1}{\mathbf{w}} + \overset{2}{\mathbf{w}}, \phi, T) \tag{23}$$

Proof. Let $\mathbf{w}$ be a certain solution of equation (2). Let us prove that $\mathbf{w}$ can be represented in the form (23). Using the identity

$$\Delta\mathbf{w} = graddiv\mathbf{w} - rotrot\mathbf{w}$$

from equation (2) we obtain

$$\mathbf{w} = \frac{k_7}{k_2}\text{graddiv}\mathbf{w} - \frac{k_6}{k_2}\text{rotrot}\mathbf{w} - \frac{k_3}{k_2}\text{grad}T.$$

Let

$$\overset{1}{\mathbf{w}} = \frac{k_7}{k_2}\text{graddiv}\mathbf{w} - \frac{k_3}{k_2}\text{grad}T, \tag{24}$$

$$\overset{2}{\mathbf{w}} = -\frac{k_6}{k_2}\text{rotrot}\mathbf{w}, \tag{25}$$

then $\mathbf{w} = \overset{1}{\mathbf{w}} + \overset{2}{\mathbf{w}}, \quad \text{rot}\overset{1}{\mathbf{w}} = 0 \quad \text{div}\overset{2}{\mathbf{w}} = 0.$

Taking into account the last equalities, from (25) we have

$$(\Delta - s_2^2)\overset{2}{\mathbf{w}} = 0, \quad s_2^2 = \frac{k_2}{k_6}. \tag{26}$$

Substituting the values div$\mathbf{w}$ and T into (1), we obtain

$$\overset{1}{\mathbf{w}} = -\text{grad}\left[\frac{k_3}{k_2}\vartheta + \frac{k}{k_1}\vartheta_1\right]. \tag{27}$$

Theorem 3. *The regular solution* $\mathbf{w}$, *where* $\mathbf{w} = (w_1, w_2, w_3)$, *can be represented in the form* (for details see in [20,24])

$$\mathbf{w} = -\text{grad}\left[\frac{k_3}{k_2}\vartheta + \frac{k}{k_1}\vartheta_1\right] + c\,\text{rot}\boldsymbol{\varphi}^{\mathbf{3}}(\mathbf{x}), \tag{28}$$

where

$$\begin{cases} (\Delta - s_2^2)\boldsymbol{\varphi}^{\mathbf{3}} = 0, \qquad \mathrm{div}\boldsymbol{\varphi}^{\mathbf{3}} = 0, \quad c = -\dfrac{k_6}{k_2} \\ \boldsymbol{\varphi}^{\mathbf{3}}(\mathbf{x}) = [\mathbf{x}\cdot\nabla]\varphi_3(\mathbf{x}) + rot[\mathbf{x}.\nabla]\varphi_4(\mathbf{x}), \\ \nabla = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dfrac{\partial}{\partial x_3}\right), \quad (\Delta - s_2^2)\varphi_j = 0 \qquad j = 3, 4. \end{cases} \tag{29}$$

In addition if

$$\int\limits_{S(0,a_1)} \varphi_j ds = 0,$$

where $S(0, a_1) \subset D$ is an arbitrary spherical surface of radius a_1. Between the vector $\mathbf{w}(\mathbf{x})$ and the functions $\psi, \vartheta, \ \vartheta_j, \ j = 1, 2, \ \varphi_j, \ j = 3, 4$, there exists one-to-one correspondence.

Remark. Hence, we have proved that the solution $\mathbf{w}(\mathbf{x})$ of equation (2) can be written in the form

$$\mathbf{w}(\mathbf{x}) = -\mathrm{grad}\left[\frac{k_3}{k_2}\vartheta + \frac{k}{k_1}\vartheta_1\right] + [\mathbf{x}\cdot\nabla]\varphi_4(\mathbf{x}) + c\, rot[\mathbf{x}\cdot\nabla]\varphi_3(\mathbf{x}), \tag{30}$$

From the above reasoning we have proved the following theorem:

Theorem 4. *The general solution of the system (1)-(4) admits in the domain of regularity a representation*

$$\begin{cases} \mathbf{u} = \boldsymbol{\Psi} + \dfrac{2}{2\mu - p}\mathrm{grad}\left[q_0\psi_0 + Q\vartheta_0 - \dfrac{2\mu - p}{2\mu_0} \ \dfrac{\lambda_0\psi_3}{s_3^2} + Q_1\dfrac{\vartheta_1}{s_1^2}\right], \\ \mathbf{w}(\mathbf{x}) = -\mathrm{grad}\left[\dfrac{k_3}{k_2}\vartheta + \dfrac{k}{k_1}\vartheta_1\right] + [\mathbf{x}\cdot\nabla]\varphi_4(\mathbf{x}) + c\, rot[\mathbf{x}\cdot\nabla]\varphi_3(\mathbf{x}), \\ \phi = \psi + \psi_3 + \dfrac{A\vartheta_1}{s_1^2 - s_3^2}, \quad T = \vartheta + \vartheta_1, \end{cases} \tag{31}$$

where

$$\Delta\psi_0 = \psi, \quad \Delta\vartheta_0 = \vartheta, \quad \Delta\boldsymbol{\Psi} = 0, \quad \Delta\vartheta = 0, \quad \Delta\psi = 0,$$

$$(\Delta - s_1^2)\vartheta_1 = 0, \quad (\Delta - s_3^2)\psi_3 = 0, \quad (\Delta - s_2^2)\varphi_j = 0, \quad j = 3, 4,$$

$$\mathrm{div}\boldsymbol{\Psi} = a_0\psi + a\vartheta, \quad \mathrm{div}\mathbf{w} = -\frac{k}{k_1}s_1^2\vartheta_1, \quad \mathrm{div}\mathbf{u} = q\vartheta - \frac{\varepsilon_1}{\lambda_0}\psi - \frac{\lambda_0}{\mu_0}\psi_3 + q_1\vartheta_1.$$

From relation (31), we conclude that the representation of a solution of $\mathbf{u}$ contains harmonic, biharmonic, and metaharmonic functions, while the representation of $\mathbf{w}$, ϕ

and T contains a harmonic and a metaharmonic functions.

5. Solution of the BVP 1

The vector $\mathbf{P}^{(2)}(\partial\mathbf{x},\partial\mathbf{n})\mathbf{w}$ has the form

$$\mathbf{P}^{(2)}(\partial\mathbf{x},\partial\mathbf{n})\mathbf{w}=(k_5+k_6)\frac{\partial\mathbf{w}}{\partial\mathbf{n}}+k_4\mathbf{n}\,\mathrm{div}\mathbf{w}+k_5[\mathbf{n}\cdot\mathrm{rot}\mathbf{w}],\quad \mathbf{n}(\mathbf{x})=\frac{\mathbf{x}}{\rho}. \tag{32}$$

If we use the following identities

$$\begin{aligned}&\frac{\partial}{\partial\mathbf{n}}\mathrm{grad}h(\mathbf{x})=\frac{1}{\rho}\mathrm{grad}\left[\left(\rho\frac{\partial}{\partial\rho}-1\right)h(\mathbf{x}\right],\\&\frac{\partial}{\partial\mathbf{n}}\mathrm{rot}h(\mathbf{x})=\frac{1}{\rho}\mathrm{rot}\left[\left(\rho\frac{\partial}{\partial\rho}-1\right)h(\mathbf{x}\right],\end{aligned} \tag{33}$$

the vector $\mathbf{P}^{(2)}(\partial\mathbf{x},\partial\mathbf{n})\mathbf{w}$ takes the form

$$\begin{aligned}&\mathbf{P}^{(2)}(\partial\mathbf{x},\partial\mathbf{n})\mathbf{w}\\&=\frac{k_5+k_6}{\rho}\left\{\mathrm{grad}\left(\rho\frac{\partial}{\partial\rho}-1\right)\left[-\frac{k_3}{k_2}\vartheta-\frac{k}{k_1}\vartheta_1\right]\right.\\&\left.+c\,\mathrm{rot}\left[\left(\rho\frac{\partial}{\partial\rho}-1\right)\boldsymbol{\varphi}^3(\mathbf{x})\right]\right\}-\frac{k_4ks_1^2}{\rho k_1}\mathbf{x}\vartheta_1+k_5\frac{1}{\rho}[\mathbf{x}\cdot\boldsymbol{\varphi}^{(3)}].\end{aligned} \tag{34}$$

It is easily seen that, by direct calculation from (31) and (34) we obtain

$$\begin{aligned}&(\mathbf{x}\cdot\mathbf{u})=(\mathbf{x}\cdot\boldsymbol{\Psi})+\frac{2}{2\mu-p}\rho\frac{\partial}{\partial\rho}\left[q_0\psi_0+Q\vartheta_0-\frac{2\mu-p}{2\mu_0}\frac{\lambda_0\psi_3}{s_3^2}+Q_1\frac{\vartheta_1}{s_1^2}\right],\\&\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{u}]_k=\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\boldsymbol{\Psi}]_{\mathbf{k}}\\&+\frac{2}{2\mu-p}\sum_{k=1}^{3}\frac{\partial^2}{\partial S_k^2(x)}\left[q_0\psi_0+Q\vartheta_0-\frac{2\mu-p}{2\mu_0}\frac{\lambda_0\psi_3}{s_3^2}+Q_1\frac{\vartheta_1}{s_1^2}\right]\\&\sum_{k=1}^{3}\frac{\partial u_k}{\partial S_k(x)}=\sum_{k=1}^{3}\frac{\partial\Psi_k}{\partial S_k(x)},\end{aligned} \tag{35}$$

$$(\mathbf{x}\cdot\mathbf{P}^{(2)}(\partial\mathbf{x},\partial\mathbf{n})\mathbf{w})=(k_5+k_6)\rho\frac{\partial^2}{\partial\rho^2}\left[-\frac{k_3}{k_2}\vartheta-\frac{k}{k_1}\sum_{j=1}^{2}\vartheta_j\right]$$

$$+\frac{s_1^2k}{k_1}k_4\rho\vartheta_1+c(k_5+k_6)\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\sum_{k=0}^{3}\frac{\partial^2\varphi_3}{\partial S_k^2(x)},$$

$$\sum_{k=0}^{3}\frac{\partial}{\partial S_k(x)}\Big[\mathbf{x}\cdot\mathbf{P}^{(2)}(\partial\mathbf{x},\partial\mathbf{n})\mathbf{w})\Big]_k$$

$$=(k_5+k_6)\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\sum_{k=0}^{3}\frac{\partial^2}{\partial S_k^2(x)}\left[-\frac{k_3}{k_2}\vartheta-\frac{k}{k_1}\sum_{j=1}^{2}\vartheta_j\right]$$

$$-\left\{c(k_5+k_6)\left[\rho\frac{\partial^2}{\partial\rho^2}+\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right]+k_5\rho\right\}\sum_{k=0}^{3}\frac{\partial^2\varphi_3}{\partial S_k^2(x)},$$

$$\sum_{k=0}^{3}\frac{\partial}{\partial S_k(x)}\Big[\mathbf{P}^{(2)}(\partial\mathbf{x},\partial\mathbf{n})\mathbf{w})\Big]_k=k_5\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\sum_{k=0}^{3}\frac{\partial^2\varphi_4}{\partial S_k^2(x)},$$

$$\phi=\psi+\psi_3+\frac{A\vartheta_1}{s_1^2-s_3^2},\quad T=\vartheta+\vartheta_1.$$

Let us replace functions $(\mathbf{x}\cdot\mathbf{\Psi})$ and $\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{\Psi}]_k$ with functions ϑ and ψ. To this end, taking into account the following identities

$$\Delta(\mathbf{x}\cdot\mathbf{\Psi})=2\mathrm{div}\ \mathbf{\Psi}=2[a_0\psi+a\vartheta],$$

$$\sum_{j=1}^{3}\frac{\partial}{\partial S_k}[\mathbf{x}\cdot\mathbf{g}]_j=\rho^2\mathrm{div}\ \mathbf{g}-\left(\rho\frac{\partial}{\partial\rho}+1\right)(\mathbf{x}\cdot\mathbf{g}),$$

we obtain

$$\begin{aligned}&(\mathbf{x}\cdot\mathbf{\Psi})=\Omega+2\left[a_0\psi_0+a\vartheta_0\right],\\&\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{\Psi}]_k=\rho^2\left[a_0\psi+a\vartheta\right]-\left(\rho\frac{\partial}{\partial\rho}+1\right)(\mathbf{x}\cdot\mathbf{\Psi}),\end{aligned}\tag{36}$$

where Ω is an arbitrary harmonic function $\Delta\Omega=0$.

We seek solutions to equations (1)-(4) with boundary conditions

$$\mathbf{u}^+=\mathbf{F}(y),\quad\Big(\mathbf{P}^{(2)}(\partial\mathbf{x},\mathbf{n})\mathbf{w}\Big)^+=\mathbf{f}(\mathbf{y}),\quad\psi^+=f_4(y),\quad T^+=f_5(\mathbf{y}),\quad\mathbf{y}\in S,$$

in the form (31), where

$$\vartheta(\mathbf{x})=\sum_{n=0}^{\infty}\frac{\rho^n}{R^n}Y_n(\xi,\eta),\quad \Omega=\sum_{n=0}^{\infty}\frac{\rho^n}{R^n}Z_{4n}(\xi,\eta),$$

$$\vartheta_1(\mathbf{x})=\sum_{n=0}^{\infty}\phi_n^{(1)}(is_1\rho)Z_{1n}(\xi,\eta),$$

$$\psi_3(\mathbf{x})=\sum_{n=0}^{\infty}\phi_n^{(1)}(is_3\rho)Z_{2n}(\xi,\eta),\tag{37}$$

$$\varphi_j(\mathbf{x})=\sum_{m=0}^{\infty}\phi_n^{(1)}(is_2\rho)Y_{jn}(\xi,\eta),\quad j=3,4,$$

$$\sum_{k=1}^{3}\frac{\partial\Psi_k}{\partial S_k(x)}=\sum_{n=0}^{\infty}\frac{\rho^n}{R^n}Z_{3n}(\xi,\eta),\quad \psi=\sum_{m=0}^{\infty}\frac{\rho^m}{R^m}Z_n(\xi,\eta),\quad \rho<R,$$

where $Z_n,\ Z_{jn},\ Y_n$ *and* Y_{jn} are the unknown spherical harmonics of order n

$$\phi_m^{(1)}(l\rho)=\frac{\sqrt{R}J_{m+\frac{1}{2}}(l\rho)}{\sqrt{\rho}J_{m+\frac{1}{2}}(lR)}.$$

Taking into account (37), we can write the particular solutions of equations $\Delta\vartheta_0=\vartheta$ and $\Delta\psi_0=\psi$ in the following form

$$\vartheta_0(\mathbf{x})=\frac{1}{2}\sum_{n=0}^{\infty}\frac{\rho^2}{3+2n}\left(\frac{\rho}{R}\right)^n Y_n(\xi,\eta).\tag{38}$$

$$\psi_0(\mathbf{x})=\frac{1}{2}\sum_{n=0}^{\infty}\frac{\rho^2}{3+2n}\left(\frac{\rho}{R}\right)^n Z_n(\xi,\eta).\tag{39}$$

Substituting (36),(37),(38) and (39) into (35), we obtain the following system of algebraic equations:

$$\begin{aligned}
&Z_{4n}+\frac{R^2}{3+2n}\left(a_0+\frac{n+2}{2\mu-p}q_0\right)Z_n+\frac{R^2}{3+2n}\left(a+\frac{n+2}{2\mu-p}Q\right)Y_n\\
&+\frac{2RQ_1}{2\mu-p}\left[\frac{\partial}{\partial\rho}\phi_n^{(1)}(is_1\rho)\right]_{\rho=R}\frac{Z_{1n}}{s_1^2}-\frac{\lambda_0R}{\mu_0s_3^2}\left[\frac{\partial}{\partial\rho}\phi_n^{(1)}(is_3\rho)\right]_{\rho=R}Z_{2n}=h_{1n}^+,\\
&-(n+1)Z_{4n}+\frac{nR^2}{3+2n}\left[a_0-\frac{n+1}{2\mu-p}q_0\right]Z_n+\frac{nR^2}{3+2n}\left[a-\frac{n+1}{2\mu-p}Q\right]Y_n\\
&+\frac{n(n+1)\lambda_0}{\mu_0s_3^2}Z_{2n}-\frac{2n(n+1)}{2\mu-p}\frac{Q_1}{s_1^2}Z_{1n}=h_{2n}^+,\\
&Z_{3n}=h_{3n}^+,
\end{aligned}\tag{40}$$

$$-(k_5+k_6)\left[\frac{n(n-1)}{k_2R}k_3Y_n+R\frac{k}{k_1}\left(\frac{\partial^2}{\partial\rho^2}\phi_n^{(1)}(is_1\rho)\right)_{\rho=R}Z_{1n}\right]$$

$$-\frac{k_4k}{k_1}Rs_1^2Z_{1n}-c(k_5+k_6)n(n+1)\left[\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\phi_n^{(1)}(is_2\rho)\right]_{\rho=R}Y_{3n}=h_{4n}^+,$$

$$(k_5+k_6)n(n+1)\left[\frac{(n-1)k_3}{k_2R}Y_n+\frac{k}{k_1}\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\phi_n^{(1)}(is_1\rho)Z_{1n}\right]_{\rho=R}$$

$$+n(n+1)\left[c(k_5+k_6)\left(R\frac{\partial^2}{\partial\rho^2}+\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\phi_n^{(1)}(is_2\rho)+k_5R\right]_{\rho=R}Y_{3n}=h_{5n}^+,$$

$$-k_5n(n+1)\left[\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\phi_n^{(1)}(is_2\rho)\right]_{\rho=R}Y_{4n}=h_{6n}^+,\quad h_{60}^+=0,\quad h_{50}^+=0,$$

$$Z_n+Z_{2n}+\frac{A}{s_1^2-s_3^2}Z_{1n}=h_{7n}^+,\quad Y_n+Z_{1n}=h_{8n}^+.$$

By virtue of Theorem 1, we get the following result: the system (40) for $n\geq 0$ is uniquely solvable.

6. Solution of the BVP 2

Quite similarly as above, we can investigate Problem 2 for an elastic space with a spherical cavity.

Let us assume that functions ϑ, ϑ_j, $j=1,2$, φ_j, $j=3,4$, ψ, Ω and $\sum\limits_{k=1}^{3}\dfrac{\partial\Psi_k}{\partial S_k(x)}$ are sought in the form

$$\begin{aligned}
&\vartheta(\mathbf{x})=\sum_{n=0}^{\infty}\left(\frac{R}{\rho}\right)^{n+1}Y_n(\xi,\eta),\quad \vartheta_1(\mathbf{x})=\sum_{n=0}^{\infty}\phi_m(is_1\rho)Z_{1n}(\xi,\eta),\\
&\psi_3(\mathbf{x})=\sum_{n=0}^{\infty}\phi_m(is_3\rho)Z_{2n}(\xi,\eta),\quad \varphi_j(\mathbf{x})=\sum_{m=0}^{\infty}\phi_m(is_2\rho)Y_{jm}(\xi,\eta),\quad j=3,4,\\
&\Omega=\sum_{n=0}^{\infty}\left(\frac{R}{\rho}\right)^{n+1}Z_{4n}(\xi,\eta),\quad \sum_{k=1}^{3}\frac{\partial\Psi_k}{\partial S_k(x)}=\sum_{n=0}^{\infty}\left(\frac{R}{\rho}\right)^{n+1}Z_{3n}(\xi,\eta),\\
&\psi=\sum_{n=0}^{\infty}\left(\frac{R}{\rho}\right)^{n+1}Z_n(\xi,\eta),\quad \rho>R,
\end{aligned}\tag{41}$$

where

$$\Phi_n(\lambda_k\rho)=\frac{\sqrt{R}H^{(1)}_{n+\frac12}(\lambda_k\rho)}{\sqrt{\rho}H^{(1)}_{n+\frac12}(\lambda_kR)}\qquad k=1,2,$$

$H^{(1)}_{n+\frac{1}{2}}(\lambda\rho)$ is Hankel's function.

The particular solutions of equations $\Delta\vartheta_0 = \vartheta$ and $\Delta\psi_0 = \psi$ have the following form

$$\begin{aligned}\vartheta_0(\mathbf{x}) &= \frac{\rho^2}{2}\sum_{n=0}^{\infty}\frac{Y_n(\vartheta,\eta)}{(1-2n)}\left(\frac{R}{\rho}\right)^{n+1}, \quad \rho > R,\\ \psi_0(\mathbf{x}) &= \frac{\rho^2}{2}\sum_{n=0}^{\infty}\frac{Z_n}{(1-2n)}\left(\frac{R}{\rho}\right)^{n+1}, \quad \rho > R.\end{aligned} \tag{42}$$

Substituting (41) and (42) into (35), passing to the limit as $\rho \to R$, for the determination of unknown functions we arrive at the following system of algebraic equations:

$$Z_{4n} + \frac{R^2}{1-2n}\left(a_0 + \frac{1-n}{2\mu-p}q_0\right)Z_n + \frac{R^2}{1-2n}\left(a + \frac{1-n}{2\mu-p}Q\right)Y_n$$

$$-\frac{\lambda_0 R}{\mu_0 s_3^2}\left[\frac{\partial}{\partial\rho}\phi_n(is_3\rho)\right]_{\rho=R} Z_{2n} + \frac{2RQ_1}{2\mu-p}\left[\frac{\partial}{\partial\rho}\phi_n(is_1\rho)\right]_{\rho=R}\frac{Z_{1n}}{s_1^2} = h^-_{1n},$$

$$nZ_{4n} - \frac{R^2(n+1)Z_n}{(1-2n)}\left[a_0 + \frac{q_0 n}{2\mu-p}\right] - \frac{R^2(n+1)Y_n}{(1-2n)}\left[a + \frac{nQ}{2\mu-p}\right]$$

$$+\frac{n(n+1)}{\mu_0 s_3^2}Z_{2n} - \frac{2n(n+1)Q_1}{(2\mu-p)s_1^2}Z_{1n} = h^-_{2n}, \quad Z_{3n} = h^-_{3n},$$

$$-\frac{(k_5+k_6)(n+1)(n+2)}{Rk_2}k_3Y_n - R\frac{k(k_5+k_6)}{k_1}\left[\frac{\partial^2}{\partial\rho^2}\phi_n(is_1\rho)Z_{1n}\right]_{\rho=R}$$

$$-\frac{kk_4}{k_1}Rs_1^2 Z_{1n} - c(k_5+k_6)n(n+1)\left[\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\phi_n(is_2\rho)\right]_{\rho=R} Y_{3n} = h^-_{4n},$$

$$(k_5+k_6)n(n+1)\left[\frac{n+2}{k_2 R}k_3 Y_n + \frac{k}{k_1}\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\phi_n(is_1\rho)Z_{1n}\right]_{\rho=R}$$

$$+n(n+1)\left[c(k_5+k_6)\left(R\frac{\partial^2}{\partial\rho^2} + \frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\phi_n(is_2\rho) + k_5 R\right]_{\rho=R} Y_{3n} = h^-_{5n},$$

$$-k_5 n(n+1)\left[\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\phi_n(is_2\rho)\right]_{\rho=R} Y_{4n} = h^-_{6n}, \quad h^-_{50} = 0, \quad h^-_{60} = 0,$$

$$Z_n + Z_{2n} + \frac{A}{s_1^2 - s_3^2}Z_{1n} = h^-_{7n}, \quad Y_n + Z_{1n} = h^-_{8n}. \tag{43}$$

According to Theorem 1 we conclude that system (43) for $n \geq 0$ is uniquely solvable.

7. Conclusions

The main results of this work can be formulated as follows:

1. The general solution of the system of equations in the considered theory is presented by means of elementary (harmonic, meta-harmonic and bi-harmonic) functions.

2. Analytical (exact) solutions of the Neumann type BVPs are obtained for the sphere with voids and microtemperatures.

3. The obtained solution is represented as absolutely and uniformly convergent series.

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