ABOUT SOME SOLUTIONS OF SYSTEM OF EQUATIONS OF STEADY VIBRATIONS FOR THERMOELASTIC MATERIALS WITH VOIDS

Bitsadze L.

Abstract. In this paper the 2D linear theory of steady vibrations of thermoelastic materials with voids is considered. The representation of common decision of the system of equations in the considered theory is obtained. The fundamental and some other matrices of singular solutions are constructed in terms of elementary (meta-harmonic) functions. Some basic properties of single-layer and double-layer potentials are also established.

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1. Introduction

In this paper the 2D linear theory of steady vibrations for thermoelastic materials with voids is considered. The linear theory of thermoelastic porous materials with voids is the generalization of the classical theory of elasticity. This theory is used for investigated various types of geological and biological materials for which classical theory of elasticity is not adequate. Porous materials with voids have applications in many fields of engineering, such as the petroleum industry, material science and biology. This theory enables us to analyze the behaviour of elastic porous materials which can be found in engineering, such as rock and soil, bone, the manufactured porous materials. The voids are assumed to contain nothing of mechanical or energetic significance.

The non-linear version of elastic materials with voids was proposed by Nunziato and Cowin [1] and the linear version was developed by Cowin and Nunziato [2] to study mathematically the mechanical behaviour of porous solids. Another version of the linear theory, called the dilatation theory of elasticity, was independently proposed by Markov [3]. Ieşan in [4] established a variational theory for thermoelastic materials with voids. In [5,6] Ciarletta and Scalia studied a linear thermoelastic theory of materials with voids, and established uniqueness and reciprocal theorems. In [7] Ieşan and Quintanilla have developed the theory of Nunziato and Cowin for thermoelastic deformable materials with double porosity structure.

Many problem are investigated by several researchers in the elastic materials with the microstructure. Some of these results are presented in [8-17] and in references therein.

In this paper the 2D linear theory of steady vibrations of thermoelastic materials with voids is considered. The representation of common decision of the system of equations in the considered theory is obtained. The fundamental and some other matrixes of singular solutions are constructed in terms of elementary (meta-harmonic) functions. Some basic properties of single-layer and double-layer potentials are also established

2. Basic equations

Let $\mathbf{x} = (x_1, x_2)$ be a point of the Euclidean 2D dimensional space E^2 . $\partial \mathbf{x} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$. Let D be a bounded 2D domain (surrounded by the curve S.) Let us assume that the isotropic material with voids occupies the domain D. Bitsadze L.

The basic system of equations of motion in the linear theory thermoelasticity with voids, for isotropic materials can be written as [4]

$$\begin{cases} \mu \Delta \mathbf{u}' + (\lambda + \mu) \operatorname{graddiv} \mathbf{u}' + b \operatorname{grad} \varphi' - \beta \operatorname{grad} \theta' + \varrho \mathbf{f}' = \varrho \frac{\partial^2 \mathbf{u}'}{\partial t^2}, \\ \alpha \Delta \varphi' - b \operatorname{div} \mathbf{u}' - \xi \varphi + m \theta' + \varrho l = \varrho \kappa \frac{\partial^2 \varphi'}{\partial t^2}, \\ k \Delta \theta' - \beta T_0 \frac{\partial}{\partial t} \operatorname{div} \mathbf{u}' - a T_0 \frac{\partial \theta'}{\partial t} - m T_0 \frac{\partial \varphi'}{\partial t} = -\varrho s, \end{cases}$$
(1)

where $\mathbf{u}' := (u'_1, u'_2)^{\top}$ is the displacement vector in a solid, φ is the change of volume fraction, θ is the temperature, λ , μ , β , α , ξ , m, a, k are constitutive coefficients, ϱ is the density, \mathbf{f} is the body force vector, l is the extrinsic equilibrated body force and k is the equilibrated inertia, s is the extrinsic heat supply, $T_0 = const > 0$ is the absolute temperature in the reference state, Δ is the 2D Laplace operator. The superscript $(.)^{\top}$ denotes transposition operation.

As in the classical theory of thermoelasticity, we assume that $\mathbf{u}', \varphi', \theta', f'$ to have a harmonic time variation, that is

$$\begin{aligned} \mathbf{u}' &= Re[\mathbf{u}(x,\omega)exp(-i\omega t)] \quad \varphi' &= Re[\varphi(x,\omega)exp(-i\omega t)], \\ f' &= Re[\varphi(x,\omega)exp(-i\omega t)], \quad \theta' &= Re[\theta(x,\omega)exp(-i\omega t)] \end{aligned}$$

then from (1) we obtain the following system of equations of steady vibrations in the linear theory of thermoelasticity for isotropic materials with voids

$$\begin{cases} \mu \Delta \mathbf{u} + (\lambda + \mu) \operatorname{graddiv} \mathbf{u} + b \operatorname{grad} \varphi - \beta \operatorname{grad} \theta + \varrho \mathbf{f} = -\varrho \omega^2 \mathbf{u}, \\ \alpha \Delta \varphi - b \operatorname{div} \mathbf{u} - \xi \varphi + m \theta + \varrho l = -\varrho \kappa \omega^2 \varphi, \\ k \Delta \theta + i \omega \beta T_0 \operatorname{div} \mathbf{u} + i \omega a T_0 \theta + i \omega m T_0 \varphi = -\varrho s \end{cases}$$
(2)

where Re[f] denotes the real part of f and ω is a oscillation frequency.

Let us consider the basic homogeneous system of equations of steady vibrations in the linear theory of thermoelasticity for isotropic materials with voids

$$\begin{cases}
(\mu\Delta + \rho\omega^2)\mathbf{u} + (\lambda + \mu)\operatorname{graddiv}\mathbf{u} + b\operatorname{grad}\varphi - \beta\operatorname{grad}\theta = 0, \\
(\alpha\Delta + b_0)\varphi - b\operatorname{div}\mathbf{u} + m\theta = 0, \\
(k\Delta + b_1)\theta + b_2\operatorname{div}\mathbf{u} + b_3\varphi = 0.
\end{cases}$$
(3)

where

$$b_0 = \kappa \rho \omega^2 - \varsigma, \quad b_1 = a T_0 i \omega, \quad b_2 = \beta T_0 i \omega, \quad b_3 = m T_0 i \omega$$

We introduce the matrix differential operator

$$\mathbf{A}(\partial_{\mathbf{x}},\omega) = \parallel A_{lj}(\partial x) \parallel_{4\times 4}, \ l,j=1,2,3,4,$$

where

$$\begin{aligned} A_{lj} &:= \delta_{lj}(\mu\Delta + \rho\omega^2) + (\lambda + \mu)\frac{\partial^2}{\partial x_l \partial x_j}, \quad l, j = 1, 2, \\ A_{j3} &:= b\frac{\partial}{\partial x_j}, \quad A_{j4} := -\beta\frac{\partial}{\partial x_j}, \quad j = 1, 2, \\ A_{3j} &:= -b\frac{\partial}{\partial x_j} \quad j = 1, 2, \quad A_{33} := \alpha\Delta + b_0, \\ A_{34} &:= m, \quad A_{4j} := b_2\frac{\partial}{\partial x_j}, \quad A_{43} = b_3, \quad A_{44} = k\Delta + b_1, \end{aligned}$$

 $\delta_{\alpha\gamma}$ is the Kronecker symbol.

It easily seen that system (3) can be rewritten in the following form

$$\mathbf{A}(\partial_{\mathbf{x}},\omega)\mathbf{U}=0,\tag{4}$$

where $\mathbf{U} = (\mathbf{u}, \varphi, \theta)$ is fourth component vector-function.

We also consider the equation

$$\widetilde{\mathbf{A}}(\partial_{\mathbf{x}},\omega)\mathbf{U} = \mathbf{A}^{\top}(-\partial_{\mathbf{x}},\omega)\mathbf{U} = 0.$$
(5)

where $\mathbf{A}^{\top}(\partial_{\mathbf{x}}, \omega)$ is the transpose of matrix $\mathbf{A}(\partial_{\mathbf{x}}, \omega)$.

Definition. A vector-function $\mathbf{U} = (\mathbf{u}, \varphi, \theta)^{\top}$ defined in the domain D is called regular if

$$\mathbf{U} \in C^2(D) \cap C^1(\overline{D})$$

and the vector **U** additionally should satisfy the following conditions at the infinity:

$$\mathbf{U}(\mathbf{x}) = o(1), \qquad \frac{\partial \mathbf{U}}{\partial x_j} = (O|\mathbf{x}|^{-2}), \qquad |\mathbf{x}|^2 = x_1^2 + x_2^2 >> 1, \qquad j = 1, 2.$$

3. The basic fundamental matrix

In this section we will construct the fundamental solution of the system (4) explicitly, which consists of four metaharmonic functions. For this we introduce the matrix differential operator $\mathbf{B}(\partial \mathbf{x}, \omega)$ consisting of cofactors of elements of the matrix \mathbf{A}^{\top} divided on $k\mu\mu_0\alpha$:

$$\mathbf{B}(\partial_{\mathbf{x}}, \omega) = \frac{1}{k\mu\mu_{0}\alpha} \| B_{lj}(\partial x) \|_{5\times 5}, \ l, j = 1, 2, 3, 4,$$

where

$$B_{ij} = \delta_{ij}k\mu_0\alpha(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2) - A_{12}\frac{\partial^2}{\partial x_i\partial x_j},$$

$$A_{12} = (k\Delta + b_1)[(\lambda + \mu)(\alpha\Delta + b_0) + b^2]$$

$$-m[b_3(\lambda + \mu) - bb_2] + \beta[bb_3 + b_2(\alpha\Delta + b_0)],$$

$$\begin{split} B_{j3} &= -\mu(\Delta + \lambda_4^2)[b(k\Delta + b_1) + \beta b_3]\frac{\partial}{\partial x_j}, \ j = 1, 2, \\ B_{j4} &= \mu(\Delta + \lambda_4^2)[\beta(\alpha\Delta + b_0) + mb]\frac{\partial}{\partial x_j}, \ j = 1, 2, \\ B_{3j} &= \mu(\Delta + \lambda_4^2)[b(k\Delta + b_1) + mb_2]\frac{\partial}{\partial x_j}, \ j = 1, 2, \\ B_{33} &= \mu(\Delta + \lambda_4^2)[(k\Delta + b_1)(\mu_0\Delta + \rho\omega^2) + b_2\beta\Delta], \\ B_{34} &= -\mu(\Delta + \lambda_4^2)[(m\mu_0 - b\beta)\Delta + m\rho\omega^2], \\ B_{4j} &= -\mu(\Delta + \lambda_4^2)[b_2(\alpha\Delta + b_0)_2 + bb_3]\frac{\partial}{\partial x_j}, \ j = 1, 2, \\ B_{43} &= -\mu(\Delta + \lambda_4^2)[b_3(\mu_0\Delta + \rho\omega^2) - bb_2\Delta], \\ B_{44} &= \mu(\Delta + \lambda_4^2)[(\alpha\Delta + b_0)(\mu_0\Delta + \rho\omega^2) + b^2\Delta]. \end{split}$$

Substituting $\mathbf{U} = \mathbf{B} \boldsymbol{\Psi}$ into (4) we obtain

$$\mu_0 k \alpha \mu (\Delta + \lambda_1^2) (\Delta + \lambda_2^2) (\Delta + \lambda_3^2) (\Delta + \lambda_4^2) \Psi = 0, \qquad (6)$$

where $\lambda_4^2 = \frac{\rho \omega^2}{\mu}$ and λ_j^2 , j = 1, 2, 3 are roots of the equation $k\alpha\mu_0\xi^3 - a_1\xi^2 + a_2\xi - \varrho\omega^2(b_0b_1 - mb_3) = 0,$ $a_1 = \mu_0(\alpha b_1 + kb_0) + \alpha k \rho \omega^2 + \alpha \beta b_2 + kb^2,$ $a_2 = \mu_0(b_0b_1 - mb_3) + (\alpha b_1 + kb_0)\varrho\omega^2 + b_1b^2 + mbb_2 + \beta(bb_3 + b_0b_2).$

We assume that the values λ_j^2 are distinct and different from zero. The solution of the equation (6) can be represented as

$$\Psi = -\sum_{j=1}^{4} d_j \varphi_j,\tag{7}$$

where

$$\begin{aligned} (\Delta + \lambda_m^2)\varphi_m &= 0, \\ \varphi_m &= \frac{\pi}{2i} H_0^{(1)}(\lambda_m r), \\ d_1 &= \frac{1}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)(\lambda_1^2 - \lambda_4^2)}, \quad d_2 &= \frac{1}{(\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_4^2)}, \\ d_3 &= \frac{1}{(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)(\lambda_3^2 - \lambda_4^2)}, \quad d_4 &= \frac{1}{(\lambda_4^2 - \lambda_2^2)(\lambda_4^2 - \lambda_2^2)(\lambda_4^2 - \lambda_3^2)}, \end{aligned}$$

 $H_0^{(1)}(\lambda_m r)$ is Hankel's function of the first kind with the index 0

Substituting (7) into $\mathbf{U} = \mathbf{B} \Psi$, we obtain the matrix of fundamental solutions $\Gamma(\mathbf{x}-\mathbf{y})$ for the equation (4)

$$\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y},\omega) = \| \Gamma_{kj}(\mathbf{x}-\mathbf{y}) \|_{4\times 4}$$
(8)

where

$$\begin{split} \Gamma_{lm} &= \frac{\delta_{lm}}{\mu} \varphi_4 + \frac{\partial^2}{\partial x_l \partial x_m} \left[\frac{\varphi_4}{\mu \lambda_4^2} + \frac{1}{\mu_0 \alpha k} \sum_{j=1}^3 d_j (\lambda_j^2 - \lambda_4^2) m_j \varphi_j \right], \quad l, m = 1, 2, \\ m_j &= \alpha b_1 + k b_0 - k \alpha \lambda_j^2 - \frac{b_0 b_1 - m b_3}{\lambda_j^2}, \\ \Gamma_{i3} &= \frac{1}{\mu_0 \alpha k} \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [-b k \lambda_j^2 + \beta b_3 + b b_1] \frac{\partial \varphi_j}{\partial x_i}, \quad i = 1, 2, \\ \Gamma_{i4} &= -\frac{1}{\mu_0 \alpha k} \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [-\lambda_j^2 \alpha \beta + m b + \beta b_0] \frac{\partial \varphi_j}{\partial x_i}, \\ \Gamma_{3i} &= -\frac{1}{\mu_0 \alpha k} \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [-b k \lambda_j^2 + m b_2 + b b_1] \frac{\partial \varphi_j}{\partial x_i}, \\ \Gamma_{33} &= -\frac{1}{\mu_0 \alpha k} \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [-b k \lambda_j^2 + m b_2 + b b_1] \frac{\partial \varphi_j}{\partial x_i}, \\ \Gamma_{34} &= \frac{1}{\mu_0 \alpha k} \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [m \varrho \omega^2 - (m \mu_0 - b \beta) \lambda_j^2] \varphi_j, \end{split}$$

$$\begin{split} \Gamma_{4i} &= \frac{1}{\mu_0 \alpha k} \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [-\lambda_j^2 \alpha b_2 + b b_3 + b_2 b_0] \frac{\partial \varphi_j}{\partial x_i}, \\ \Gamma_{43} &= \frac{1}{\mu_0 \alpha k} \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [b_3 \varrho \omega^2 - (b_3 \mu_0 - b b_2) \lambda_j^2] \varphi_j, \\ \Gamma_{44} &= -\frac{1}{\mu_0 \alpha k} \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) \{\lambda_j^4 \mu_0 \alpha - \lambda_j^2 [\alpha \rho \omega^2 + b_0 \mu_0 + b^2] + b_0 \rho \omega^2 \} \varphi_j. \end{split}$$

Clearly

$$\frac{\pi}{2i}H_0^{(1)}(\lambda r) = \ln|\mathbf{x} - \mathbf{y}| - \frac{\lambda^2}{4}|\mathbf{x} - \mathbf{y}|^2 \ln|\mathbf{x} - \mathbf{y}| + const + O(|\mathbf{x} - \mathbf{y}|^2).$$

It is evident that all elements of $\Gamma(\mathbf{x}-\mathbf{y},\omega)$ are single-valued functions on the whole plane and they have a logarithmic singularity at most.

By applying the methods, as in the classical theory of elasticity, we can directly prove the following: (for details see in [18])

Theorem 1. The element of the matrix $\Gamma(\mathbf{x}-\mathbf{y},\omega)$ has a logarithmic singularity as $\mathbf{x} \rightarrow \mathbf{y}$ and each column of the matrix $\Gamma(\mathbf{x}-\mathbf{y},\omega)$, considered as a vector, is a solution of the system (4) at every point \mathbf{x} if $\mathbf{x} \neq \mathbf{y}$.

Let us we consider the matrix $\widetilde{\Gamma}(\mathbf{x}) := \Gamma^{\top}(-\mathbf{x})$. The following basic properties of $\widetilde{\Gamma}(\mathbf{x})$ may be easily verified:

Theorem 2. Each column of the matrix $\widetilde{\Gamma}(\mathbf{x}-\mathbf{y},\omega)$, considered as a vector, satisfies the associated system $\widetilde{A}(\partial \mathbf{x})\widetilde{\Gamma}(\mathbf{x}-\mathbf{y},\omega) = 0$, at every point \mathbf{x} if $\mathbf{x} \neq \mathbf{y}$ and the elements of the matrix $\widetilde{\Gamma}(\mathbf{x}-\mathbf{y},\omega)$ have a logarithmic singularity as $\mathbf{x} \to \mathbf{y}$.

4. Singular matrix of solutions

Let $P(\partial \mathbf{x}, \mathbf{n})$ be the stress operator in the linear theory of thermoelasticity for materials with voids and $P(\partial \mathbf{x}, \mathbf{n})\mathbf{U}$ is the stress vector which acts on an element of the arc with the normal $\mathbf{n} = (n_1, n_2)$

$$P(\partial \mathbf{x}, \mathbf{n})\mathbf{U} = T(\partial \mathbf{x}, \mathbf{n})\mathbf{u} + \mathbf{n}(b\varphi - \beta\theta), \tag{9}$$

where $T(\partial \mathbf{x}, \mathbf{n})$ is the stress operator in the classical theory of elasticity

$$\begin{split} \boldsymbol{T}(\partial_{\mathbf{x}},\mathbf{n}) &= \begin{pmatrix} \mu \frac{\partial}{\partial \mathbf{n}} + (\lambda + \mu)n_1 \frac{\partial}{\partial x_1} & (\lambda + \mu)n_1 \frac{\partial}{\partial x_2} + \mu \frac{\partial}{\partial s} \\ (\lambda + \mu)n_2 \frac{\partial}{\partial x_1} - \mu \frac{\partial}{\partial s} & \mu \frac{\partial}{\partial \mathbf{n}} + (\lambda + \mu)n_2 \frac{\partial}{\partial x_2} \end{pmatrix}, \\ \\ & \frac{\partial}{\partial \mathbf{n}} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial s} = n_2 \frac{\partial}{\partial x_1} - n_1 \frac{\partial}{\partial x_2}. \end{split}$$

Let us introduce the following matrix differential operators of dimension 4x4

$$\boldsymbol{R}(\partial_{\mathbf{x}},\mathbf{n}) = \begin{pmatrix} T_{11}(\partial_{\mathbf{x}},\mathbf{n}) & T_{12}(\partial_{\mathbf{x}},\mathbf{n}) & bn_1 & -\beta n_1 \\ T_{21}(\partial_{\mathbf{x}},\mathbf{n}) & T_{22}(\partial_{\mathbf{x}},\mathbf{n}) & bn_2 & -\beta n_2 \\ 0 & 0 & \alpha \frac{\partial}{\partial \mathbf{n}} & 0 \\ 0 & 0 & 0 & k \frac{\partial}{\partial \mathbf{n}} \end{pmatrix},$$

$$\widetilde{\boldsymbol{R}}(\partial_{\mathbf{x}}, \mathbf{n}) = \begin{pmatrix} T_{11}(\partial_{\mathbf{x}}, \mathbf{n}) & T_{12}(\partial_{\mathbf{x}}, \mathbf{n}) & bn_1 & -b_2n_1 \\ T_{21}(\partial_{\mathbf{x}}, \mathbf{n}) & T_{22}(\partial_{\mathbf{x}}, \mathbf{n}) & bn_2 & -b_2n_2 \\ 0 & 0 & \alpha \frac{\partial}{\partial \mathbf{n}} & 0 \\ 0 & 0 & 0 & k \frac{\partial}{\partial \mathbf{n}} \end{pmatrix}.$$

Applying the operator $\mathbf{R}(\partial \mathbf{x}, \mathbf{n})$ to the matrix $\mathbf{\Gamma}(\mathbf{x-y}, \omega)$, we obtain

$$\boldsymbol{R}(\partial_{\mathbf{x}},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y},\omega) = \|R_{pq}\|_{4\times 4},$$

where the elements ${\cal R}_{pq}$ are the following

$$\begin{split} R_{11} &= \frac{\partial \varphi_4}{\partial n} + \left[-\rho \omega^2 n_1 + 2\mu \frac{\partial}{\partial s} \frac{\partial}{\partial x_2} \right] \frac{\partial \Psi_{11}}{\partial x_1}, \\ \Psi_{11} &= \frac{\varphi_4}{\lambda_4^2} - \frac{1}{\mu_0 \alpha k} \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) m_j \varphi_j, \\ R_{22} &= \frac{\partial \varphi_4}{\partial n} - \left[\rho \omega^2 n_2 + 2\mu \frac{\partial}{\partial s} \frac{\partial}{\partial x_1} \right] \frac{\partial \Psi_{11}}{\partial x_2}, \\ R_{12} &= \frac{\partial \varphi_4}{\partial s} + \left[-\rho \omega^2 n_1 + 2\mu \frac{\partial}{\partial s} \frac{\partial}{\partial x_2} \right] \frac{\partial \Psi_{11}}{\partial x_2}, \\ R_{21} &= -\frac{\partial \varphi_4}{\partial s} - \left[\rho \omega^2 n_2 + 2\mu \frac{\partial}{\partial s} \frac{\partial}{\partial x_1} \right] \frac{\partial \Psi_{11}}{\partial x_1}, \\ R_{3j} &= \alpha \frac{\partial \Gamma_{3j}}{\partial n}, \quad R_{4j} = k \frac{\partial \Gamma_{4j}}{\partial n}, \quad R_{33} = \alpha \frac{\partial \Gamma_{33}}{\partial n}, \quad R_{34} = \alpha \frac{\partial \Gamma_{34}}{\partial n}, \\ R_{43} &= k \frac{\partial \Gamma_{43}}{\partial n}, \quad R_{44} = k \frac{\partial \Gamma_{44}}{\partial n}, \end{split}$$

$$\begin{split} R_{13} &= \frac{\mu}{\mu_0 \alpha k} \left[-n_1 \lambda_4^2 + 2 \frac{\partial}{\partial s} \frac{\partial}{\partial x_2} \right] \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [-bk\lambda_j^2 + \beta b_3 + bb_1] \varphi_j, \\ R_{23} &= \frac{\mu}{\mu_0 \alpha k} \left[-n_2 \lambda_4^2 + 2 \frac{\partial}{\partial s} \frac{\partial}{\partial x_1} \right] \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [-bk\lambda_j^2 + \beta b_3 + bb_1] \varphi_j, \\ R_{14} &= \frac{\mu}{\mu_0 \alpha k} \left[n_1 \lambda_4^2 - 2 \frac{\partial}{\partial s} \frac{\partial}{\partial x_2} \right] \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [-\alpha \beta \lambda_j^2 + mb + \beta b_0] \varphi_j, \\ R_{24} &= \frac{\mu}{\mu_0 \alpha k} \left[n_2 \lambda_4^2 + 2 \frac{\partial}{\partial s} \frac{\partial}{\partial x_1} \right] \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [-\alpha \beta \lambda_j^2 + mb + \beta b_0] \varphi_j. \end{split}$$

Similarly, applying the operator $\widetilde{R}(\partial \mathbf{x}, \mathbf{n})$ to the matrix $\widetilde{\Gamma}(\mathbf{x}-\mathbf{y}, \omega) = \Gamma^T(\mathbf{y}-\mathbf{x}, \omega)$, we obtain

$$\boldsymbol{R}(\partial_{\mathbf{x}},\mathbf{n})\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y},\omega) = \|\hat{R}_{pq}\|_{4\times4},$$

where

$$\begin{split} \widetilde{R}_{pq} &= R_{pq}, \quad p, q = 1, 2, \quad \widetilde{R}_{3j} = -\alpha \frac{\partial \Gamma_{j3}}{\partial n}, \quad \widetilde{R}_{4j} = -k \frac{\partial \Gamma_{j4}}{\partial n}, \\ \widetilde{R}_{33} &= \alpha \frac{\partial \Gamma_{33}}{\partial n}, \quad \widetilde{R}_{34} = \alpha \frac{\partial \Gamma_{43}}{\partial n}, \quad \widetilde{R}_{44} = k \frac{\partial \Gamma_{44}}{\partial n}, \quad \widetilde{R}_{43} = k \frac{\partial \Gamma_{34}}{\partial n}, \\ \widetilde{R}_{13} &= \frac{\mu}{\mu_0 \alpha k} \left[n_1 \lambda_4^2 + 2 \frac{\partial}{\partial s} \frac{\partial}{\partial x_2} \right] \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [-kb\lambda_j^2 + bb_1 + mb_2] \varphi_j, \\ \widetilde{R}_{23} &= \frac{\mu}{\mu_0 \alpha k} \left[n_2 \lambda_4^2 - 2 \frac{\partial}{\partial s} \frac{\partial}{\partial x_1} \right] \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [-kb\lambda_j^2 + bb_1 + mb_2] \varphi_j, \\ \widetilde{R}_{14} &= \frac{\mu}{\mu_0 \alpha k} \left[n_1 \lambda_4^2 - 2 \frac{\partial}{\partial s} \frac{\partial}{\partial x_2} \right] \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [-\alpha b_2 \lambda_j^2 + bb_3 + b_0 b_2] \varphi_j, \\ \widetilde{R}_{24} &= \frac{\mu}{\mu_0 \alpha k} \left[n_2 \lambda_4^2 + 2 \frac{\partial}{\partial s} \frac{\partial}{\partial x_1} \right] \sum_{j=1}^4 d_j (\lambda_4^2 - \lambda_j^2) [-\alpha b_2 \lambda_j^2 + bb_3 + b_0 b_2] \varphi_j. \end{split}$$

Let $[\mathbf{R}(\partial_{\mathbf{y}}, \mathbf{n})\mathbf{\Gamma}(\mathbf{y}-\mathbf{x}, \omega)]^{\top}$, be the matrix which we get from $[\mathbf{R}(\partial_{\mathbf{x}}, \mathbf{n})\mathbf{\Gamma}(\mathbf{x}-\mathbf{y}, \omega)]$ by transposition of the columns and rows and the variables \mathbf{x} and \mathbf{y} (analogously $\left[\tilde{\mathbf{R}}(\partial \mathbf{y}, \mathbf{n})\tilde{\mathbf{\Gamma}}(\mathbf{y}-\mathbf{x}, \omega)\right]^{\top}$). Let us introduce the following single-layer and double-layer potentials :

the vector-functions defined by the equalities

$$\mathbf{V}(\mathbf{x};\mathbf{g}) = \frac{1}{\pi} \int_{S} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y}, \omega) \mathbf{g}(\mathbf{y}) d_{\mathbf{y}} S,$$

$$\widetilde{\boldsymbol{V}}(\mathbf{x};\mathbf{g}) = \frac{1}{\pi} \int_{S} \boldsymbol{\Gamma}^{\top}(\mathbf{y} - \mathbf{x}, \omega) \mathbf{g}(\mathbf{y}) d_{\mathbf{y}} S$$

will be called single- layer potentials, while the vector-functions defined by the equalities

$$\begin{split} \mathbf{W}(\mathbf{x};\mathbf{h}) &= \frac{1}{\pi} \int_{S} \left[\mathbf{R}(\partial_{\mathbf{y}},\mathbf{n}) \mathbf{\Gamma}(\mathbf{y}-\mathbf{x},\omega) \right]^{\top} \mathbf{h}(\mathbf{y}) d_{\mathbf{y}} S, \\ \widetilde{\mathbf{W}}(\mathbf{x};\mathbf{h}) &= \frac{1}{\pi} \int_{S} \left[\widetilde{\mathbf{R}}(\partial_{\mathbf{z}},\mathbf{n}) \mathbf{\Gamma}^{\top}(\mathbf{y}-\mathbf{x},\omega) \right]^{\top} \mathbf{h}(\mathbf{y}) d_{\mathbf{y}} S \end{split}$$

will be called double layer potentials. Here \mathbf{g} and \mathbf{h} are the continuous (or Hölder continuous) vectors and S is a closed Lyapunov curve.

By applying the methods, as in the classical theory of elasticity, we can state the following:(for details see in [18]).

Theorem 3. The vectors $\widetilde{\mathbf{V}}(\mathbf{x}; \mathbf{g})$ and $\mathbf{W}(\mathbf{x}; \mathbf{h})$ are the solutions of the system $\widetilde{\mathbf{A}}(\partial_{\mathbf{x}})\mathbf{U} = \mathbf{0}$ at any point \mathbf{x} and $\mathbf{x} \neq \mathbf{y}$. The vectors $\mathbf{V}(\mathbf{x}; \mathbf{g})$ and $\widetilde{\mathbf{W}}(\mathbf{x}; \mathbf{h})$ are the solutions of the system $\mathbf{A}(\partial_{\mathbf{x}})\mathbf{U} = \mathbf{0}$ at any point \mathbf{x} and $\mathbf{x} \neq \mathbf{y}$. The elements of the matrices $[\mathbf{R}(\partial_{\mathbf{y}}, \mathbf{n})\mathbf{\Gamma}(\mathbf{y} - \mathbf{x}\omega)]^{\top}$ and $[\widetilde{\mathbf{R}}(\partial_{\mathbf{y}}, \mathbf{n})\mathbf{\Gamma}^{\top}(\mathbf{x} - \mathbf{y})\omega]^{\top}$ contain a singular part, which is integrable in the sense of the Cauchy principal value.

Remark. By using the above-mentioned method, it is possible to construct explicitly the fundamental and singular matrices of solutions of the systems of equations in the modern linear theories of elasticity, thermoelasticity and poroelasticity.

5. A representation of general solutions

Theorem 4. If $U := (u, \varphi, \theta)$ is a regular solution of the homogeneous system (3) then u, divu, φ and θ satisfy the conditions

$$\begin{cases} (\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)(\Delta + \lambda_4^2)\boldsymbol{u} = 0, \\ (\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\boldsymbol{\Psi} = 0, \end{cases}$$
(10)

where $\boldsymbol{\Psi} = (\operatorname{div} \boldsymbol{u}, \varphi, \theta).$

Proof. Let $\mathbf{U} = (\mathbf{u}, \varphi, \theta)$ be a regular solution of the equation (3). Upon taking the divergence operation, from (3) we get

$$\begin{cases} (\mu_0 \Delta + \rho \omega^2) \operatorname{div} \mathbf{u} + b \Delta \varphi - \beta \Delta \theta = 0, \\ (\alpha \Delta + b_0) \varphi - b \operatorname{div} \mathbf{u} + m \theta = 0, \\ (k \Delta + b_1) \theta + b_2 \operatorname{div} \mathbf{u} + b_3 \varphi = 0. \end{cases}$$
(11)

Rewrite the latter system as follows

$$D(\Delta)\Psi := \begin{pmatrix} \mu_0 \Delta + \rho \omega^2 & b\Delta & -\beta\Delta \\ -b & \alpha\Delta + b_0 & m \\ b_2 & b_3 & k\Delta + b_1 \end{pmatrix} \Psi = 0,$$

By the direct evaluation, we get

$$detD = k\mu_0\alpha(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2).$$

Clearly, from the system (11) it follows that

$$\begin{cases} (\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2) \operatorname{div} \mathbf{u} = 0, \\ (\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\varphi = 0, \\ (\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\theta = 0, \end{cases}$$
(12)

Further, applying the operator $(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)$ to equation (3)₁, and using the last relations we obtain

$$(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)(\Delta + \lambda_4^2)\mathbf{u} = 0,$$
(13)

where

$$\lambda_4^2 = \frac{\varrho \omega^2}{\mu}.$$

The last formulas prove the theorem.

Theorem 5. The regular solution $U = (u, \varphi, \theta)$ of the system (3) admits in the domain of regularity a representation

$$\boldsymbol{U} = (\mathbf{\dot{u}} + \mathbf{\ddot{u}}, \varphi, \theta), \tag{14}$$

where $\overset{1}{\mathbf{u}}$, and $\overset{2}{\mathbf{u}}$ are the regular vectors, satisfying the conditions

$$(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)^{\mathbf{l}} = 0, \quad rot^{\mathbf{l}} = 0,$$
$$(\Delta + \lambda_4^2)^{\mathbf{l}} = 0, \quad \operatorname{div}^{\mathbf{l}} = 0$$

and the functions div $\boldsymbol{u}, \ \varphi, \ \theta$ can be replaced by the functions $\vartheta_j, \ j = 1, 2, 3$

$$\operatorname{div} \boldsymbol{u} = \sum_{j=1}^{3} A_{j} \vartheta_{j}, \quad \varphi = \sum_{j=1}^{3} B_{j} \vartheta_{j}, \quad \theta = \sum_{j=1}^{3} \vartheta_{j}, \quad (15)$$
$$A_{j} = \frac{(k\lambda_{j}^{2} - b_{1})(b_{0} - \alpha\lambda_{j}^{2}) + mb_{3}}{b_{2}(b_{0} - \alpha\lambda_{j}^{2}) + bb_{3}},$$
$$B_{j} = \frac{b(k\lambda_{j}^{2} - b_{1}) - mb_{2}}{b_{2}(b_{0} - \alpha\lambda_{j}^{2}) + bb_{3}},$$

respectively, where ϑ_j is the solution of the scalar equation

$$(\Delta + \lambda_j^2)\vartheta_j = 0.$$

Proof. It is easily checked that the expressions (15) satisfy the Eqs: $(3)_2$ and $(3)_3$. Let $\mathbf{U} = (\mathbf{u}, \varphi, \vartheta)$ be a regular solution of system (3). Using the identity

$$\Delta \mathbf{w} = \operatorname{graddiv} \mathbf{w} - \operatorname{rotrot} \mathbf{w},\tag{16}$$

from Eq. (3) we obtain

$$\mathbf{u} = -\frac{\mu_0}{\rho\omega^2} \operatorname{graddiv} \mathbf{u} - \frac{1}{\rho\omega^2} \operatorname{grad}(b\varphi - \beta\theta) + \frac{\mu}{\rho\omega^2} \operatorname{rotrot} \mathbf{u},$$

Let

$$\mathbf{\overset{1}{u}} := -\frac{\mu_0}{\rho\omega^2} \operatorname{graddiv} \mathbf{u} - \frac{1}{\rho\omega^2} \operatorname{grad}(b\varphi - \beta\theta), \tag{17}$$

$$\mathbf{\hat{u}} := \frac{\mu}{\rho \omega^2} \operatorname{rotrot} \mathbf{u}.$$
(18)

Clearly

$$\mathbf{u} = \mathbf{\hat{u}} + \mathbf{\hat{u}}, \quad \operatorname{rot}\mathbf{\hat{u}} = 0, \quad \operatorname{div}\mathbf{\hat{u}} = 0.$$
(19)

Using the identity $\Delta \mathbf{\hat{u}}^2 = -\text{rotrot}\mathbf{\hat{u}}^2$, from (18) we obtain

$$\left(\Delta + \lambda_4^2\right)^2 = 0. \tag{20}$$

Keeping in mind (15) from (17) we obtain

$$\label{eq:u_states} \begin{matrix} 1 \\ u = - \mathrm{grad} \sum_{j=1}^3 \frac{A_j}{\lambda_j^2} \vartheta_j. \end{matrix}$$

Thus we have the following representation of the general solution of system (3)

$$\mathbf{u} = -\text{grad} \sum_{j=1}^{3} \frac{A_j}{\lambda_j^2} \vartheta_j + \mathbf{u}^2, \quad \varphi = \sum_{j=1}^{3} B_j \vartheta_j, \quad \theta = \sum_{j=1}^{3} \vartheta_j,$$

$$A_j = \frac{(k\lambda_j^2 - b_1)(b_0 - \alpha\lambda_j^2) + mb_3}{b_2(b_0 - \alpha\lambda_j^2) + bb_3}, \quad B_j = \frac{b(k\lambda_j^2 - b_1) - mb_2}{b_2(b_0 - \alpha\lambda_j^2) + bb_3}.$$
(21)

6. Conclusions

In this paper the 2D linear theory of steady vibrations of thermoelasticity for materials with voids is considered and the following results are obtained:

1. The fundamental and singular matrices of solutions of the system of equations of steady vibrations in the 2D linear theory of thermoelasticity for isotropic materials with voids are constructed explicitly in terms of elementary (meta-harmonic) functions.

2. Some basic properties for single and double layer potentials are established.

3. The general solution of the system of steady vibrations in the linear theory of thermoelasticity for isotropic materials with voids is constructed by means four arbitrary metaharmonic functions.

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Author's address:

L. Bitsadze I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University 2, University St., Tbilisi 0186 Georgia E-mail: lamarabitsadze@yahoo.com