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CLOSED CONVEX SHELLS

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Abstract. If Ω is a closed convex shell, then $S : x_3 = 0$ is an ovaloid. It is proved that in this case the equation of equilibrium may have only the unique regular solution and hence, the corresponding homogenous equation has no non-zero solution on S.

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Mixed forms of stress-strain relations are given in the form

$$\sigma^i_j = \lambda\theta g^i_j + 2\mu e^i_j \quad (i = j = 1, 2, 3) \tag{1}$$

where σ^i_j and e^i_j are the mixed components, respectively, of stress and strain tensors, θ is the cubical dilatation which will be written as

$$\theta = e^i_i = \theta' + e^3_3, \quad \theta' = e^\alpha_\alpha, \quad (\alpha = 1, 2) \tag{2}$$

when $j = 3$ from (1) we have

$$\sigma^\alpha_3 = 2\mu e^\alpha_3, \quad \sigma^3_3 = \lambda\theta + 2\mu e^3_3 = \lambda\theta' + (\lambda + 2\mu)e^3_3. \tag{3}$$

from (3)

$$e^\alpha_3 = \frac{1}{2\mu}\sigma^\alpha_3, \quad e^3_3 = -\frac{\lambda}{\lambda + 2\mu}\theta' + \frac{1}{\lambda + 2\mu}\sigma^3_3. \tag{4}$$

By inserting (4) into (2) we obtain

$$\theta = \frac{\lambda'}{\lambda}\theta' + \frac{\lambda}{\lambda + 2\mu}\sigma^3_3, \tag{5}$$

where $\lambda' = \dfrac{2\lambda\mu}{\lambda + 2\mu}$.

Substituting expression (5) into (1) we get

$$\sigma^i_j = T^i_j + Q^i_j = \left(\lambda'\theta' + \frac{\lambda}{\lambda + 2\mu}\sigma^3_3\right) g^i_j + 2\mu e^i_j$$

where

$$T^\alpha_\beta = \lambda'\theta' g^\alpha_\beta + 2\mu e^\alpha_\beta, \quad Q^\alpha_\beta = \sigma'\sigma^3_3 g^\alpha_\beta, \quad T^i_3 = 0, \quad Q^i_3 = \sigma^i_3,$$

and $\sigma' = \dfrac{\lambda}{\lambda + 2\mu}$.

The vector $\boldsymbol{T}^\alpha$ satisfies the condition $\boldsymbol{n}\boldsymbol{T}^\alpha = 0$ and is therefore called the tangential stress field and the vector $\boldsymbol{Q}^i$ will be called the transverse stress field.

The vectorial equation of equilibrium has the form

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\boldsymbol{\sigma}^i) + \boldsymbol{\Phi} = 0,$$

$$(\sqrt{g} = \sqrt{a}\vartheta, \quad \vartheta = 1 - 2Hx_3 + Kx_3^2)$$

where H and K are respectively, middle and principal curvatures of the surface S, may be written as

$$\frac{1}{\sqrt{g}}[\partial_\alpha\sqrt{g}\boldsymbol{T}^\alpha) + \partial_i(\sqrt{g}\boldsymbol{Q}^i)] + \boldsymbol{\Phi} = 0 \tag{6}$$

Let the surface $\hat{S} : x^3 = const$ be the neutral surface of a non-shallow shell. Then $\boldsymbol{T}^\alpha = 0$, i.e. $T^{\alpha\beta} = 0$ (on $\hat{S}$), and equation (6) becomes

$$\frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{a}\vartheta\boldsymbol{Q}^\alpha) + \partial_3(\vartheta\boldsymbol{\sigma}^3) + \vartheta Phi = 0,$$

or

$$\left[\nabla_\alpha(\vartheta\boldsymbol{Q}^\alpha) + \partial_3(\vartheta\boldsymbol{\sigma}^3) + \vartheta\boldsymbol{\Phi}\right]_{x^3=c} = 0; \quad (-h \le x^3 = x_3 \le h)$$

where $2h$ is the thickness of shell and

$$\boldsymbol{Q}^\alpha = \sigma'\sigma_3^3\boldsymbol{r}^\alpha + \sigma_3^\alpha\boldsymbol{n}, \quad \nabla_\alpha(\cdot) = \frac{1}{\sqrt{a}}\partial_\alpha(\sqrt{a}(.)).$$

Denoting the stress forces acting on the face surfaces S^+ and S^- by $\overset{(+)}{\boldsymbol{P}}$ and $\overset{(-)}{\boldsymbol{P}}$ we have

$$\overset{(+)}{\boldsymbol{P}} = -(\boldsymbol{\sigma}^3)_{x^3=h}, \quad \overset{(-)}{\boldsymbol{P}} = (\boldsymbol{\sigma}^3)_{x^3=-h}, \tag{7}$$

If we approximately represent $\boldsymbol{\sigma}^3$ by the formula

$$\boldsymbol{\sigma}^3(x^1,x^2,x^3) \cong \overset{(0)}{\boldsymbol{\sigma}}(x^1,x^2) + x^3\overset{(1)}{\boldsymbol{\sigma}}(x^1,x^2).$$

From (7) we get

$$\begin{aligned}\boldsymbol{\sigma}^3(x^1,x^2,x^3) &\cong -\frac{1}{2}\left[\overset{(+)}{\boldsymbol{P}} - \overset{(-)}{\boldsymbol{P}} + \frac{x^3}{h}(\overset{(+)}{\boldsymbol{P}} + \overset{(-)}{\boldsymbol{P}})\right] \\ &= -\frac{1}{2}\left[\frac{h+x^3}{h}(\overset{(+)}{\boldsymbol{P}} - \overset{(-)}{\boldsymbol{P}}) + \frac{2x^3}{h}\overset{(-)}{\boldsymbol{P}}\right]\end{aligned}$$

or

$$\boldsymbol{\sigma}^3(x^1,x^2,x^3) = -\frac{1}{2}\left[\frac{h+x^3}{h}(p^\alpha\boldsymbol{r}_\alpha + p^3\boldsymbol{n}) + \frac{2x^3}{h}\overset{(-)}{\boldsymbol{P}}\right],$$

$$p^\alpha = \overset{(+)}{P^\alpha} - \overset{(-)}{P^\alpha}, \quad p = \overset{(+)}{P^3} - \overset{(-)}{P^3}.$$

Then to define the vector field $\overset{(+)}{\boldsymbol{P}}$ we have the equation

$$\{\nabla_\alpha(\sigma' A^\alpha_\beta p \boldsymbol{r}^\beta + A p^\alpha \boldsymbol{n}) + B(p\boldsymbol{n} + p^\alpha \boldsymbol{r}_\alpha) + \tilde{\boldsymbol{\Phi}}\}_{x^3=c} = 0 \tag{8}$$

where

$$A^\alpha_\beta = \frac{h+c}{h}[a^\alpha_\beta + c(b^\alpha_\beta - 2Ha^\alpha_\beta)], \quad A = \frac{h+c}{h}\vartheta(c)$$

$$B = \frac{1}{h}[1 - 2Hh + 2(Kh - 2H)c + 3kc^2],$$

$$\tilde{\boldsymbol{\Phi}} = -2\vartheta(c)\boldsymbol{\Phi}(c) + \nabla_\alpha\{\sigma' \frac{2c}{h}[a^\alpha_\beta + c(b^\alpha_\beta - 2Ha^\alpha_\beta)] \overset{(-)}{p^3} \boldsymbol{r}^\beta + \frac{2c}{h}\vartheta(c) \overset{(-)}{p^\alpha} \boldsymbol{n}\}$$

$$+\frac{2}{h}[\vartheta(c) + 2(kc - h)] \overset{(-)}{\boldsymbol{p}}.$$

From (8) we have

$$\sigma'\nabla_\alpha(A^\alpha_\beta p) + (Ba_{\alpha\beta} - Ab_{\alpha\beta})p^\alpha + \tilde{\Phi}_\beta = 0, \quad \tilde{\Phi}_\beta = \tilde{\boldsymbol{\phi}}\boldsymbol{r}_\beta, \tag{9}$$

$$\nabla_\alpha(Ap^\alpha) + (\sigma' A^\alpha_\beta b^\beta_\alpha + B)p + \tilde{\Phi}_3 = 0, \quad (\tilde{\Phi}_3 = \tilde{\boldsymbol{\Phi}}\boldsymbol{n}). \tag{10}$$

From the system of equation (9) we have

$$p^\alpha = \overset{(+)}{P^\alpha} - \overset{(-)}{P^\alpha} = -\tilde{d}^{\alpha\beta}[\nabla_\gamma(A^\gamma_\beta p) + \tilde{\Phi}_\beta], \quad p = \overset{(+)}{P^3} - \overset{(-)}{P^3} \tag{11}$$

where

$$\hat{d}^{\alpha\beta} = \frac{1}{\Delta}[(B - 2AH)a^{\alpha\beta} + Ab^{\alpha\beta}] \quad \hat{F}_\beta = -[\tilde{\Phi}_\beta + \nabla_\alpha(A^\alpha_\beta p)],$$

$$\Delta = B^2 - 2ABH + A^2K.$$

Inserting expressions (11) into (10) we obtain the equation

$$\sigma'\nabla_\alpha[A\tilde{d}^{\alpha\beta}\nabla_\gamma(A^\gamma_\beta p)] - (B + \sigma' A^\alpha_\beta b^\beta_\alpha)p + \Phi = 0. \tag{12}$$

It is easily seen that equation (12) is of the elliptic type.

Thus, if the surface $x^3 = c$ is neutral then the stresses $\overset{(+)}{\boldsymbol{P}}$ and $\overset{(-)}{\boldsymbol{P}}$ applied to the face surfaces, must satisfy the vector equations (9) and (10). This means that the stresses $\overset{(+)}{\boldsymbol{P}}$ and $\overset{(-)}{\boldsymbol{P}}$ cannot be prescribed arbitrarily both at the same time. However there are problems when this does not occur. For example, in aircraft or submarine apparatus the force $\overset{(-)}{\boldsymbol{P}}$ acting on the inner face surface S^- may be assumed to be prescribed, but the force $\overset{(+)}{\boldsymbol{P}}$ acting on the external face surface S^+ is not, in general, assigned beforehand. The same situation occurs on dams. One face surface of the dam is free from stresses and the other is under the hydrodynamic load, a variable which is generally difficult to define exactly at any moment of time.

For closed convex shells when $x^3 = c$ is the middle surface (i.e. $c = 0 \Rightarrow x^3 = 0$) the homogenous equation (12) may be written in the form

$$\nabla_\alpha(d^{\alpha\beta}\nabla_\beta u) - d^2 u = 0, \tag{13}$$

where

$$d^{\alpha\beta} = \frac{h[a^{\alpha\beta}(1-2hH) + hb^{\alpha\beta}]}{1 - 2Hh + Kh^2 + 4hH(2Hh-1)},$$

$$d^2 = \frac{1}{\sigma' h}[1 - 2h(1-\sigma')H] > 0, \quad \left(\sigma' = \frac{\lambda}{\lambda + 2\mu}\right)$$

$$a^{\alpha\beta} = \boldsymbol{r}^\alpha \boldsymbol{r}^\beta, \quad \boldsymbol{r}^\alpha \boldsymbol{r}_\beta = a^\alpha_\beta = \sigma^\alpha_\beta, \quad 2H = b^1_1 + b^2_2, \quad K = b^1_1 b^2_2 - b^2_1 b^1_2,$$

$$I = a_{\alpha\beta}dx^\alpha dx^\beta, \quad a_{\alpha\beta} = \boldsymbol{r}_\alpha \boldsymbol{r}_\beta,$$

$$II = b_{\alpha\beta}dx^\alpha dx^\beta, \quad b_{\alpha\beta} = -\boldsymbol{n}_\alpha \boldsymbol{r}_\beta.$$

Let u be the regular solution (13) on $S(x^3 = 0)$, i.e. u is the continuous function of the point of the surface S and has continuous partial derivatives with respect to Gausian coordinates of this surface. We represent the surface S as $S = S_1 \cup S_2$, where S_1 and S_2 are parts of the surface with no common points $S_1 \cap S_2 = \cap = \emptyset$. Let L be the common boundary of S_1 and S_2. Denote the tangential normal to L by l directed to S_1. Multiplying both sides of equation (13) by u, we may rewrite it as

$$\nabla_\alpha(ud^{\alpha\beta}\nabla_\beta u) - d^{\alpha\beta}\nabla_\alpha u \nabla_\beta u - d^2u^2 = 0.$$

Integrating this equality with respect to the surfaces S_1 and S_2, and then applying Green's formula, we have

$$\int_L ul_\alpha d^{\alpha\beta}\nabla_\beta u ds - \iint_{S_1} (d^{\alpha\beta}\nabla_\alpha u \nabla_\beta u + d^2u^2)dS_1 = 0,$$

$$-\int_L ul_\alpha d^{\alpha\beta}\nabla_\beta u ds - \iint_{S_2} (d^{\alpha\beta}\nabla_\alpha u \nabla_\beta u + d^2u^2)dS_2 = 0.$$

By adding these equalities we obtain

$$\iint_s (d^{\alpha\beta}\nabla_\alpha u \nabla_\beta u + d^2 u)dS = 0. \tag{14}$$

Since $d^{\alpha\beta}\nabla_\alpha u \nabla_\beta u \geq 0$, $d^2 > 0$ from (14) is follows that $u = 0$, which was to be proved.

The problem under consideration is thus reduced to the determination of the globally regular particular solution of the non-homogeneous equation

$$\sigma'\nabla_\alpha(d^{\alpha\beta}\nabla_\beta p) - \frac{1}{h}[1 - 2(1-\sigma')hH]p + \Phi = 0. \tag{15}$$

It remains to show that if equation (15) has globally regular solution, then the middle surface $S : x^3 = 0$ of the shell is neutral. To do this we have to show first that the tangential stress field vanishes on S, i.e. it should be shown that the equation

$$\frac{1}{\sqrt{a}}\partial_\alpha(\sqrt{a}\boldsymbol{T}^\alpha) \equiv \frac{1}{\sqrt{a}}\partial_\alpha(\sqrt{a}T^{\alpha\beta}\boldsymbol{r}_\beta) = 0 \tag{16}$$

has no globall solution, except trivial $T_{\alpha\beta} = 0$. It is evident since, with respect to isometric-conjugate coordinates x, y, equation (16) is equivalent to the homogeneous generalized Cauchy-Riemann equation

$$\partial_{\bar{z}}w - B\bar{w} = 0, \quad (z = x + iy)$$

where

$$w = \frac{1}{2}aK^{\frac{1}{4}}(T^{11} - T^{22} - 2iT^{12}), \quad T^{11} + T^{22} = 0. \tag{17}$$

The complex stress function w is continuous on the whole plane E of the complex variable $z = x + iy$ and at infinity satisfies the condition

$$w = 0(|z|^{-4}).$$

This implies, in view of the generalized Liouville theorem, that $w = 0$. then from (17) it follows that $T^{\alpha\beta} = 0$, which was to be proved.

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