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OSCILLATION OF SOLUTIONS OF SECOND ORDER ALMOST LINEAR DIFFERENCE EQUATIONS

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Abstract. We study oscillatory properties of solutions of a difference equation of the form

$$\Delta^{(2)} u(k) + F(u)(k) = 0,$$

where $\Delta u(k) = u(k+1) - u(k)$, $\Delta^{(2)} = \Delta \circ \Delta$, $F : \ell(N) \to \ell(N)$ ($\ell(N)$ denote the set of functions $u : N \to R$). In the paper, sufficient conditions are established for all proper solutions of the above equation to be oscillatory.

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1. Introduction

For any $k \in N$ denote $N_k = \{k, k+1, \dots\}$. Let $\tau : N \to N$, $\lim\limits_{k \to +\infty} \tau(k) = +\infty$. Denote by $V(\tau)$ the set of operators $F : \ell(N) \to \ell(N)$ (By $\ell(N)$ denote the set of functions $u : N \to R$) satisfying the condition: $F(x)(k) = F(y)(k)$ holds for any $k \in N$ and $x, y \in \ell(N)$ provided that $x(s) = y(s)$ for $s \in N_{\tau(k)}$.

This work is dedicated to the study of oscillatory properties of solutions of a difference equation of the form

$$\Delta^{(2)} u(k) + F(u)(k) = 0. \tag{1.1}$$

For any $k_0 \in N$ we denote by $H_{k_0,\tau}$ the set of all functions $u \in \ell(N)$ - satisfying $u(k) > 0$ or $u(k) < 0$ for $k \geq k_*$, where $k_* = \min\{\tau_*(k_0), k_0\}$, $\tau_*(k) = \inf\{\tau(s) : s \in N_k\}$.

It will always be assumed that the condition

$$F(u)(k)\, u(k) \geq 0 \quad \text{for} \quad k \in N_{k_0}, \;\; u \in H_{k_0,\tau} \tag{1.2}$$

is fulfilled.

Let $k_0 \in N$. A function $u : N_{k_0} \to R$ is said to be a proper solution of equation (1.1) if

$$\sup\big\{|u(s)| : s \in N_k\big\} > 0 \quad \text{for} \quad k \in N_{k_0}$$

and there exist a function $u_* \in \ell(N)$ such that $u_*(k) = u(k)$ for $k \in N_{k_0}$ and the equality

$$\Delta^{(2)} u_*(k) + F(u_*)(k) = 0$$

holds for $k \in N_{k_0}$.

Definition 1.1 A proper solution $u : N_{k_0} \to R$ of equation (1.1) is said to be oscillatory if for any $k \in N_{k_0}$, there are $n_1, n_2 \in N_k$ such that $u(n_1)\, u(n_2) \leq 0$. Otherwise, the proper solution is called nonoscillatory.

The present paper is devoted to the problem of oscillation of (1.1). As to second order linear and nonlinear difference equations, they are studied well enough in [1–7].

2. A necessary conditions for the existence of a positive solution

The result obtained in this section is very important for establishing sufficient conditions of oscillation of all proper solutions of equation (1.1). Below the following notation will be used.

Let $k_0 \in N$. Denote by U_{k_0} the set of all proper solutions of (1.1) satisfying the condition $u(k) > 0$ for $k \in N_{k_0}$. Everywhere we assumed that the inequality

$$\big|F(u)(k)\big| \geq \sum_{j=1}^{m} \sum_{s=\sigma_i(k)}^{\tau_i(k)} |u(s)|^{\mu_i(s)} \Delta_s r_i(s,k) \ \text{ for } \ k \in N_{k_0}, \ u \in H_{k_0\tau} \tag{2.1}$$

holds, where

$$\sigma_i;\, \tau_i : N \to N, \ \ \mu_i : N \to (0,+\infty), \ \ \sigma_i(k) \leq k, \ \ \sigma_i(k) \leq \tau_i(k), \ \ \lim_{k\to+\infty} \sigma_i(k) = +\infty,$$

$r_i : N^2 \to R$, $r_i(\cdot,k)$ are nondecreasing functions $(i = 1,\dots,m)$.

Theorem 2.1 *Let $F \in V(\tau)$, conditions* (1.2), (2.1) *be fulfilled,*

$$\sum_{k=1}^{+\infty} \bigg(\sum_{j=1}^{m} \sum_{s=\sigma_j(k)}^{\tau_i(k)} s^{\mu_i(s)} \Delta_s r_i(s,k) \bigg) = +\infty, \tag{2.2}$$

$$\sum_{k=1}^{+\infty} \bigg(k \sum_{j=1}^{m} \sum_{s=\sigma_j(k)}^{\tau_i(k)} \Delta_s r_i(s,k) \bigg) = +\infty \tag{2.3}$$

and

$$\liminf_{k\to+\infty} \mu_j(k) > 0 \quad (j = 1,\dots,m). \tag{2.4}$$

Moreover, let $U_{k_0} \neq \emptyset$ for some $k_0 \in N$. Then there exist $\lambda_0 \in [0,1]$ such that

$$\limsup_{\varepsilon\to0+} \bigg(\liminf_{k\to+\infty} k^{1-\lambda_0+h_{2\varepsilon}(\lambda_0)} \sum_{i=k}^{+\infty} \frac{i^{-h_\varepsilon(\lambda_0)}}{i(1+i)} \sum_{\ell=1}^{i} \ell \bigg(\sum_{j=1}^{m} \sum_{s=\sigma_j(\ell)}^{\widetilde{\tau}_j(\ell)} \times s^{(\lambda_0-h_{2\varepsilon}(\lambda_0))\mu_j(s)+h_\varepsilon(\lambda_0)\mu(s)} \Delta_s r_j(s,\ell) \bigg) \bigg) \leq 1,$$

where

$$\mu(k) = \min\big\{1, \mu_1(k),\dots,\mu_m(k)\big), \quad h_\varepsilon(\lambda) = h_{1\varepsilon}(\lambda) + h_{2\varepsilon}(\lambda),$$
$$h_{1\varepsilon} = \begin{cases} 0 & \textit{for } \lambda = 0, \\ \varepsilon & \textit{for } \lambda \in (0,1], \end{cases} \qquad h_{2\varepsilon} = \begin{cases} 0 & \textit{for } \lambda = 1, \\ \varepsilon & \textit{for } \lambda \in [0,1), \end{cases} \tag{2.5}$$
$$\widetilde{\tau}_j(k) = \min\big\{k, \tau_j(k)\big\} \quad (j = 1,\dots,m).$$

Theorem 2.2 *Let $F \in V(\tau)$, conditions* (1.2), (2.1)–(2.4), *be fulfilled and $U_{k_0} \neq \emptyset$ for some $k_0 \in N$. Then there exist $\lambda_0 \in [0,1]$ such that*

$$\limsup_{\varepsilon\to 0+}\bigg(\liminf_{k\to+\infty} k^{(1-\frac{1}{\mu(k)})-\lambda_0-h_{2\varepsilon}(\lambda_0)}\sum_{i=k}^{k-1}\big(\sigma(i)\big)^{h_\varepsilon(\lambda_0)}\sum_{\ell=1}^{+\infty}\bigg(\sum_{j=1}^{m} \times\sum_{s=\sigma_j(k)}^{\tau_j(\ell)} s^{(\lambda_0-h_{1\varepsilon}(\lambda_0))\mu_j(s)}\Delta_s r_j(s,\ell)\bigg)\bigg)\le 1,$$

where functions μ, $h_{1\varepsilon}$, $h_{2\varepsilon}$, h_ε are given by (2.5) *and σ is a nondecreasing function satisfying the condition*

$$\lim_{k\to+\infty}\sigma(k)=+\infty,\ \ \sigma(k)\le\min\big\{k,\sigma_i(k): i=1,\dots,m\big\},\ \ \sigma(N_k)\supset\bigcup_{j=1}^{m}\sigma_j(N_k)$$

for any $k\in N$.

3. Sufficient conditions for oscillation

Theorem 3.1 *Let $F \in V(\tau)$, conditions* (1.2), (2.1)–(2.4), *be fulfilled and for any $\lambda \in [0,1]$*

$$\limsup_{\varepsilon\to 0+}\bigg(\liminf_{k\to+\infty} k^{1-\lambda+h_{2\varepsilon}(\lambda)}\sum_{i=k}^{+\infty}\frac{i^{-h_\varepsilon(\lambda)}}{i(1+i)}\sum_{\ell=1}^{i}\ell\bigg(\sum_{j=1}^{m}\sum_{s=\sigma_j(\ell)}^{\widetilde{\tau}_j(\ell)} s^{(\lambda-h_{1\varepsilon}(\lambda))\mu_j(s)} \times s^{h_\varepsilon(\lambda)\mu(s)}\Delta_s r_j(s,\ell)\bigg)\bigg)>1,$$

Then any proper solution of equation (1.1) *is oscillatory, where functions μ, $h_{1\varepsilon}$, $h_{2\varepsilon}$, h_ε and $\widetilde{\tau}_j$ are given by* (2.5).

Theorem 3.2 *Let $F \in V(\tau)$, conditions* (1.2), (2.1)–(2.4) *be fulfilled and for any $\lambda \in [0,1]$*

$$\limsup_{\varepsilon\to 0+}\bigg(\liminf_{k\to+\infty} k^{(1-\frac{1}{\mu(k)})\lambda_\varepsilon(\lambda)-\lambda_0-h_{2\varepsilon}(\lambda)}\sum_{i=1}^{k-1}\big(\sigma(i)\big)^{h_\varepsilon(\lambda)}\sum_{\ell=1}^{+\infty}\bigg(\sum_{j=1}^{m} \times\sum_{s=\sigma_j(\ell)}^{\widetilde{\tau}_j(\ell)} s^{(\lambda-h_{1\varepsilon}(\lambda))\mu_j(s)}\Delta_s r_j(s,\ell)\bigg)\bigg)>1.$$

Then any proper solution of equation (1.1) *is oscillatory, where functions $h_{1\varepsilon}$, $h_{2\varepsilon}$, h_ε and $\widetilde{r}_j$ are given by* (2.5).

Corollary 3.1 Let $F \in V(\tau)$, conditions (1.2), (2.1)–(2.4) be fulfilled and for any $\lambda \in [0,1]$ there exist $\delta > 1$, such that

$$\liminf_{\varepsilon\to 0+}\bigg(\liminf_{k\to+\infty} k^{-\lambda-h_{1\varepsilon}(\lambda)}\sum_{i=1}^{k} i\bigg(\sum_{j=1}^{m}\sum_{s=\sigma_j(i)}^{\widetilde{\tau}_j(i)} s^{(\lambda-h_{2\varepsilon}(\lambda))\mu_j(s)+h_\varepsilon(\lambda)\mu(s)} \times\Delta_s r_j(s,i)\bigg)\bigg)>(1-\lambda)\delta.$$

Then any proper solution of equation (1.1) is oscillatory, where functions $h_{1\varepsilon}$, $h_{2\varepsilon}$, h_ε and $\widetilde{\tau}_j$ are given by (2.5).

Corollary 3.2 Let $F \in V(\tau)$, conditions (1.2), (2.1)–(2.4) be fulfilled and for any $\lambda \in [0,1]$ there exist $\delta > 1$, such that

$$\liminf_{\varepsilon\to 0+}\left(\liminf_{k\to+\infty} k^{-1}\sum_{i=1}^{k} i^{2-\lambda-h_{2\varepsilon}(\lambda)}\left(\sum_{j=1}^{m}\sum_{s=\sigma_j(i)}^{\widetilde{\tau}_j(i)} s^{(\lambda+h_{2\varepsilon}(\lambda))\mu_j(s)+h_\varepsilon(\lambda)\mu(s)} \times \Delta_s r_j(s,i)\right)\right) > \lambda(1-\lambda)\delta.$$

Then any proper solution of equation (1.1) is oscillatory, where functions $h_{1\varepsilon}$, $h_{2\varepsilon}$, h_ε, μ and $\widetilde{\tau}_j(i)$ are given by (2.5).

Corollary 3.3 Let $F \in V(\tau)$, conditions (1.2), (2.1)–(2.4) be fulfilled and there exist $\alpha_i \in (0,+\infty)$ $(i = 1,\dots,m)$ such that

$$\liminf_{k\to+\infty}\frac{\sigma_i(k)}{k^{\alpha_i}} > 0 \quad (i = 1,\dots,m). \tag{3.1}$$

Moreover, if for any $\lambda \in [0,1]$

$$\limsup_{\varepsilon\to 0+}\left(\liminf_{k\to+\infty} k^{(1-\frac{1}{\mu(k)})h_\varepsilon(\lambda)-\lambda-h_{2\varepsilon}(\lambda)}\sum_{i=1}^{k-1} i^{\alpha h_\varepsilon(\lambda)} \times \sum_{\ell=1}^{+\infty}\left(\sum_{j=1}^{m}\sum_{s=\sigma_j(i)}^{\tau_j(\ell)} s^{(\lambda-h_{1\varepsilon}(\lambda))\mu_j(s)}\Delta_s r_j(s,\ell)\right)\right) > 1,$$

then any proper solution of equation (1.1) is oscillatory, where functions $h_{1\varepsilon}$, $h_{2\varepsilon}$, h_ε and μ are given by (2.5), $\alpha = \min\{1,\alpha_1,\dots,\alpha_m\}$.

Corollary 3.4 Let $F \in V(\tau)$, conditions (1.2), (2.1)–(2.4) and (3.1) be fulfilled and for any $\lambda \in [0,1]$ there exist $\delta > 1$, such that

$$\limsup_{\varepsilon\to 0+}\left(\liminf_{k\to+\infty} k^{1-\lambda+\alpha h_\varepsilon(\lambda)-h_{2\varepsilon}(\lambda)}\sum_{\ell=k}^{+\infty} \times \left(\sum_{j=1}^{m}\sum_{s=\sigma_j(\ell)}^{\tau_j(\ell)} s^{(\lambda-h_{1\varepsilon}(\lambda))\mu_j(s)}\Delta_s r_j(s,\ell)\right)\right) > \delta\,\lambda.$$

Then any proper solution of equation (1.1) is oscillatory, where functions $h_{1\varepsilon}$, $h_{2\varepsilon}$, h_ε and α are given by (2.5).

Corollary 3.5 Let $F \in V(\tau)$, conditions (1.2), (2.1)–(2.4) and (3.1) be fulfilled and for any $\lambda \in [0,1]$ there exist $\delta > 1$, such that

$$\limsup_{\varepsilon\to 0+}\left(\liminf_{k\to+\infty} k^{1+(\alpha-1)h_\varepsilon(\lambda)}\sum_{\ell=k}^{+\infty}\left(\sum_{j=1}^{m} \times \sum_{s=\sigma_j(i)}^{\tau_j(i)} s^{(\lambda-h_{1\varepsilon}(\lambda))\mu_j(s)}\Delta_s r_j(s,\ell)\right)\right) \geq \delta\,\lambda(1-\lambda).$$

Then any proper solution of equation (1.1) is oscillatory, where functions $h_{1\varepsilon}$, $h_{2\varepsilon}$, h_ε and α are given by (2.5) and (3.1).

Corollary 3.6 Let $F \in V(\tau)$, $k_0 \in N$, α_i, $\beta_i \in (0,+\infty)$, $d_i \in R$, $\alpha_i < \beta_i \le 1$, $p_i; N \to R_+$ $(i = 1,\dots,m)$, conditions (1.2) be fulfilled and

$$\big|F(u)(k)\big| \ge \sum_{i=1}^{m} p_i(k) \sum_{s=[\alpha_i k]}^{[\beta_i k]} \big|u(s)\big|^{1+\frac{d_i}{\log_2 s}} \quad \text{for} \quad k \in N_{k_0}, \ \ u \in H_{k_0}. \tag{3.2}$$

Then the condition

$$\liminf_{k\to+\infty} k^{-1} \sum_{i=1}^{k} i^3 \Big(\prod_{\ell=1}^{m} p_\ell(i)\Big)^{\frac{1}{m}} > \max\left\{ \frac{\lambda(1-\lambda)(1+\lambda)2^{-\frac{\lambda}{m}\sum\limits_{\ell=1}^{m} d_\ell}}{\prod\limits_{\ell=1}^{m} (\beta_\ell^{1+\lambda} - \alpha_\ell^{1+\lambda})^{\frac{1}{m}}} : \lambda \in [0,1] \right\} \tag{3.3}$$

is sufficient for oscillation of a proper solutions of (1.1).

Corollary 3.7 Let $F \in V(\tau)$ and for some $k_0 \in N$ condition (3.2) be fulfilled. Then the condition

$$\liminf_{k\to+\infty} k \sum_{i=1}^{+\infty} i \Big(\prod_{\ell=1}^{m} p_\ell(i)\Big)^{\frac{1}{m}} > \max\left\{ \frac{\lambda(1-\lambda)(1+\lambda)2^{-\frac{\lambda}{m}\sum\limits_{\ell=1}^{m} d_\ell}}{\Big(\prod\limits_{\ell=1}^{m} (\beta_\ell^{1+\lambda} - \alpha_\ell^{1+\lambda})\Big)^{\frac{1}{m}}} : \lambda \in [0,1] \right\} \tag{3.4}$$

is sufficient for oscillation of a proper solution of (1.1).

Remark 3.1 For $m = 1$ the conditions (3.3) and (3.4) are optimal conditions. Here we will give an example illustrating that condition (3.3) for $m = 1$ is an optimal condition. That is (3.3) cannot be replaced by the inequality

$$\liminf_{k\to+\infty} k^{-1} \sum_{i=1}^{+\infty} i^3 \Big(\prod_{\ell=1}^{m} p_\ell(i)\Big)^{\frac{1}{m}} \ge \max\left\{ \frac{\lambda(1-\lambda)(1+\lambda)2^{-\frac{\lambda}{m}\sum\limits_{\ell=1}^{m} d_\ell}}{\prod\limits_{\ell=1}^{m} (\beta_\ell^{1+\lambda} - \alpha_\ell^{1+\lambda})^{\frac{1}{m}}} : \lambda \in [0,1] \right\}. \tag{3.5}$$

Let $m = 1$, $d \in R$ and $\alpha < \beta \le 1$. Denote

$$c_0 = \max\left\{ \frac{\lambda(1-\lambda)(1+\lambda)2^{-\lambda d}}{\beta^{1+\lambda} - \alpha^{1+\lambda}} : \lambda \in [0,1] \right\} \tag{3.6}$$

and let λ_0 be the point where attains the maximum. Consider the equation

$$\Delta^{(2)} u(k) + p(k) \sum_{s=[\alpha k]}^{[\beta k]} \Big(\big(u(s)\big)^{1+\frac{d_i}{\log_2 s}} \operatorname{sign} u(s) \Big) = 0 \tag{3.7}$$

where

$$p(k) = \frac{-\Delta^{(2)}k^{\lambda_0}}{\sum\limits_{s=[\alpha k]}^{[\beta k]} s^{\lambda_0\left(1+\frac{d}{\log_2 s}\right)}} = \frac{-\Delta^{(2)}k^{\lambda_0}}{2^{d\lambda_0}\sum\limits_{s=[\alpha k]}^{[\beta k]} s^{\lambda_0}}\,. \tag{3.8}$$

It is obvious that

$$\Delta^{(2)}k^{\lambda_0} = \frac{\lambda_0(\lambda_0-1)}{k^{2-\lambda_0}} + \frac{o(1)}{k^{2-\lambda_0}}. \tag{3.9}$$

On the other hand we have

$$\begin{aligned}\sum_{s=[\alpha k]}^{[\beta k]} s^{\lambda_0} &= \sum_{s=[\alpha k]}^{[\beta k]} s^{\lambda_0}\int_s^{s+1} d\xi \le \sum_{s=[\alpha k]}^{[\beta k]}\int_s^{s+1}\xi^{\lambda_0}d\xi \\ &= \int_{[\alpha k]}^{[\beta k]+1}\xi^{\lambda_0}d\xi = \frac{1}{1+\lambda_0}\Big(\big([\beta k]+1\big)^{1+\lambda_0} - [\alpha k]^{1+\lambda_0}\Big) \\ &= \frac{k^{\lambda_0+1}}{1+\lambda_0}\big(\beta^{1+\lambda_0}-\alpha^{1+\lambda_0}\big)\big(1+o(1)\big). \end{aligned}\tag{3.10}$$

and

$$\begin{aligned}\sum_{s=[\alpha k]}^{[\beta k]} s^{\lambda_0} &= \sum_{s=[\alpha k]}^{[\beta k]} s^{\lambda_0}\int_{s-1}^{ds} d\xi \ge \sum_{s=[\alpha k]}^{[\beta k]}\int_{s-1}^{s}\xi^{\lambda_0}d\xi \\ &= \int_{[\alpha k]-1}^{[\beta k]}\xi^{\lambda_0} = \frac{1}{\lambda_0+1}\Big([\beta k]^{\lambda_0+1} - \big([\alpha k]-1\big)^{\lambda_0+1}\Big) \\ &= \frac{k^{\lambda_0+1}}{1+\lambda_0}\big(\beta^{1+\lambda_0}-\alpha^{1+\lambda_0}\big)\big(1+o(1)\big). \end{aligned}\tag{3.11}$$

By (3.10) and (3.11) we have

$$k^{-\lambda_0-1}\sum_{s=[\alpha k]}^{[\beta k]} s^{\lambda_0} = \frac{1}{1+\lambda_0}\big(\beta^{1+\lambda_0}-\alpha^{1+\lambda_0}\big)\big(1+0(1)\big).$$

According to (3.6), (3.8) and (3.9)

$$p(k) = \frac{\lambda_0(1-\lambda_0)2^{-\lambda_0 d}}{k^3\frac{1}{1+\lambda}(\beta^{\lambda_0+1}-\alpha^{\lambda_0+1})} + \frac{o(1)}{k^3} = \frac{c_0}{k^3} + \frac{o(1)}{k^3}\,.$$

Therefore

$$\liminf_{k\to+\infty} k^{-1}\sum_{i=1}^{k} i^3 p(i) = c_0$$

i.e. condition (3.5) holds, but equation (3.7) has a positive solution $u(k)=k^{\lambda_0}$.

4. Difference equations with deviating arguments

Throughout this section it is assumed that instead of (2.1) the inequality

$$\big|F(u)(k)\big| \ge \sum_{i=1}^{m} p_i(k)\big|u(\delta_i(k))\big|^{\mu_i(k)} \quad \text{for} \quad k\ge k_0,\ \ u\in H_{k_0,\tau} \tag{4.1}$$

holds with $k_0 \in N$ sufficiently large. Here we assume that

$$\begin{gathered}\delta_i : N \to N, \quad \lim_{k\to+\infty} \delta_i(k) = +\infty \ \ (i = 1, \dots, m), \quad p_i : N \to R_+, \\ \mu_i : N \to (0, +\infty), \quad \liminf_{k\to+\infty} \mu_i(k) > 0 \quad (i = 1, \dots, m).\end{gathered} \tag{4.2}$$

Theorem 4.1 *Let $F \in V(\tau)$, conditions* (1.2), (4.1), (4.2) *be fulfilled*

$$\sum_{k=1}^{+\infty} \sum_{i=1}^{m} p_i(k) \big(\delta_i(k)\big)^{\mu_i(k)} = +\infty, \tag{4.3}$$

$$\sum_{k=1}^{+\infty} k \sum_{i=1}^{m} p_i(k) = +\infty \tag{4.4}$$

and for any $\lambda \in [0, 1]$ there exist $\delta > 1$ such that

$$\limsup_{\varepsilon\to 0+} \left(\liminf_{k\to+\infty} k^{-1} \sum_{i=1}^{k} i^{2-\lambda-h_{2\varepsilon}(\lambda)} \sum_{j=1}^{m} p_j(i) \big(\widetilde{\tau}_j(i)\big)^{(\lambda-h_{2\varepsilon}(\lambda))\mu_j(i)+h_\varepsilon(\lambda)\mu(i)} \right) > \delta\lambda(1-\lambda).$$

Then any proper solution of equation (1.1) *is oscillatory, where functions $h_{1\varepsilon}$, $h_{2\varepsilon}$, h_ε, μ are given by* (2.5) *and*

$$\widetilde{\tau}_j(i) = \min\big\{i, \delta_j(i) : j = 1, \dots, m\big\}. \tag{4.5}$$

Theorem 4.2 *Let $F \in V(\tau)$, conditions* (1.2), (3.1), (4.1)–(4.4) *be fulfilled and for any $\lambda \in [0, 1]$ there exist $\delta > 1$ such that*

$$\limsup_{\varepsilon\to 0+} \left(\liminf_{k\to+\infty} k^{1+(\alpha-1)h_\varepsilon(\lambda)} \sum_{i=k}^{+\infty} \left(\sum_{j=1}^{m} p_j(i) \big(\widetilde{\tau}_j(i)\big)^{(\lambda-h_{1\varepsilon}(\lambda))\mu_j(i)} \right) \right) > \delta\lambda(1-\lambda),$$

then any proper solution of equation (1.1) *is oscillatory, where functions h_1, h_2, h_3, μ and $\widetilde{\tau}$ are given by* (2.5) *and* (4.5).

Corollary 4.1 Let $F \in V(\tau)$, $\alpha_i \in (0, +\infty)$, $d_i \in R$, $p_i : N \to R_+$, $(i = 1, \dots, m)$, conditions (1.2), (4.4) and

$$\big|F(u)(k)\big| \ge \sum_{j=1}^{m} p_j(k) \big|u(\alpha_j k)\big|^{1+\frac{d_j}{\log_2 \alpha_j \cdot k}} \ \text{ for } \ k \ge k_0, \ \ u \in H_{k_0,\tau}, \tag{4.6}$$

be fulfilled. Moreover, if for any $\lambda \in [0, 1]$ there exist $\delta > 1$ such that

$$\liminf_{k\to+\infty} k^{-1} \sum_{i=1}^{k} i^2 \sum_{j=1}^{m} p_j(i)(\alpha_j)^\lambda 2^{\lambda d_j} > \delta\lambda(1-\lambda),$$

then any proper solution of equation (1.1) is oscillatory.

Corollary 4.2 Let $F \in V(\tau)$, conditions (1.2), (4.4), (4.6) be fulfilled and for any $\lambda \in [0, 1]$ there exist $\delta > 1$ such that

$$\liminf_{k\to+\infty} k \sum_{i=k}^{+\infty} \sum_{j=1}^{m} p_j(i)(\alpha_j)^\lambda 2^{d_j\lambda} > \delta\lambda(1-\lambda).$$

Then any proper solution of equation (1.1) is oscillatory.

Corollary 4.3 Let $F \in V(\tau)$, conditions (1.2), (4.4), (4.6) be fulfilled and

$$\liminf_{k\to+\infty} k^{-1} \sum_{i=1}^{k} i^2 \left(\prod_{j=1}^{m} p_j(i)\right)^{\frac{1}{m}} > \max\left\{ \frac{\lambda(1-\lambda)2^{-\frac{\lambda}{m}\sum\limits_{j=1}^{m} d_j}}{\left(\prod\limits_{j=1}^{m} \alpha_j\right)^{\frac{\lambda}{m}}} : \lambda \in [0,1] \right\}.$$

Then any proper solution of equation (1.1) is oscillatory.

Corollary 4.4 Let $F \in V(\tau)$, conditions (1.2), (4.4), (4.6) be fulfilled and

$$\liminf_{k\to+\infty} k \sum_{i=k}^{+\infty} \left(\prod_{j=1}^{m} p_j(i)\right)^{\frac{1}{m}} > \max\left\{ \frac{\lambda(1-\lambda)2^{-\frac{\lambda}{m}\sum\limits_{j=1}^{m} d_j}}{\left(\prod\limits_{j=1}^{m} \alpha_j\right)^{\frac{\lambda}{m}}} : \lambda \in [0,1] \right\}.$$

Then any proper solution of equation (1.1) is oscillatory.

R E F E R E N C E S

1. Agarval R. P. Difference equations and inequalities. *II Edition, Mercel Dekker, New York*, (2000).

2. Agarval R. P., Grace S. R. and O'Regam D. Oscillation theory for Difference and Functional differential equations. *Kluwer, Dordrecht*, (2000).

3. Cheng S. S., Yan T. Ch. and Li H. J. Oscillation criteria for second order difference equations. *Funkcialaj Eqvacioj*, **34** (1991), 223–239.

4. Koplatadze R. G. Oscillation of linear difference equations with deviating arguments. *Comp. Math. Appl.*, **42** (2001), 477-486.

5. Koplatadze R., Kvinikadze G. and Stavroulakis I. Oscillation of second order linear difference equations with deviating arguments. *Adv. Math. Sci. Appl.* **12**, 1 (2002), 217-226.

6. Koplatadze R., Kvinikadze G. Necessary conditions for existence of positive solutions of second order linear difference equations and sufficient conditions for oscillation of solutions. *J. Nonlinear Oscillations*, **12**, 2 (2009), 180-194.

7. Koplatadze R. G. and Pinelas S. Oscillation of nonlinear difference equations with delayed argument Commun. *Appl. Anal.* **16**, 1 (2012), 87-95.

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