Seminar of I. Vekua Institute of Applied Mathematics REPORTS, Vol. 43, 2017

ON ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF HIGHER ORDER DIFFERENCE EQUATION

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Abstract. The present paper deals with the oscillation problem of solutions of the higher order Emden-Fowler type difference equation. We gave sufficient conditions that difference Equation has Property A or Property B.

Keywords and phrases: Property A, Property B, oscillation.

AMS subject classification (2010): 34K11.

1. Introduction

Consider the higher order nonlinear difference equation

$$\Delta^{(n)}u(k) + p(k)|u(\sigma(k))|^{\lambda}\mathrm{sign}(u(\sigma(k))) = 0, \tag{1.1}$$

where $n \geq 2$, $0 < \lambda < 1$, $p : N \to R$, $\sigma : N \to N$.

$$\lim_{k\to\infty} \sigma(k) = +\infty. \tag{1.2}$$

Here

$$\Delta u(k) = \Delta^{(1)}u(k) = u(k+1) - u(k), \quad \Delta^{(i)}u(k) = \Delta \circ \Delta^{(i-1)}u(k) \quad i = 2, 3, \dots, n.$$

It will always be assumed that either the condition

$$p(k) \geq 0, \quad \text{for} \quad k \in N, \tag{1.3}$$

or

$$p(k) \leq 0, \quad \text{for} \quad k \in N \tag{1.4}$$

holds.

2. Sufficient conditions of nonexistence of monotone solutions

For each $k \in N$ denote $N_k = \{k, k+1, \dots\}$.

Definition 2.1. Let $k_0 \in N$. A function $u : N_{k_0} \to R$ is said to be a proper solution of Equation (1.1), if

$$\sup\{|u(k)| : k \geq s\} > 0, \quad \text{for any} \quad s \geqslant k_0,$$

and there exists the function $\overline{u} : N \to R$ such that $\overline{u}(k) \equiv u(k)$ on N_{k_0} and the equality

$$\Delta^{(n)}\overline{u}(k) + p(k)\,|\overline{u}(\sigma(k))|^{\lambda}\, sign\overline{u}(\sigma(k)) = 0$$

holds for $k \in N_{k_0}$.

Definition 2.2. Let $k_0 \in N$. A proper solution $u : N_{k_0} \to R$ of equation (1.1) is said to be oscillatory if for any $k \in N_{k_0}$ there are $k_1, k_2 \in N_{k_0}$ such that $u(k_1)u(k_2) \leq 0$. Otherwise the solution is called nonoscillatory.

Definition 2.3. We say that equation (1.1) has Property A if any its proper solutions are oscillatory when n is even and either is oscillatory or satisfies

$$|\Delta^{(i)}u(k)| \downarrow 0, \quad \text{for} \quad k \uparrow +\infty \quad i = 0, 1, \dots, n-1, \tag{2.1}$$

when n is odd.

Definition 2.4. We say that equation (1.1) has Property B if any of its proper solutions are oscillatory or satisfies either (2.1) or

$$|\Delta^{(i)}u(k)| \uparrow +\infty, \quad \text{for} \quad k \uparrow +\infty \quad i = 0, 1, \dots, n-1, \tag{2.2}$$

when n is even and either is oscillatory or satisfies (2.2) when n is odd.

In the present paper we give sufficient condition for equation (1.1) to have Property A or B. Analogously problem for higher and second order difference equations is considered in [1-4].

Similar problems for functional differential equations were considered in [5-7].

For any $k_0 \in N$ denote by $U_{k_0,l}$ the set of solutions $u : N_{k_0} \to R$ of equation (1.1) which satisfies the condition:

$$\begin{aligned} \Delta^{(i)}u(k) > 0, \quad &\text{for} \quad k \geq k_0 \quad i = 0, \dots, l-1, \\ (-1)^i \Delta^{(i)}u(k) \geq 0, \quad &\text{for} \quad k \geq k_0 \quad i = l, \dots, n. \end{aligned}$$

Theorem 2.1. *Let conditions (1.2), (1.3) ((1.4)) be fulfilled, ,$l \in \{1, 2, \dots, n-1\}$, $l+n$ is odd ($l+n$ is even) and*

$$\sum_{j=1}^{+\infty} j^{n-l-1} \sigma^{\lambda(l-1)}(j) \widetilde{\sigma}^{\lambda}(j)\, |p(j)| = +\infty. \tag{2.3}$$

Then for any $k_0 \in N$, $U_{k_0,l} = \varnothing$, where

$$\widetilde{\sigma}(k) = \begin{cases} \sigma(k) & if \quad \sigma(k) \leq k \\ k & if \quad \sigma(k) > k \end{cases}.$$

3. Difference equations with property A

Theorem 3.1 *Let conditions* (1.2), (1.3) *and for any $l \in \{1, \dots, n-1\}$ with $l+n$ is odd* (2.3) *be fulfilled. Moreover, if*

$$\sum_{k=1}^{\infty} k^{n-1} p(k) = +\infty,$$

when n is odd, then Equation (1.1) *has Property A.*

Theorem 3.2 *Let conditions* (1.2), (1.3) *be fulfilled and $\sigma(k) \leq k$ for $k \in N$. Moreover, if*

$$\sum_{k=1}^{+\infty} \sigma^{\lambda(n-1)}(k) p(k) = +\infty,$$

then equation (1.1) *has Property A.*

4. Difference equations with property B

Theorem 4.1 *Let conditions* (1.2), (1.4) *and for any* $l \in \{1, \ldots, n-1\}$, $l+n$ *is even* (2.3) *be fulfilled. Moreover, if*

$$\sum_{k=1}^{+\infty} k^{n-1}|p(k)| = +\infty,$$

when n *is even and*

$$\sum_{k=1}^{+\infty} (\sigma(k))^{\lambda(n-1)}\,|p(k)| = +\infty, \tag{4.1}$$

then equation (1.1) *has Property B.*

Theorem 4.2. *Let conditions* (1.2), (1.4) *be fulfilled and* $\sigma(k) \le k$ *for any* $k \in N$. *Then equation (1.1) to have Property B it is sufficient that condition (4.1) had been fulfilled.*

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Received 07.09.2017; revised 20.10.2017; accepted 15.11.2017

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