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ABOUT ONE PROBLEM OF THE PLANE THEORY OF ELASTICITY WITH A PARTIALLY UNKNOWN BOUNDARY

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Abstract. In the present paper we consider the problem of finding a partially unknown boundary of the plane theory of elasticity for a rectangular domain which is weakened by an equally strong contour (the unknown part of the boundary). The unknown part of the boundary is assumed to be free from external force, and to the remaining part of the rectangular boundary are applied the same absolutely smooth rigid punches subjected to the action of external normal contractive forces with the given principal vectors.

For solution of the problem using the method of complex analysis and Kolosov-Muskhelishvili's potentials and the equation of the equally strong contour are constructed effectively (analytically).

Keywords and phrases: Equally strong contour, Kolosov-Muskhelishvili's formulas, conformal mapping.

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Introduction

The boundary value problems of the plane theory of elasticity and plate bending with a partially unknown boundary or which is the same the problems of finding an equally strong contour belong to the broad problems of optimization of elastic bodies and where always was in the focus of attention of many scientists. Different methods were introduced for researching these problems and among them one of important is the method of complex analysis. Analogous problems of plane elasticity are considered in [1-10].

1. Statement of the problem

Let a homogeneous Isotropic plate on a plane $z = x + iy$ of a complex variable occupy a domain S. S is a rectangular from which is cuttings the unknown part of the boundary (see Fig. 1). The unknown part of the boundary (an equally strong contour) is assumed to be free from external force, and to the remaining part of the rectangular boundary are applied the same absolutely smooth rigid punches subjected to the action of external normal contractive forces with the given principal vectors P and Q, respectively.

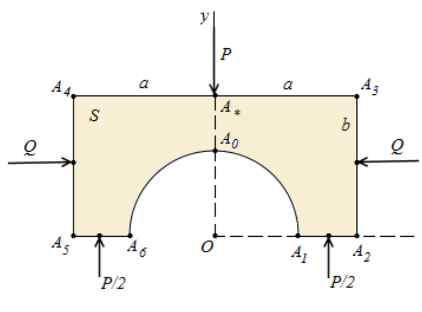

Fig. 1.

Consider the problem: Find an elastic equilibrium of the plate and analytic form of an unknown contour under the condition that the tangential normal stress takes on the contour value $\sigma_\vartheta = k = const$.

2. Solution of the problem

By $L = L_1 \cup L_0$ we denote the boundary of the domain S consisting of rectilinear segments $L_1 = \overset{5}{\underset{k=1}{\cup}} L_1^{(k)} \equiv \overset{5}{\underset{k=1}{\cup}} A_k A_{k+1}$ and arc $L_0 = A_6 A_1$.

On the basis of the well-known Kolosov-Muskhelishvili formulas [11] the problem under consideration is reduced to finding two holomorphic in S functions $\varphi(z)$ and $\psi(z)$ with the following boundary conditions L:

$$\mathrm{Re}\left[e^{-\alpha(t)}(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)})\right] = C(t), \quad t \in L_1, \tag{1}$$

$$\mathrm{Re}\left[e^{-i\alpha(t)}(\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}\right] = 2\mu v_n(t), \ \ t \in L_1, \tag{2}$$

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = 0, \quad t \in L_0, \tag{3}$$

$$\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} = 0, \quad t \in L_0, \tag{4}$$

$$\mathrm{Re}\left[\varphi'(t)\right] = \frac{k}{4}, \quad t \in L_0, \tag{5}$$

where $\alpha(t)$ is the angle lying between the ox-axis and the external normal to the boundary L_1 at the point $t \in L_1$, and $\alpha(t) = \alpha_k = -\dfrac{\pi}{2} + \dfrac{\pi(k-1)}{2}$, $t \in L_1^{(k)}$, $(k = \overline{1,5})$. $C(t)$ and $v_n(t)$ are the piecewise constant functions $C(t) = \mathrm{Re} \int\limits_{A_1}^{t} iN(s_0) \exp i[\alpha(t_0) - \alpha(t)] ds_0$, $N(s_0)$ is a normal stress; $v_n(t) = v_n^k(t) = const$, $t \in L_1^{(k)}$, $(k = 1,5)$ and $v_n(t) = 0$, $t \in L_0$.

Summing up the equalities (1) and (2), differentiating with respect to the arc abscissa s and taking into account the fact that the functions $c(t)$ and $v_n(t)$ are piecewise constant, we obtain

$$\mathrm{Im}\varphi'(t) = 0, \ \ t \in L_1. \tag{6}$$

The conditions (5) and (6) can be written in the following form

$$\mathrm{Re}\left[\varphi'(t) - \frac{k}{4}\right] = 0, \ t \in L_0; \mathrm{Im}\left[\varphi'(t) - \frac{k}{4}\right] = 0 \ \ t \in L_1 \tag{7}$$

It is proved that problem (7) has a unique solution

$$\varphi(z) = \frac{k}{4} z \tag{8}$$

(an arbitrary constant of integration is assumed to be equal to zero).

By virtue of relation (7), boundary condition (1), for the functions

$$\phi_1(z) = \frac{k}{2} z - \psi(z) + iQ \tag{9}$$

can be written in the following forms:

$$
\begin{aligned}
&\mathrm{Re}\phi_1(t) = \tfrac{P}{2} + ka, \ \ t \in A_2A_3; \ \ \mathrm{Im}\phi_1(t) = 0, \ \ t \in A_3A_4;\\
&\mathrm{Re}\phi_1(t) = -\tfrac{P}{2} - ka; \ \ t \in A_4A_5,\\
&\mathrm{Im}\phi_1(t) = Q, \ \ t \in A_1A_2 \cup A_5A_6 \cup L_0.
\end{aligned} \tag{10}
$$

From the given conditions and simple transformation for the functions

$$
\phi_2(z) = \frac{\varkappa - 1}{4} kz + \psi(z) - iQ - i\frac{\varkappa + 1}{4} kb \tag{11}
$$

we obtain the following boundary value problem:

$$
\begin{aligned}
&\mathrm{Re}\ \phi_2(t) = -\frac{P}{2} + \frac{\varkappa - 3}{4} ka, \ \ t \in A_2A_3;\\
&\mathrm{Im}\phi_2(t) = 0, \ \ t \in A_3A_4;\\
&\mathrm{Re}\ \phi_2(t) = \frac{P}{2} - \frac{\varkappa - 3}{4} ka; \ \ t \in A_4A_5;\\
&\mathrm{Im}\phi_2(t) = -Q - \tfrac{\varkappa+1}{4} kb, \ \ t \in A_5A_2,
\end{aligned} \tag{12}
$$

($2a$ and b are the length of the sides of the rectangle).

Consider the problem (10). Let the function $z = \omega(\zeta)$ map conformally the upper half-plane ($\mathrm{Im}\zeta$) onto the domain S. By a_k we denote preimages of the points A_k ($k = \overline{1,6}$) and assume that $\omega(A_*) = \infty$ (A_* is the middle point the side A_3A_4); $a_6 = -1$; $a_1 = 1$. Moreover, owing to the symmetry, we may assume that $a_2 = \delta_0$; $a_5 = -\delta_0$; $a_3 = \delta$; $a_2 = -\delta$.

We will seek a bounded at infinity solution of problem (10) of the class $h(a_1, ..., a_6)$ (regarding this class see [12]). The indices of the problems of the given class are equal to -2.

The necessary and sufficient condition for the existence of a bounded at infinity solution of problem (10) has the form

$$
H_1 \int_{-\delta}^{-\delta_0} \frac{dt}{\chi_1(t)} + iH_2 \int_{-\delta_0}^{\delta_0} \frac{dt}{\chi_1(t)} + H_3 \int_{\delta_0}^{\delta} \frac{dt}{\chi_1(t)} = 0 \tag{13}
$$

where $H_1 = \dfrac{-P}{2} - ka$; $H_2 = Q;$; $H_3 = \dfrac{P}{2} + ka$; $\chi_1(\xi) = \sqrt{(\zeta^2 - \delta^2)(\zeta^2 - 1)}$, and the solution itself is given by the formula

$$
\Phi_{10}(\zeta) = \frac{\chi_1(\zeta)}{\pi i} \left[H_1 \int_{-\delta}^{-\delta_0} \frac{dt}{\chi_1(t)(t - \zeta)} + iH_2 \int_{-\delta_0}^{\delta_0} \frac{dt}{\chi_1(t)(t - \zeta)} + H_3 \int_{\delta_0}^{\delta} \frac{dt}{\chi_1(t)(t - \zeta)} \right], \tag{14}
$$

where $\Phi_{10}(\zeta) = \phi_1[\omega(\zeta)]$.

Similarly, The necessary and sufficient condition for the existence of a bounded at infinity solution of problem (12) has the form

$$
H_4 \int_{-\delta}^{-\delta_0} \frac{dt}{\chi_1(t)} + iH_5 \int_{-\delta_0}^{\delta_0} \frac{dt}{\chi_1(t)} + H_6 \int_{\delta_0}^{\delta} \frac{dt}{\chi_1(t)} = 0. \tag{15}
$$

and such a solution is represented by the formula

$$\Phi_{20}(\zeta) = \frac{\chi_1(\zeta)}{\pi i}\left[H_4\int\limits_{-\delta}^{-\delta_0}\frac{dt}{\chi_1(t)(t-\zeta)} + iH_5\int\limits_{-\delta_0}^{\delta_0}\frac{dt}{\chi_1(t)(t-\zeta)} + H_6\int\limits_{\delta_0}^{\delta}\frac{dt}{\chi_1(t)(t-\zeta)}\right], \quad (16)$$

where $\Phi_{20}(\zeta) = \phi_2[\omega(\zeta)]$; $H_4 = \frac{P}{2} - \frac{\varkappa-3}{4}ka$; $H_5 = -Q - \frac{\varkappa+1}{4}kb$; $H_6 = -\frac{P}{2} + \frac{\varkappa-3}{4}ka$.

Having found the fuctions $\Phi_{10}(\zeta)$ and $\Phi_{20}(\zeta)$, by virtue of (9) and (11), we can define the fuctions $\omega(\zeta)$ and $\psi_0(\zeta) = \psi[\omega(\zeta)]$ by the formulas

$$\omega(\zeta) = \frac{4}{(\varkappa+1)k}[\Phi_{10}(\zeta) + \Phi_{20}(\zeta)] + ib, \quad (17)$$

$$\psi_0(\zeta) = \frac{2}{\varkappa+1}[\Phi_{20}(\zeta) - \frac{\varkappa-1}{2}\Phi_{10}(\zeta)] + iQ + \frac{kb}{2}. \quad (18)$$

The equation for the part A_6A_1 of the unknown contour can be obtained from the image of the function $\omega(\xi)$ for $\xi \in [-1;\ 1]$.

The integrals appearing in formulas (13)-(16) are the first and third kind elliptic integral and have the forms (see [1])

$$I_1 = \int\limits_{-\delta}^{-\delta_0}\frac{dt}{\chi_1(t)} = \frac{-2i}{\delta+\delta_0}F\left[\frac{\pi}{2}; \frac{\delta-\delta_0}{\delta+\delta_0}\right]$$

$$I_2 = \int\limits_{-\delta}^{-\delta_0}\frac{dt}{\chi_1(t)(t-\xi)} = \frac{-2i}{(\xi^2-\delta_0^2)(\delta+\delta_0)}\left\{-2\delta_0\Pi\left[\frac{\pi}{2}; \frac{(\delta-\delta_0)(\xi-\delta_0)}{(\delta+\delta_0)(\xi+\delta_0)}; \frac{\delta-\delta_0}{\delta+\delta_0}\right]\right.$$
$$\left. + (\xi+\delta_0)F\left[\frac{\pi}{2}; \frac{\delta-\delta_0}{\delta+\delta_0}\right]\right\}$$

$$I_3 = \int\limits_{-\delta}^{-\delta_0}\frac{dt}{\chi_1(t)} = \frac{2}{\delta+\delta_0}F\left[\frac{\pi}{2}; \frac{2\sqrt{\delta_0\delta}}{\delta+\delta_0}\right];$$

$$I_4 = \int\limits_{-\delta_0}^{\delta_0}\frac{dt}{\chi_1(t)(t-\xi)} = \frac{2}{(\xi+\delta_0)(\xi+\delta)(\delta+\delta_0)}\left\{(\delta-\delta_0)\Pi\left[\frac{\pi}{2}; \frac{2\delta_0(\xi+\delta)}{(\delta+\delta_0)(\xi+\delta_0)}; \frac{2\sqrt{\delta_0\delta}}{\delta+\delta_0}\right]\right.$$
$$\left. + (\xi+\delta_0)F\left[\frac{\pi}{2}; \frac{2\sqrt{\delta_0\delta}}{\delta+\delta_0}\right]\right\},$$

$$I_5 = \int\limits_{\delta_0}^{\delta}\frac{dt}{\chi_1(t)} = \frac{2i}{\delta+\delta_0}F\left[\frac{\pi}{2}; \frac{\delta-\delta_0}{\delta+\delta_0}\right];$$

$$I_6 = \int\limits_{-\delta_0}^{\delta_0}\frac{dt}{\chi_1(t)(t-\xi)} = \frac{-2i}{(\xi^2-\delta^2)(\delta+\delta_0)}\left\{2\delta\Pi\left[\frac{\pi}{2}; \frac{(\delta_0-\delta)(\xi+\delta)}{(\delta_0+\delta)(\xi-\delta)}; \frac{\delta-\delta_0}{\delta+\delta_0}\right]\right.$$
$$\left. + (\xi-\delta)F\left[\frac{\pi}{2}; \frac{\delta-\delta_0}{\delta+\delta_0}\right]\right\},$$

where $F(\varphi;k) = \int\limits_0^{\varphi} \dfrac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$ is the elliptic integral of the first kind, $\Pi(\varphi;n;k) = \int\limits_0^{\varphi} \dfrac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}$ is the elliptic integral of the third kind.

If the approximations

$$F\left[\frac{\pi}{2};k\right] = \frac{\pi}{2}\frac{4+k^2}{4}; \quad \Pi\left[\frac{\pi}{2};n;k\right] = \frac{\pi}{2}\left[1+\frac{k^2+2n}{4}\right],$$

are satisfied, then conditions (13) and (15) take the forms

$$\begin{aligned} &(p+2ka)\left(4+\frac{(\delta-\delta_0)^2}{(\delta+\delta_0)^2}\right) + Q\left(4+\frac{4\delta_0\delta}{(\delta+\delta_0)^2}\right) = 0, \\ &\left(p-\frac{\varkappa-3}{2}ka\right)\left(4+\frac{(\delta-\delta_0)^2}{(\delta+\delta_0)^2}\right) + \left(Q+\frac{\varkappa+1}{4}kb\right)\left(4+\tfrac{4\delta_0\delta}{(\delta+\delta_0)^2}\right) = 0. \end{aligned} \tag{19}$$

The necessary and sufficient condition for the existence of a solution of system (19) has the form

$$(p+2ka)\left(Q+\frac{\varkappa+1}{4}kb\right) + Q\left(-p+\frac{\varkappa-3}{2}ka\right) = 0$$

Thus to find k we obtain the formula

$$K = -\left(\frac{P}{2a}+\frac{Q}{b}\right), \tag{20}$$

(sign $(-)$ indicates that we have the contractive stress).

After that for determination of the parameters δ_0 and δ we have obtained one condition

$$(P+2ka)(4(\delta+\delta_0)^2 + (\delta-\delta)^2) + 4Q((\delta+\delta_0)^2 + \delta_0\delta) = 0. \tag{21}$$

Let us $\alpha = \dfrac{a}{b}$, $x = \dfrac{\delta_0}{\delta}$. Owing to the condition (20) from (21) we obtain the quadratic equation

$$(5\alpha-2)x^2 + 6(\alpha-1)x + 5\alpha - 2 = 0. \tag{22}$$

We assume that $\alpha \neq \dfrac{2}{5}$, then equation (22) is reduced in the following form

$$x^2 + 6\frac{\alpha-1}{5\alpha-2}x + 1 = 0. \tag{23}$$

The necessary and sufficient condition for the existence of a unique solution of problem (23) from the segment $(0;1)$ is

$$\begin{cases} \dfrac{\alpha-1}{5\alpha-2} < 0, \\ \dfrac{8\alpha-5}{5\alpha-2} < 0, \end{cases}$$

and for α we obtain the estimate

$$\frac{2}{5} < \alpha < \frac{5}{8},$$

so the length of the sides of the rectangle must satisfy the condition

$$\frac{4}{5} < \frac{2\alpha}{b} < \frac{10}{8}.$$

In particular if we assume that $b = 2a$ (i.e., when the preset rectangular is a square), then $x_0 = \dfrac{\delta_0}{\delta} = 3 - \sqrt{8}$. So if δ is given we obtain δ_0 and from formula (17) for $\xi \in [-1, 1]$, the part of the unknown equally strong contour is defined.

By use the boundary values of Cauchy type integral from (14), (16) and (17) for $\xi \in [-1, 1]$ we have

$$\omega(\xi) = -\frac{\chi_1(\xi)}{\pi i}\left[a\int\limits_{-\delta}^{-\delta_0}\frac{dt}{\chi_1(t)(t-\xi)} + ib\int\limits_{-\delta_0}^{\delta_0}\frac{dt}{\chi_1(t)(t-\xi)} - a\int\limits_{\delta_0}^{\delta}\frac{dt}{\chi_1(t)(t-\xi)}\right].$$

It is easy to see that $\omega(\xi)$ satisfy the condition $\omega(-\xi) = \omega(\xi)$, so we can consider only $\xi \in [0, 1]$.

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R E F E R E N C E S

1. Cherepanov G. P. Inverse problems of the plane theory of elasticity (Russian). *J. Appl. Math. Mech.* **38**, 6 (1974), 915931 (1975). Translated from *Prikl. Mat. Meh.* **38**, 6 (1974), 963-979.

2. Banichuk N. V. Optimization of Forms of Elastic Bodies (Russian). *Nauka, Moscow,* 1980.

3. Bantsuri R. On one mixed problem of the plane theory of elasticity with a partially unknown boundary. *Proc. A. Razmadze Math. Inst.*, **140** (2006), 9-16.

4. Bantsuri R. Solution of the mixed problem of plate bending for a multi-connected domain with partially unknown boundary in the presence of cyclic symmetry. *Proc. A. Razmadze Math. Inst.*, **145** (2007), 9-22.

5. Odishelidze N., Criado-Aldeanueva F. Some axially symmetric problems of the theory of plane elasticity with partially unknown boundaries. *Acta Mechanica*, **199** (2008), 227-240.

6. Mzhavanadze Sh. V. Inverse problems of elasticity theory in the presence of cyclic symmetry (Russian). *Soobshch. Akad. Nauk Gruzin. SSR* **113**, 1 (1984), 53-56.

7. Kapanadze G. A. On a problem of bending a plate for a doubly connected domain with partially unknown boundary (Russian). *Prikl. Mat. Mekh.*, **71**, 1 (2007), 33-42; *translation in J. Appl. Math. Mech.*, **71**, 1 (2007), 30-39.

8. Kapanadze G. On one problem of the plane theory of elasticity with a partially unknown boundary. *Proc. A. Razmadze Math. Inst.*, **143** (2007), 61-71.

9. Kapanadze G. A. The problem of plate bending for the finite doubly-connected domain with a partially unknown boundary. (Russian) *Prikl. Mekh.*, **39**, 5 (2003), 121-126.

10. Bantsuri R., Kapanadze, G. The problem of finding a full-strength contour inside the polygon. *Proc. A. Razmadze Math. Inst.*, **163** (2013), 1-7.

11. Muskhelishvili N. I. Some basic problems of the mathematical theory of elasticity. *Nauka, Moscow,* 1966.

12. Muskhelishvili N. I. Singular integral equations. *Nauka, Moscow*, 1968.

13. Keldysh M. V., Sedov L. I. Effective solution of some boundary value problems for harmonic functions. *Dokl. Akad. Nauk SSSR*, **16**, 1 (1937), 7-10.

14. Prudnikov A. P., Brychkov, Yu. A., Marichev, O. I. Integrals and Series. Elementary Functions. *Nauka, Moscow,* 1981.

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