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ON THE WELL-POSSEDNESS OF THE CAUCHY PROBLEM FOR ONE CLASS OF NEUTRAL QUASI-LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Theorems on the continuous dependence of a solution on perturbations of the initial data and nonlinear term of the right-hand side are given for the functional differential equations with the discontinuous initial condition whose right-hand sides are linear with respect to the prehistory of the phase velocity. The perturbations of initial data (initial moment, initial vector, initial functions, initial matrix, delay in the phase coordinates) are small in the standard norm, the perturbation of the nonlinear term right-hand side is small in the integral sense.

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Let $= I = [a, b]$ be a finite interval and let $\mathbb{R}^n$ be the n-dimensional vector space of points $(x^1, \dots, x^n)^\top$, where $\top$ is the sign of transposition. Suppose that $O \subset \mathbb{R}^n$ is an open set and let E_f be the set of functions $f : I \times O^2 \times \mathbb{R}^n$ which satisfy the following conditions: for each fixed $(x, y) \in O^2$ the function $f(\,\cdot\,, x, y) : I \times \mathbb{R}^n$ is measurable; for each $f \in E_f$ and compact set $K \subset O$ there exist functions $M_{f,K}(t), L_{f,K}(t) \in L(I, \mathbb{R}_+)$, where $\mathbb{R}_+ = [0, +\infty)$ such that for almost all $t \in I$

$$|f(t, x, y)| \le M_{f,K}(t) \;\; \forall\, (x, y) \in K^2,$$
$$\big|f(t, x_1, y_1) - f(t, x_2, y_2)\big| \le L_{f,K}(t) \sum_{i=1}^{2} |x_i - y_i| \;\; \forall\, (x_i, y_i) \in K^2, \;\; i = 1, 2.$$

We introduce the topology in E_f by the following basis of neighborhoods of zero:

$$\big\{V_{K,\delta} : \; K \subset O \text{ is a compact set and } \delta > 0 \text{ is an arbitrary number}\big\},$$

where

$$V_{K,\delta} = \big\{\delta f \in E_f : \; \Delta(\delta f; K) \le \delta\big\},$$
$$\Delta(\delta f; K) = \sup\bigg\{\bigg|\int_{t'}^{t''} \delta f(t, x, y)\, dt\bigg| : \; t', t'' \in I, \; x, y \in K\bigg\}.$$

Let D be the set of continuously differentiable scalar functions (delay functions) $\tau(t)$, $t \in [a, +\infty)$, satisfying the conditions

$$\tau(t) \le t, \;\; \dot{\tau}(t) > 0, \;\; t \in [a, +\infty),$$
$$\inf\big\{\tau(a) : \; \tau \in D\big\} := \widehat{\tau} > -\infty, \quad \sup\big\{\tau^{-1}(b) : \; \tau \in D\big\} := \widehat{\gamma} > +\infty,$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$. Let E_φ be the space of piecewise-continuous functions $\varphi(t) \in \mathbb{R}^n$, $t = I_1 = [\widehat{\tau}, b]$, with finitely many first kind discontinuities equipped with the norm $\|\varphi\|_{I_1} = \sup\{\varphi(t) :\ t \in I_1\}$. Let $\Phi_1 = \{\varphi \in E_\varphi :\ \operatorname{cl}\varphi(I_1) \subset O\}$ denote the set of initial functions of trajectories, where $\varphi(I_1) = \{\varphi(t) :\ t \in I_1\}$; we denote by Φ_2 the set of bounded measurable functions $v : I_1 \to \mathbb{R}^n$ and $v(t)$ is called the initial function of the trajectory derivative. Let $\mathbb{R}^{n\times n}$ be the space of matrices $A = (a_{ij})_{i,j=1}^n$, $|A|^2 = \sum\limits_{i,j=1}^n |a_{ij}|^2$. Let Λ be the space of continuous matrix functions $A : I \to \mathbb{R}^{n\times n}$, $\|A\| = \sup\{|A(t) :\ t \in I\}$. We denote by μ the collection of initial data

$$(t_0, \tau, x_0, A, \varphi, v) \in [a, b) \times D \times O \times \Lambda \times \Phi_1 \times \Phi_2$$

and by f the nonlinear term of right-hand side.

To each element $\mu = (t_0, \tau, x_0, A, \varphi, v, f) \in \mathfrak{M} = [\mathfrak{a}, \mathfrak{b}) \times \mathfrak{D} \times \mathfrak{O} \times \mathfrak{f} \times {}_1 \times {}_2 \times \mathfrak{E}_\mathfrak{f}$ we assign the linear with respect to the phase velocity (quasi-linear) neutral functional differential equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f\big(t, x(t), x(\tau(t))\big) \tag{1}$$

with the initial condition

$$x(t) = \varphi(t), \quad \dot{x}(t) = v(t), \ \ t \in [\widehat{\tau}, t_0), \quad x(t_0) = x_0. \tag{2}$$

Here $\sigma \in D$ are fixed delay function in the phase velocity with $\sigma(t) < t$. The symbol $\dot{x}(t)$ on the interval $[\widehat{\tau}, t_0)$ is not connected with the derivative of the function $\varphi(t)$.

Definition 1. Let $\mu = (t_0, \tau, x_0, A, \varphi, v, f) \in \mathfrak{M}$. A function $x(t) = x(t; \mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is a called the solution of equation (1) with the initial condition (2) or the solution corresponding to the element μ and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies condition (2), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

To formulate the main results, we introduce the following sets:

$$W(K; \alpha) = \Big\{\delta f \in E_f :\ \text{there exist } M_{\delta f,K}(t), L_{\delta f,K} \in L(I, \mathbb{R}_+) \\ \text{such that } \int\limits_I \big[M_{\delta f,K}(t) + L_{\delta f,K}(t)\big]\, dt \le \alpha\Big\},$$

where $K \subset O$ is a compact set and $\alpha > 0$ is a fixed number independent of δf. Furthermore,

$$\begin{aligned}
B(t_{00}; \delta) &= \big\{t_0 \in I :\ |t_0 - t_{00}| < \delta\big\},\\
B_1(x_{00}; \delta) &= \big\{x_0 \in O : |x_0 - x_{00}| < \delta\big\},\\
V(\tau_0; \delta) &= \big\{\tau \in D :\ \|\tau - \tau_0\|_{I_2} < \delta\big\},\\
V_1(A_0; \delta) &= \big\{A \in \Lambda :\ \|A - A_0\|_I < \delta\big\},\\
V_2(\varphi_0; \delta) &= \big\{\varphi \in \Phi_1 :\ \|\varphi - \varphi_0\|_{I_1} < \delta\big\},\\
V_3(v_0; \delta) &= \big\{v \in \Phi_2 :\ \|v - v_0\|_{I_1} < \delta\big\},
\end{aligned}$$

where $t_{00} \in [a, b)$ and $x_{00} \in O$ are fixed points, $\tau \in D$, $\varphi_0 \in \Phi_1$, $v_0 \in \Phi_2$ are fixed functions, $\delta > 0$ is a fixed number, $I_2 = [a, \widehat{\gamma}]$.

Theorem 1. *Let $x_0(t)$ be the solution corresponding to the element $\mu_0 = (t_{00}, \tau_0, x_{00}, A_0, \varphi_0, v_0, f_0) \in \mathfrak{M}$ and defined on $[\widehat{\tau}, t_{10}]$, $t_{10} < b$, and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \operatorname{cl}\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following assertions hold:*

- *there exist numbers* $\delta_i > 0$, $i = 0, 1$, *such that, to each element*

$$\begin{aligned}\mu = (t_0, \tau, x_0, A, \varphi, v, f_0 + \delta f) &\in V(\mu_0; K_1, \delta_0, \alpha)\\ &= B(t_{00}; \delta_0) \times V(\tau_0; \delta_0) \times B_1(x_{00}; \delta_0) \times V_1(A_0; \delta_0)\\ &\qquad \times V_2(\varphi_0; \delta_0) \times V_3(v_0; \delta_0) \times \big[f_0 + (W(K_1; \alpha) \cap V_{K_1, \delta_0})\big],\end{aligned}$$

  there corresponds the solution $x(t; \mu)$ *defined on the interval* $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ *and satisfying the condition* $x(t; \mu) \in K_1$*;*

- *for an arbitrary* $\varepsilon > 0$ *there exists a number* $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0]$ *such that the following inequality holds for any* $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$*:*

$$|x(t; \mu) - x(t; \mu_0)| \le \varepsilon \ \textit{ for all } \ t \in [\theta, t_{10} + \delta_0], \ \ \theta = \max\{t_{00}, t_0\};$$

- *for an arbitrary* $\varepsilon > 0$ *there exists a number* $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0]$ *such that the following inequality holds for any* $\mu \in V(\mu_0; K_1, \delta_3, \alpha)$*:*

$$\int\limits_{\widehat{\tau}}^{t_{10}+\delta_1} |x(t; \mu) - x(t; \mu_0)|\, dt \le \varepsilon.$$

It is clear that the solution $x(t; \mu_0)$ is the continuation of the solution $x_0(t)$.

In the space $E_\mu - \mu_0$, where $E_\mu = \mathbb{R} \times D \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \times E_\varphi \times \Phi_2 \times E_f$, we introduce the following set of variations:

$$\begin{aligned}\mathfrak{J} = \Big\{\delta\mu = (\delta t_0, \delta\tau, \delta x_0, \delta A, \delta\varphi, \delta v, \delta f) \in E_\mu - \mu_0 :\ |\delta t_0| \le \beta,\\ \|\delta\tau\|_{I_2} \le \beta,\ |\delta x_0| \le \beta,\ \|\delta A\|_I \le \beta,\ \|\delta\varphi\|_{I_1} \le \beta,\ \|\delta v\|_{I_1} \le \beta,\\ \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i,\ |\lambda_i| \le \beta,\ i = 1, \dots, k\Big\},\end{aligned}$$

where $\beta > 0$ is a fixed number and $\delta f_i \in E_f - f_0$, $i = 1, \dots, k$, are fixed functions.

Theorem 2. *Let* $x_0(t)$ *be the solution corresponding to the element* $\mu_0 \in \mathfrak{M}$ *and defined on* $[\widehat{\tau}, t_{10}]$, $t_{i0} \in (a, b)$, $i = 0, 1$. *Let* $K_1 \subset O$ *be a compact set containing a certain neighborhood of the set* K_0. *Then the following assertions hold:*

- *there exist numbers* $\varepsilon_1 > 0$, $\delta_1 > 0$ *such that, for an arbitrary* $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times \mathfrak{J}$, *we have* $\mu_0 + \varepsilon\delta\mu \in \mathfrak{M}$ *and the solution* $x(t; \mu_0 + \varepsilon\delta\mu)$ *defined on the interval* $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ *corresponds to this element. Moreover,* $x(t; \mu_0 + \varepsilon\delta\mu) \in K_1$*;*

- $\lim\limits_{\varepsilon\to 0} \sup \big\{|x(t; \mu_0 + \varepsilon\delta\mu) - x(t; \mu_0)| :\ t \in [\theta, t_{10} + \delta_1]\big\} = 0$, *where* $\theta = \max\{t_{00}, t_0 + \varepsilon t_0\}$ *and*

$$\lim_{\varepsilon\to 0} \int\limits_{\widehat{\tau}}^{t_{10}+\delta_1} |x(t; \mu_0 + \varepsilon\delta\mu) - x(t; \mu_0)|\, dt = 0 \ \ \textit{uniformly for} \ \ \delta\mu \in \mathfrak{J}.$$

This theorem is a simple corollary of Theorem 1.

Finally, we note that theorems analogous to Theorem 1 for various classes of neutral equations are given in [1-4].

R E F E R E N C E S

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