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ON THE WELL-POSSEDNESS OF THE CAUCHY PROBLEM FOR NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH CONCENTRED AND DISTRIBUTED VARIABLES DELAYS

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Abstract. In the present paper, for the differential equation with concentred and distributed variables delays, continuity of a solution is proved with respect to perturbations of the initial data and the right-hand side of equation. Under initial data we imply the collection of the initial moment, the initial and delay functions. Perturbations of the initial data and right-hand side of equation are small in a standard norm and in the integral sense, respectively.

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Let $I=[a,b]$ be a finite interval and let $\mathbb{R}^n$ be an n-dimensional vector space of points $x=(x^1,\dots,x^n)^\top$, where $\top$ denotes transposition. Suppose that $O\subset\mathbb{R}^n$ is an open set, and E_f is the space of functions $f: I\times O^2\times\mathbb{R}^n\to\mathbb{R}^n$ satisfying conditions: for each fixed $(x^1,x^2,x^3)\in O^2\times\mathbb{R}^n$ the function $f(\,\cdot\,,x^1,x^2,x^3): I\to\mathbb{R}^n$ is measurable; for each $f\in E_f$ and compact set $K\subset O$ there exist functions $M_{f,K}(t)$, $L_{f,K}(t)\in L_1(I;\mathbb{R}_+)$, $\mathbb{R}_+=[0;+\infty)$ such that for almost all $t\in I$,

$$|f(t,x^1,x^2,x^3)|\le M_{f,K}(t)\ \ \forall\,(x^1,x^2,x^3)\in K^2\times\mathbb{R}^n,$$

$$\big|f(t,x^1,x^2,x^3)-f(t,y^1,y^2,y^3)\big|\le L_{f,K}(t)\sum_{i=1}^{3}|x_i-y_i|$$
$$\forall\,(x^1,y^1,x^2,y^2)\in K^4\ \ \forall(x^3,y^3)\in\mathbb{R}^n\times\mathbb{R}^n.$$

Two functions $f_1,f_2\in E_f$ are said to be equivalent if for every fixed $(x^1,x^2,x^3)\in O^2\times\mathbb{R}^n$ and for almost all $t\in I$, $f_1(t,x^1,x^2,x^3)-f_2(t,x^1,x^2,x^3)=0$.

The equivalence classes of functions of the space E_f compose a vector space which is also denoted by E_f; these classes are called functions and denoted by f again. We introduce a topology in E_f using the following base of neighborhood of the origin $\{V_{K,\delta}:\ K\subset O$ is a compact set and $\delta>0$ is an arbitrary number$\}$, where

$$V_{K,\delta}=\big\{\delta f\in E_f:\ H(\delta f;K)\le\delta\big\}$$

and

$$H(\delta f:K)=\sup\bigg\{\bigg|\int_{t'}^{t''}\delta f(t,x^1,x^2,x^3)\,dt\bigg|:\ t',t''\in I,\ x^1,x^2\in K^2,\ x^3\in\mathbb{R}^n\bigg\}.$$

Let D be the set of continuous differentiable scalar functions (delay functions) $\tau(t)$, $t\in[a,+\infty)$, satisfying the conditions

$$\tau(t)\le t,\ \ \dot\tau(t)>0,\ \ \inf\{\tau(a):\ \tau\in D\}:=\widehat\tau>-\infty.$$

Let $C(I_1)$ be the space of continuous functions $\varphi(t) \in \mathbb{R}^n$, $t = I_1 = [\widehat{\tau}, b]$ equipped with the norm $\|\varphi\|_{I_1} = \sup\{\varphi(t) : \ t \in I_1\}$. By $\Phi = \{\varphi \in C(I_1) : \ \varphi(t) \in O, \ t \in I_1\}$ we denote the set of initial functions.

To each element $\mu = (t_0, \tau, \theta, \varphi, f) \in \Lambda = [a, b) \times D^2 \times \Phi \times E_f$ we assign the functional differential equation with concentrated and distributed delay (prehistory) on the interval $[\tau(t), t]$

$$\dot{x}(t) = f\Bigg(t, x(t), x(\tau(t)), \int\limits_{\theta(t)}^{t} \sigma(s, x(s))\, ds\Bigg) \tag{1}$$

with the continuous initial condition

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0], \tag{2}$$

where $\sigma(s, x)$, $(s, x) \in I_1 \times O$ is a given function satisfying the following conditions: for each fixed $x \in O$ the function $\sigma(\,\cdot\,, x) : I \to \mathbb{R}^n$ is measurable; for each compact set $K \subset O$ there exist functions $M_K(t), L_K(t) \in L_1(I; \mathbb{R}_+)$ such that for almost all $t \in I_1$

$$|\sigma(t, x)| \le M_K(t), \quad |\sigma(t, x) - \sigma(t, y)| \le L_K|x - y| \ \ (x, y) \in K^2.$$

Definition 1. Let $\mu = (t_0, \tau, \theta, \varphi, f) \in \Lambda$. A function $x(t) = x(t, \mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (\tau_0, b]$, is called a solution of (1) with the initial condition (2), or a solution corresponding to the element μ and defined on the interval $[\widehat{\tau}, t_1]$, if it satisfies (2), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies (1) almost everywhere on $[t_0, t_1]$.

To formulate the main result we introduce the following sets:

$$W(K; \alpha) = \Bigg\{\delta f \in E_f : \ \exists\, M_{\delta f, K}(t), L_{\delta f, K}(t) \in L_1(I; \mathbb{R}_+), \ \int\limits_a^b (M_{\delta f, K}(t) + L_{\delta f, K}(t))\, dt \le \alpha\Bigg\},$$

where $K \subset O$ is a compact set and $\alpha > 0$ is a fixed number not dependent on δf: the set $W(K; \alpha)$ is called the set of perturbations of the right side of (1);

$$B(t_{00}; \delta) = \big\{t_0 \in I : \ |t_0 - t_{00}| \le \delta\big\}, \quad V(\tau_0; \delta) = \big\{\tau \in D : \ \|\tau - \tau_0\|_I < \delta\big\},$$
$$V_1(\theta_0; \delta) = \big\{\theta \in D : \ \|\theta - \theta_0\|_I < \delta\big\}, \quad V_2(\varphi_0; \delta) = \big\{\varphi \in \Phi : \ \|\varphi - \varphi_0\|_I < \delta\big\},$$

where $t_{00} \in [a, b)$ is a fixed point, $\tau_0, \theta_0 \in D$ and $\varphi_0 \in \phi$ are fixed functions: $\delta > 0$ is a fixed number.

Theorem 1. *Let $x_0(t)$ be the solution corresponding to $\mu_0 = (t_{00}, \tau_0, \theta_0, \varphi_0, f_0) \in \Lambda$ and defined on $[\widehat{\tau}, t_{10}]$, $t_{10} < b$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \varphi_0(I_1) \cup X([t_{00}, t_{10}])$. Than the following conditions hold:*

- *there exist numbers $\delta_i > 0$, $i = 0, 1$, such that for each element*

$$\begin{aligned}\mu &= (t_0, \tau, \theta, A, \varphi, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha)\\ &= B(t_{00}; \delta_0) \times V(\tau_0; \delta_0) \times V_1(\theta_0; \delta) \times V_2(\varphi_0; \delta_0) \times \big[f_0 + (W(K_1; \alpha) \cap V_{K_1, \delta_0})\big]\end{aligned}$$

  corresponds solution $x(t; \mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ and satisfied the condition $x(t; \mu) \in K_1$;

- *for arbitrary* $\varepsilon > 0$ *there exists a number* $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ *such that the following inequality holds for any* $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$*:*
$$|x(t;\mu) - x(t;\mu_0)| \le \varepsilon \ \ \forall\, t \in [\gamma, t_{10} + \delta_1], \ \ \gamma = \max\{t_0, t_{00}\};$$
- *for arbitrary* $\varepsilon > 0$ *there exists a number* $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ *such that the following inequality holds for any* $\mu \in V(\mu_0; K_1, \delta_3, \alpha)$*:*
$$\int_{\widehat{\tau}}^{t_{10}+\delta_1} |x(t;\mu) - x(t;\mu_0)|\, dt \le \varepsilon.$$

Obviously, the solution $x(t;\mu_0)$ *is the continuation of the* $x_0(t)$ *and to the element* $\mu = (t_0, \tau, \theta, \varphi, \delta_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha)$ *corresponds the perturbed functional differential equation*
$$\dot{x}(t) = f_0\bigg(t, x(t), x(\tau(t)), \int_{\theta(t)}^{t} \sigma(s, x(s))\, ds\bigg) + \delta f\bigg(t, x(t), x(\tau(t)), \int_{\theta(t)}^{t} \sigma(s, x(s))\, ds\bigg)$$
with the perturbed initial condition $x(t) = \varphi(t)$*,* $t \in (\widehat{\tau}, t_0]$*.*

Theorem is proved by the method given in [1].

In the space $E_\mu = \mathbb{R} \times D^2 \times C(I_1) \times E_f$ we introduce the set of variations:
$$\Delta = \Big\{\delta\mu = (\delta t_0, \delta\tau, \delta\theta, \delta\varphi, \delta f) \in E_\mu - \mu_0 : \ |\delta t_0| \le \beta, \ \|\delta\tau\|_I \le \beta,$$
$$\|\delta\theta\|_I \le \beta, \ \|\delta\varphi\|_I \le \beta, \ \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, \ |\lambda_i| \le \beta, \ i = 1, \dots, k\Big\},$$
where $\beta > 0$ is a fixed number and $\delta f_i \in E_f - f_0$, $i = 1, \dots, k$, are fixed functions.

Theorem 2. *Let* $x_0(t)$ *be the solution corresponding to* $\mu_0 = (t_{00}, \tau_0, \theta_0, \varphi_0, f_0) \in \Lambda$ *and defined on* $[\widehat{\tau}, t_{10}]$*,* $t_{i0} \in [a, b]$*,* $i = 0, 1$*. Let* $K_1 \subset O$ *be a compact set containing a certain neighborhood of the set* K_0*. Then the following conditions hold:*

- *there exist numbers* $\varepsilon_1 > 0$*,* $\delta_1 > 0$ *such that for arbitrary* $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Delta$ *the element* $\mu_0 + \varepsilon\delta\mu \in \Delta$ *and there corresponds the solution* $x(t; \mu_0 + \varepsilon\delta\mu)$ *defined on the interval* $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$*. Moreover,* $x(t; \mu_0 + \varepsilon\delta\mu) \in K_1$*;*
- *the following relations are fulfilled*
$$\lim_{\varepsilon \to 0} \sup \big\{|x(t; \mu_0 + \varepsilon\delta\mu) - x(t;\mu_0)| : \ t \in [\gamma, t_{10} + \delta_1]\big\} = 0,$$
$$\lim_{\varepsilon \to 0} \int_{\widehat{\tau}}^{t_{10}+\delta_1} \big|x(t; \mu_0 + \varepsilon\delta\mu) - x(t;\mu_0)\big|\, dt = 0$$
uniformly for $\delta\mu \in \Delta$*, where* $\gamma = \max\{t_{00}, t_{00} + \varepsilon\delta t_0\}$*.*

Theorem 2 is a simple corollary of Theorem 1.

Theorems on the continuous dependence of the solution when a perturbation of the right-hand side is small in the integral sense, were proved for various classes of ordinary differential equations in [2-4] and for differential equations with concentrated delay, in [5-9].

R E F E R E N C E S

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