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THE ISOMETRIC SYSTEM OF COORDINATES AND THE COMPLEX FORM OF THE SYSTEM OF EQUATIONS FOR THE NON-SHALLOW AND NONLINEAR THEORY OF SHELLS

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Abstract. In this paper, the 3-D geometrically and physically nonlinear theories of non-shallow shells are considered. The isometrical system of coordinates is of special interest, since in this system we can obtain bases equations of the theory of shells in a complex form. This circumstance makes is possible to apply the methods developed by N. Muskhelishvili and his disciples by means of the theory of functions of a complex variable and integral equations

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1. Introduction

The refined theory of shells is constructed by reducing the three-dimensional problems of the theory of elasticity to the two-dimensional problems [1, 2]. I. Vekua constructed several versions of the refined linear theory of thin and shallow shells, containing, the regular processes by means of the method of reduction of 3-D problems of elasticity to 2-D ones [1].

By thin and shallow shells I.Vekua means 3-D shell type elastic bodies satisfying the following conditions [3]

$$a_{\alpha}^{\beta} - x^3 b_{\alpha}^{\beta} \cong \alpha_{\alpha}^{\beta} \quad -h \leq x^3 = x_3 \leq h, \quad \alpha, \beta = 1, 2, \tag{$*$}$$

where a_{α}^{β} and b_{α}^{β} are mixed components of the metric and curvature tensors of the midsurface of the shell, x^3 is the thickness coordinate and h is the semi-thickness.

In the sequel, under non-shallow shells we wean elastic bodies free from the assumption of the type (*) or, more exactly, the bodies with the conditions

$$a_{\alpha}^{\beta} - x_3 b_{\alpha}^{\beta} \neq a_{\alpha}^{\beta} \Rightarrow |hb\beta_{\alpha}| \leq q < 1.$$

Such kind of shells are called shells with varying in thickness geometry, or non-shallow shells.

2. System of geometrically and physically nonlinear equations for non-shallow shells

We write the equation of equilibrium of an elastic shell-type body in a vector form which is convenient for reduction to the 2-D equations

$$\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g} \vec{\sigma}^i}{dx^i} + \vec{\Phi} = 0 \Rightarrow \hat{\nabla}_i \vec{\sigma}^i + \vec{\Phi} = 0, \tag{1}$$

where g is the discriminant of the metric quadratic form of the 3-D domain $\Omega, \hat{\nabla}_i$ are covariant derivatives with respect to the space coordinates x^i, $\vec{\Phi}$ is an external force, $\vec{\sigma}^i$ are the contravariant constituents of the stress vector $\vec{\sigma}_{(\overset{*}{\vec{l}})}$ acting in the area with the normal $\overset{*}{\vec{l}}$ and representable as the Cauchy formulas as follows

$$\vec{\sigma}_{(\overset{*}{\vec{l}})} = \vec{\sigma}^i \overset{*}{l}_i, \quad \overset{*}{l}_i = \overset{*}{\vec{l}}\vec{R}_i.$$

A material is said to be hyper-elastic if the stresses are obtained by means of the strain energy function

$$\sigma^{ij} = \frac{\partial \exists}{\partial e_{ij}},$$

where σ^{ij} are contravariant components of the stress tensor, $\exists$ is the strain energy function, and e_{ij} are covariant components of the strain tensor.

The theory of hyper-elasticity of the second order has the form [2, 3]

$$\begin{aligned} &\exists = \frac{1}{2}E^{ijpq}e_{ij}e_{pq} + \frac{1}{3}E^{ijpqsk}e_{ij}e_{pq}e_{sk}, \\ &e_{ij} = \frac{1}{2}(\vec{R}_i\partial_j\vec{U} + \vec{R}_j\partial_i\vec{U} + \partial_i\vec{U}\partial_j\vec{U}) \\ &\sigma^{ij} = E^{ijpq}e_{pq} + E^{ijpqsk}e_{pq}e_{sk}, \quad \vec{\sigma}^i = \sigma^{ij}(\vec{R}_j + \partial_j\vec{U}) \end{aligned} \tag{2}$$

where E^{ijpq} and E^{ijpqsk} are coefficients of elasticity of the first and second order and $\vec{U}$ is the displacement vector.

Coefficients of elasticity of the first order for isotropic elastic bodies are expressed by the two Lamé coefficients

$$E^{ijpq} = \lambda g^{ij}g^{\mu q} + \mu(g^{ip}g^{jq} + g^{iq}g^{jp}), \quad (g^{ij} = \vec{R}^i\vec{R}^j) \tag{3}$$

and coefficients of elasticity of the second order are defined by the formula

$$E^{ijpqsk} = (E_1 + E_2)g^{ij}g^{pq}g^{sk} - E_2g^{ij}g^{pk}g^{qs} + E_3g^{ip}g^{jq}g^{sk} + E_4g^{is}g^{pq}g^{jk}, \tag{4}$$

where E_1, E_2, E_3 and E_4 are modules of elasticity of the second order for isotropic elastic bodies.

Here $\vec{R}_i$ and $\vec{R}^i$ are covariant and contravariant base vectors of the space.

3. The coordinate system in a shell normally connected with a surface

Let Ω denote a shell and a domain of the space occupied by the shell. Inside the shell, we consider a smooth surface S with respect to which the shell Ω lies symmetrically. The surface S is called the midsurface of the shell Ω. To construct the theory of shells, we use more convenient coordinate system which is normally connected with the midsurface S. This means that the radius-vector $\vec{R}$ of any point of the domain Ω can be represented in the form

$$\vec{R}(x^1, x^2, x^3) = \vec{r}(x^1, x^2) + x^3\vec{n}(x^1, x^2),$$

where $\vec{R}$ and $\vec{n}$ are respectively the radius-vector and the unit vector of the normal of the surface $S(x^3 = 0)$ and (x^1, x^2) are the Gaussian parameters of the midsurfaces S.

The covariant and contravariant basis vectors $\vec{R}_i$ and $\vec{R}^i$ of the surfaces $\hat{S}(x^3 =$ const), and the corresponding basis vectors $\vec{r}_i$ and $\vec{r}^{\,i}$ of the midsurface S $(x^3 = 0)$ are connected by the following relations:

$$\vec{R}_i = A_{i.}^{\cdot j}\vec{r}_j = A_{ij}\vec{r}^{\,j}, \quad \vec{R}^i = A_{\cdot j}^{i\cdot}\vec{r}^{\,j} = A^{ij}\vec{r}_j, \quad (i, j = 1, 2, 3),$$

where

$$A_{i.}^{\cdot j} = \begin{cases} a_\alpha^\beta - x_3 b_\alpha^\beta, & i = \alpha, \quad j = \beta, \\ \delta_i^3, & j = 3, \end{cases} \qquad \vec{r}_i, \vec{r}^{\,i} = \begin{cases} \vec{r}_\alpha, \vec{r}^{\,\alpha}, & i = \alpha, \\ \vec{n}, \vec{n}, & i = 3, \end{cases}$$

$$A_{\cdot j}^{i\cdot} = \begin{cases} \dfrac{(1 - 2Hx_3)a_\beta^\alpha + x_3 b_\beta^\alpha}{1 - 2Hx_3 + Kx_3^2}, & i = \alpha, \ j = \beta, \\ \delta_i^3, \quad j = 3. \end{cases}$$

Here $(a_{\alpha\beta}, a^{\alpha\beta}, a_\alpha^\beta)$ and $(b_{\alpha,\beta}, b^{\alpha\beta}, b_\alpha^\beta)$ are the components (covariant, contravariant and mixed) of the metric and curvature tensors of the midsurface S. By H and K we denote a middle and Gaussian curvature of the surface S, where

$$2H = b_\alpha^\alpha = b_1^1 + b_2^2, \quad K = b_1^1 b_2^2 - b_2^1 b_1^2.$$

It should be noted that for the refined theory of non-shallow shells (Koiter, Naghdi, Lurie) these relations have the form

$$\vec{R}^\alpha \cong (a_\beta^\alpha + x_3 b_\beta^\alpha)\vec{r}^{\,\beta}, \quad \vec{R}_\alpha = (a_\alpha^\beta - x_3 b_\alpha^\beta)\vec{r}_\beta.$$

The main quadratic forms of the midsurface S $(x_3 = 0)$ have the forms

$$I = ds^2 = a_{\alpha\beta}dx^\alpha dx^\beta, \quad II = K_s ds^2 = b_{\alpha\beta}dx^\alpha dx^\beta,$$

where k_s is the normal courvative of the S and

$$a_{\alpha\beta} = \vec{r}_\alpha \vec{r}_\beta, \ \ b_{\alpha\beta} = -\vec{n}_\alpha \vec{r}_\beta, \ \ k_s = b_{\alpha\beta}s^\alpha s^\beta, \ \ \vec{r}_\alpha = \partial_\alpha \vec{r}, \ \ s^\alpha = \frac{dx^\alpha}{ds}.$$

It is necessary to rewrite the relation (1-4) in terms of the midsurface S of the shell Ω.

Relation (1) can be written as follows:

$$\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\vartheta\vec{\sigma}^\alpha}{\partial x^\alpha} + \frac{\partial\vartheta\vec{\sigma}^3}{\partial x^3} + \vartheta\vec{\Phi} = 0, \quad (\vartheta = 1 - 2Hx_3 + Kx_3).$$

from (2), (3), (4) we obtain

$$\vec{\sigma}^i = \sigma^{ij}(\vec{R}_j + \partial_j\vec{U}) = (E^{ijpq} + E^{ijpqsk}e_{sk})e_{pq}(\vec{R}_j + \partial_j\vec{U})$$
$$\Rightarrow \vec{\sigma}^i = \tfrac{1}{2}A_{i_1}^i[M^{i_1 j_1 p_1 q_1} + \tfrac{1}{2}M^{i_1 j_1 p_1 q_1 s_1 k_1}$$
$$\times(A_{k_1}^k \vec{r}_{s_1}\partial_k\vec{U} + A_{s_1}^s A_{k_1}^k \partial_s\vec{U}\partial_k\vec{U})]$$
$$\times(A_{p_1}^p \vec{r}_{q_1}\partial_p\vec{U} + A_{q_1}^q \vec{r}_{p_1}\partial_q\vec{U} + A_{p_1}^p A_{q_1}^q \partial_p\vec{U}\partial_q\vec{U})(\vec{r}_{j1} + A_{j1}^j\partial_j\vec{U}),$$

where

$$M^{i_1j_1p_1q_1}=\lambda a^{i_1j_1}a^{p_1q_1}+\mu(a^{i_1p_1}a^{j_1q_1}+a^{i_1q_1}a^{j_1p_1})$$

$$M^{i_1j_1p_1q_1s_1k_1}=(E_1+E_2)a^{i_1j_1}a^{p_1q_1}-E_2a^{i_1j_1}a^{p_1k_1}q^{q_1s_1}$$

$$+E_3a^{i_1p_1}a^{j_1q_1}a^{s_1k_1}+E_4a^{i_1s_1}a^{p_1q_1}a^{j_1k_1},$$

$$(a^{ij}=\vec{r}^i\vec{r}^j).$$

4. Isometric system of coordinates

The isometrical system of coordinates in the surface S is of special interest, since in this system we can obtain bases equations of the theory of shells in a complex form, which in turn, allows one for a rather wide class of problems to construct complex representation of general solutions by means of analytic functions of one variable $z=x'+ix^2$. This circumstance makes is possible to apply the methods developed by N. Muskhelishvili and his disciples by means of the theory of functions of a complex variable and integral equations [1].

The main quadratic forms in this of coordinates are of the type

$$I=ds^2=\Lambda(x^1,x^2)[(dx^1)^2+(dx^2)^2]=\Lambda(z,\bar z)dzd\bar z,\quad(\Lambda>0)$$

$$II=b_{\alpha\beta}dx^\alpha dx^\beta=\frac{1}{2}[\bar Q dz^2+2Hdzd\bar z+Qd\bar z^2],$$

where

$$Q=\frac{1}{2}(b_1^1-b_2^2+2ib_2^1),\ \ 2H=b_1^1+b_2^2.$$

Introducing the well-known differential operators

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x^1}-i\frac{\partial}{\partial x^2}\right),\ \ \frac{\partial}{\partial \bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x^1}+i\frac{\partial}{\partial x^2}\right)$$

and the notation

$$\vec{\tau}^i=\sqrt{\frac{g}{a}}\vec{\sigma}^i,\quad \vec{F}=\sqrt{\frac{g}{a}}\vec{\Phi},$$

$$\sqrt{\frac{g}{a}}=\vartheta=1-2Hx_3+Kx_3^2,$$

we obtain the following complex writing both for the system of equations of equilibrium and for "Hooke's Law"

$$\frac{1}{\Lambda}\frac{\partial}{\partial z}[\Lambda(\tau_1^1-\tau_2^2+i\tau_2^1+i\tau_1^2)]+\frac{\partial}{\partial\bar z}[\Lambda(\tau_1^1+\tau_2^2+i\tau_2^1-i\tau_1^2)]$$

$$-\Lambda(H\tau_3^+ +Q\bar\tau_3^+)+\frac{\partial\tau_+^3}{\partial x^3}+F_+=0,$$

$$\frac{1}{\Lambda}\left(\frac{\partial\Lambda\tau_3^+}{\partial z}+\frac{\partial\Lambda\bar\tau_3^+}{\partial\bar z}\right)+H(\tau_1^1+\tau_2^2)$$

$$+\mathrm{Re}[\bar Q(\tau_1^1-\tau_2^2+i\tau_2^1-i\tau_1^2)]+\frac{\partial\tau_3^3}{\partial x^3}+F_3=0,$$

where

$$\tau_1^1 - \tau_2^2 + i(\tau_2^1 + \tau_1^2) = \vec{\tau}^+\vec{r}_+ = \sqrt{\frac{g}{a}}\Big\{[\lambda\Theta + \mu(\vec{R}^+\partial_z\vec{U} + \bar{\vec{R}}^+\partial_{\bar{z}}\vec{U}$$
$$+2\partial^z\vec{U}\partial^{\bar{z}}\vec{U})](\vec{R}^+ + 2\partial^{\bar{z}}\vec{U})\vec{r}_+ + \mu[2(\vec{R}_+ + \partial^{\bar{z}}\vec{U})\partial_{\bar{z}}\vec{U}(\bar{\vec{R}}^+ + 2\partial^{\bar{z}}\vec{U})\vec{r}_+$$
$$+(\vec{R}_+\partial_3\vec{U} + 2\vec{n}\partial^{\bar{z}}\vec{U} + 2\partial^{\bar{z}}\vec{U}\partial_3\vec{U})\partial_3\vec{U}]\ \Big\},$$

$$\tau_1^1 + \tau_2^2 + i(\tau_2^1 - \tau_1^2) = \bar{\vec{\tau}}^+\bar{\vec{r}}_+ = \sqrt{\frac{g}{a}}\Big\{\lambda\Theta + \mu(\vec{R}_+\partial^z\vec{U} + \bar{\vec{R}}^+\partial_{\bar{z}}\vec{U}$$
$$+2\partial^z\vec{U}\partial_{\bar{z}}\vec{U})(\bar{\vec{R}}^+ + 2\partial^z\vec{U})\vec{r}_+ + \mu[2(\bar{\vec{R}}^+\partial_{\bar{z}}\vec{U} + \partial_z\vec{U}\partial^{\bar{z}}\vec{U})$$
$$(\vec{R}^+ + 2\partial^{\bar{z}}\vec{U})\vec{r}_+ + (\vec{R}^+\partial_3\vec{U} + 2(\vec{n} + \partial^{\bar{z}}U)\partial^z U]\partial_3\vec{U}_+]\ \Big\}$$

$$\tau_3^+ = (\vec{\tau}^1 + i\tau^2)\vec{n} = \sqrt{\frac{g}{a}}\Big\{2[\lambda\Theta + \mu(\vec{R}^+\partial_z\vec{U} + \bar{\vec{R}}^+\partial_{\bar{z}}\vec{U} + 2\partial^z\vec{U}\partial_{\bar{z}}\vec{U})$$
$$(\vec{n}\partial^{\bar{z}}\vec{U})] + \mu[2(\vec{R}^+\partial_{\bar{z}}\vec{U} + \partial^{\bar{z}}\vec{U}\partial_z\vec{U})(\vec{n}\partial^z\vec{U}) +$$
$$(\vec{R}^+\partial_3\vec{U} + 2\vec{n}\partial^{\bar{z}}\vec{U} + 2\partial^{\bar{z}}\vec{U}\partial_3\vec{U})(1 + \partial_3 U_3)]\ \Big\},$$

$$\tau_+^3 = \vec{\tau}^3\vec{r}_+ = \sqrt{\frac{g}{a}}\Big\{[\lambda\Theta + \mu(2\vec{n}\partial^3\vec{U} + \partial_3\vec{U}\partial^3\vec{U}]\partial_3\vec{U}_+$$
$$+\mu(\vec{n}\partial^{\bar{z}}\vec{U} + \frac{1}{2}\bar{\vec{R}}^+ + \partial_3\vec{U}\partial_3\vec{U}\partial_z\vec{U})(\vec{R}_+ + 2\partial_z\vec{U})\vec{r}_+$$
$$+(\vec{n}\partial^{\bar{z}}\vec{U} + \frac{1}{2}\vec{R}^+\partial_z\vec{U}\partial_3\vec{U}\partial^{\bar{z}}\vec{U})(\bar{\vec{R}}_+ + 2\partial_{\bar{z}}\vec{U})\vec{z}_+\ \Big\}$$

$$\tau_3^3 = \vec{\tau}^3\vec{n} = \sqrt{\frac{g}{a}}\Big\{[\lambda\Theta + \mu(2\vec{n}\partial^3\vec{U} + \partial_3\vec{U}\partial^3\vec{U}](1 + \partial_3\vec{U})$$
$$+2\mu[(\vec{n}\partial^z\vec{U}_+ + \frac{1}{2}\vec{R}^+\partial_3\vec{U} + \partial^z\vec{U}\partial_3\vec{U})(\vec{n}\partial_{\bar{z}}\vec{U})$$
$$+(\vec{n}\partial^{\bar{z}}\vec{U} + \frac{1}{2}\vec{R}^+\partial_3\vec{U} + \partial_3\vec{U}\partial_z\vec{U})\vec{n}\partial_z\vec{U}]\ \Big\}.$$

Then

$$\Theta = \vec{R}^+\partial_z\vec{U} + \bar{\vec{R}}^+\partial_{\bar{z}}\vec{U} + 2\partial_z\vec{U}\partial^{\bar{z}}\vec{U} + \partial_3 U_3 + \frac{1}{2}(\partial_3\vec{U})^2,$$
$$\partial^z\vec{U} = \frac{1}{2}[(\vec{R}^+\bar{\vec{R}}^+)\partial_z\vec{U}_+ + (\bar{\vec{R}}\vec{R}^+)_{\partial_{\bar{z}}}\vec{U}],$$
$$\vec{R}^+ = \vec{R}^1 + i\vec{R}^2, \quad \vec{R}_+ = \vec{R}_1 + i\vec{R}_2,$$
$$\vec{R}^+ = \vartheta^{-1}[(1 - Hx_3)\vec{r}^+ + x_3 Q\bar{\vec{r}}_+],$$
$$\vec{r}^+ = \vec{r}^1 + i\vec{r}^2, \quad \vec{r}_+ = \vec{r}_1 + i\vec{r}_2,$$
$$\vec{R}^+\vec{R}^+ = \frac{4x_3}{\Lambda}\frac{\lambda - Hx_3}{\vartheta^2}Q,$$
$$\vec{R}^+\bar{\vec{R}}^+ = \frac{2}{\Lambda}\frac{(1 - Hx_3)^2 + x_3^2 Q\bar{Q}}{\vartheta^2} = \frac{2}{\Lambda}\frac{\vartheta + 2x_3^2 Q\bar{Q}}{\vartheta^2},$$

$$\vec{R}^{+}\vec{r}_{+}=\frac{2}{\vartheta}Qx_3,\quad \bar{\vec{R}}^{+}\vec{r}_{+}=\frac{2}{\vartheta}(1-Hx_3),$$
$$\vec{r}^{+}\vec{r}^{+}=0,\quad \vec{r}^{+}\bar{\vec{r}}^{+}=\frac{2}{\Lambda},\quad \vec{r}_{+}\bar{\vec{r}}_{+}=2,$$
$$F_{+}=F_1+F_2,\quad U_{+}=U+iU_2,\quad U^{+}=U^1+iU^2.$$

We have the formulas

$$\vec{r}^{+}\partial_z\vec{U}=\frac{1}{\lambda}\partial_z U_{+}-HU_3,$$
$$\vec{r}^{+}\partial_{\bar{z}}\vec{U}=\partial_{\bar{z}}U^{+}-QU_3,$$
$$\vec{n}\partial_{\bar{z}}\vec{U}=\partial_{\bar{z}}U_3+\frac{1}{2}(\bar{Q}U_{+}+H\bar{U}_{+}).$$

The displacement vector $\vec{U}$, representable in the form

$$\vec{U}=U^{\alpha}\bar{r}_{\alpha}+U^3\vec{n}=U_{\alpha}\vec{r}^{\alpha}+U_3\vec{n}=U_{(e)}\vec{l}+U_{(s)}\vec{s}+U_3\vec{n}\quad (U_3=U^3)$$

can be rewritten as follows:

$$\vec{U}=\frac{1}{2}(U^{+}\bar{\vec{r}}_{+}+\bar{U}^{+}\vec{r}_{+})+U_3\vec{n}$$

or

$$\vec{U}=\operatorname{Im}\left[\left(U_{(l)}+iU_{(s)}\right)\frac{dz}{ds}\vec{r}_{+}\right]+U_3\vec{n}$$

where

$$U^{+}=\vec{U}\vec{r},\ U_{+}=\vec{U}\vec{r}_{+},\ U_{(\vec{l})}=\vec{U}\vec{l},\ U_s=\vec{U}\vec{s}.$$

Here $\vec{s}$ and $\vec{l}$ are the unit tangent vector and tangential normal of the midsurface $S(x_3=0)$. The expression for the unit tangent vector $\hat{\vec{s}}$ and the tangential normal $\hat{\vec{l}}$ of the surface $\hat{S}(x_3=const)$ have the forms

$$\hat{\vec{s}}=\frac{d\vec{R}}{d\hat{s}}=[(1-x_sk_s)\vec{s}+x_s\tau_s\vec{l}]\frac{ds}{d\hat{s}},$$
$$\hat{\vec{l}}=\hat{\vec{s}}\times\vec{n}=[(1-x_3k_s)\vec{l}-x_3\tau_s\vec{s}]\frac{ds}{d\hat{s}},$$

and

$$d\hat{s}=\sqrt{1-2x_3k_s+(k_s^2+l_s^2)x_3^2}ds,$$
$$\left(\hat{\vec{l}}\times\hat{\vec{s}}=\vec{n}\right)$$

where $d\hat{s}$ and ds are linear elements of the surfaces $\hat{S}$ and S, τ_s is the geodesic version of the surface S.

The formula

$$\hat{\vec{l}}\vec{R}_{\alpha}=(1-2Hx_3+Kx_3^2)(\vec{l}\vec{r}_{\alpha})\frac{ds}{d\hat{s}}.$$

which is necessary in writing the reduced basic boundary-value problems in stresses, is also valid.

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