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ON OSCILLATORY PROPERTIES OF SOLUTIONS OF ALMOST LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The differential equation

$$u^{(n)}(t) + F(u)(t) = 0$$

is considered, where $F : C(R_+; R) \to L_{\rm loc}(R_+; R)$ is a continuous mapping. In the case operator F has almost linear minorant, sufficient conditions are established for equation to have Properties **A** and **B**.

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1. Introduction

This work deals with the investigation of oscillatory properties of solutions of a functional differential equation

$$u^{(n)}(t) + F(u)(t) = 0, \tag{1.1}$$

where $F : C(R_+; R) \to L_{\rm loc}(R_+; R)$ is a continuous mapping.

Let $\tau \in C(R_+; R_+)$, $\lim\limits_{t\to+\infty} \tau(t) = +\infty$. Denote by $V(\tau)$ the set of continuous mappings F satisfying the condition $F(x)(t) = F(y)(t)$ holds for any $t \in R_+$ and $x, y \in C(R_+; R)$ provided that $x(s) = y(s)$ for $s \geq \tau(t)$. For any $t_0 \in R_+$, we denote by $H_{t_0,\tau}$ the set of all functions $u \in C(R_+; R)$ satisfying $u(t) \neq 0$ for $t \geq t_1$, where $t_1 = \min\{t_0, \tau_*(t_0)\}$, $\tau_*(t) = \inf\{\tau(s) : s \geq t\}$.

It will always be assumed that either the condition

$$F(u)(t)\, u(t) \geq 0 \quad \text{for} \quad t \geq t_0, \quad u \in H_{t_0,\tau} \tag{1.2}$$

or the condition

$$F(u)(t)\, u(t) \leq 0 \quad \text{for} \quad t \geq t_0, \quad u \in H_{t_0,\tau} \tag{1.3}$$

is fulfilled.

A function $u : [t_0, +\infty) \to R$ is said to be a proper solution of equation (1.1), if it is locally absolutely continuous along with its derivatives up to the order $n-1$ inclusive, $\sup\{|u(s)| : s \in [t, +\infty)\} > 0$ for $t \geq t_0$ and there exists a function $u_* \in C(R_+; R)$ such that $u_*(t) \equiv u(t)$ on $[t_0, +\infty)$ and the equality

$$u_*^{(n)}(t) + F(u_*)(t) = 0$$

holds for $t \in [t_0, +\infty)$. A proper solution $u : [t_0, +\infty) \to R$ of equation (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise, the solution u is said to be nonoscillatory.

Definition 1.1. We say that equation (1.1) has Property **A** if any of its proper solutions is oscillatory, when n is even and either is oscillatory or satisfies

$$\left|u^{(i)}(t)\right| \downarrow 0 \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.4}$$

when n is odd.

Definition 1.2. We say that equation (1.1) has Property **B** if any of its proper solution either is oscillatory, or satisfies either (1.4) or

$$\left|u^{(i)}(t)\right| \uparrow +\infty \quad \text{as} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.5}$$

when n is even, and either is oscillatory or satisfies (1.5) when n is odd.

A. Kneser was the first who showed the condition

$$\liminf_{t\to+\infty} t^{n/2} p(t) > 0$$

is sufficient for the equation

$$u^{(n)}(t) + p(t)\, u(t) = 0 \tag{1.6}$$

to have Property **A** [1]. This theorem for Property **A** (for Property **B**) was essentially generalized by Kondrat'ev [2] (by Chanturia [3]). Their methods was based on a comparison theorem which enables one to obtain optimal results for establishing oscillatory properties of solutions of equation (1.6). Koplatadze [4,5] proved integral comparison theorems of two types for differential equations with deviated arguments. The theorems of the first type enables one not only to generalize the above mentioned results for equations with deviated arguments, but to improve Chanturia's result concerning Property **B** even in the case of equation (1.6).

The ordinary differential equation with deviating argument

$$u^{(n)}(t) + p(t)\left|u(\sigma(t))\right|^{\mu(t)} \operatorname{sign} u(\sigma(t)) = 0 \tag{1.7}$$

is a particular case of equation (1.1) where $p \in L_{\mathrm{loc}}(R_+; R)$, $\mu \in C(R_+; (0, +\infty))$, $\sigma \in C(R_+; R)$ and $\lim\limits_{t\to+\infty} \sigma(t) = +\infty$. In case $\lim\limits_{t\to+\infty} \mu(t) = 1$, we call differential equation (1.7) almost linear, while if $\liminf\limits_{t\to+\infty} \mu(t) \neq 1$, or $\limsup\limits_{t\to+\infty} \mu(t) \neq 1$, then we call equation (1.7) the essentially nonlinear generalised Emden-Fowler type differential equation.

In the present paper developing ideas of [6,7], the both cases of Properties **A** and **B** will be studied when operator F has almost linear minorant.

Investigation of almost linear differential equations, in our opinion for the first time was carried out [6–8].

2. Almost linear functional differential equation with property A

Theorem 2.1. *Let $F \in V(\tau)$, conditions* (1.2) *and*

$$\left|F(u)(t)\right| \geq \sum_{i=1}^{n} p_i(t) \int_{\alpha_i t}^{\beta_i t} s^{\gamma_i} \left|u(s)\right|^{1+\frac{d_i}{\ln s}} ds \quad \text{for} \quad t \geq t_0 > 1, \quad u \in H_{t_0,\tau} \tag{2.1}$$

be fulfilled, where

$$p_i \in L_{\text{loc}}(R_+; R_+), \quad 0 < \alpha_i < \beta_i, \quad \gamma_i \in (-1, +\infty), \quad d_i \in R \quad (i = 1, \dots, m). \tag{2.2}$$

Then for the equation (1.1) *to have Property* **A** *it is sufficient that*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-1+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds$$
$$> \frac{1}{m} \max\left\{ \frac{-\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\frac{\lambda d}{m}}\Big(\prod\limits_{i=1}^m (1+\gamma_i+\lambda)\Big)^{\frac{1}{m}}}{\Big(\prod\limits_{i=1}^m \big(\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda}\big)\Big)^{\frac{1}{m}}} : \lambda \in [0, n-1] \right\},$$

where

$$\gamma = \sum_{i=1}^m \gamma_i, \quad d = \sum_{i=1}^m d_i. \tag{2.3}$$

Theorem 2.2. *Let $F \in V(\tau)$, conditions* (1.2), (2.1) *and* (2.2) *be fulfilled, where $\beta_i \le 1$, $d_i \in (-\infty, 0]$ $(i = 1, \dots, m)$. Then the condition*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-1+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds$$
$$> \frac{1}{m} \max\left\{ \frac{-\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\frac{\lambda d}{m}}\Big(\prod\limits_{i=1}^m (1+\gamma_i+\lambda\Big)^{\frac{1}{m}}}{\Big(\prod\limits_{i=1}^m \big(\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda}\big)\Big)^{\frac{1}{m}}} : \lambda \in [n-2, n-1] \right\}$$

is sufficient for equation (1.1) *to have Property* **A**, *where d and γ are given by* (2.3).

Theorem 2.3. *Let $F \in V(\tau)$, conditions* (1.2), (2.1) *and* (2.2) *be fulfilled, where $\alpha_i \ge 1$, $d_i \in [0, +\infty)$ $(i = 1, \dots, m)$. Then for equation* (1.1) *to have Property* **A**, *it is sufficient that the condition*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-1+\frac{\gamma}{m}} \Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}} ds$$
$$> \frac{1}{m} \max\left\{ \frac{-\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\frac{\lambda d}{m}}\Big(\prod\limits_{i=1}^m (1+\gamma_i+\lambda\Big)^{\frac{1}{m}}}{\Big(\prod\limits_{i=1}^m \big(\beta_i^{1+\gamma_i+\lambda} - \alpha_i^{1+\gamma_i+\lambda}\big)\Big)^{\frac{1}{m}}} : \lambda \in [0, 1] \right\}$$

holds when n is even and the condition

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-1+\frac{\gamma}{m}}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\left\{\frac{-\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\frac{\lambda d}{m}}\Big(\prod\limits_{i=1}^m(1+\gamma_i+\lambda\Big)^{\frac{1}{m}}}{\Big(\prod\limits_{i=1}^m\big(\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda}\big)\Big)^{\frac{1}{m}}}\right.$$
$$\left. :\lambda\in[1,2]\cup[n-2,n-1]\right\}$$

holds when n is odd, where d and γ are given by (2.3).

Theorem 2.4. *Let $F\in V(\tau)$, conditions* (1.2) *and*

$$\big|F(u)(t)\big|\geq\sum_{i=1}^m p_i(t)\big|u(\alpha_i t)\big|^{1+\frac{d_i}{\ln t}} \quad \text{for} \quad t\geq t_0>1, \quad u\in H_{t_0,\tau} \tag{2.4}$$

be fulfilled, where

$$p_i\in L_{\rm loc}(R_+;R_+), \quad \alpha_i>0, \quad d_i\in R. \tag{2.5}$$

Then for equation (1.1) *to have Property* **A**, *it is sufficient that*

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-2}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\left\{-\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m\alpha_i e^{d_i}\Big)^{-\frac{\lambda}{m}} : \lambda\in[0,n-1]\right\}.$$

Theorem 2.5. *Let $F\in V(\tau)$, conditions* (1.2), (2.4) *and* (2.5) *be fulfilled, where $\alpha_i\leq 1$ and $d_i\in(-\infty,0]$ $(i=1,\dots,m)$. Then for equation* (1.1) *to have Property* **A**, *it is sufficient that*

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-2}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\left\{-\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m\alpha_i e^{d_i}\Big)^{-\frac{\lambda}{m}} : \lambda\in[n-2,n-1]\right\}.$$

Theorem 2.6. *Let $F\in V(\tau)$, conditions* (1.2), (2.4) *and* (2.5) *be fulfilled, where $\alpha_i>1$ and $d_i\in[0,+\infty)$ $(i=1,\dots,m)$. Then for equation* (1.1) *to have Property* **A**, *it is sufficient that the condition*

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-2}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\left\{-\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m\alpha_i e^{d_i}\Big)^{-\frac{\lambda}{m}} : \lambda\in[0,1]\right\}$$

holds when n is even and

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-2}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\Big\{-\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m \alpha_i e^{d_i}\Big)^{-\frac{\lambda}{m}} : \lambda\in[1,2]\cup[n-2,n-1]\Big\}$$

holds when n is odd.

3. Almost linear functional differential equation with property B

Theorem 3.1. *Let $F\in V(\tau)$, conditions* (1.3), (2.1) *and* (2.2) *be fulfilled. Then for the equation* (1.1) *to have Property* **B** *it is sufficient that*

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-1+\frac{\gamma}{m}}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\left\{\frac{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod\limits_{i=1}^m(1+\gamma_i+\lambda)\Big)^{\frac{1}{m}}e^{-\frac{\lambda d}{m}}}{\Big(\prod\limits_{i=1}^m\big(\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda}\big)\Big)^{\frac{1}{m}}} : \lambda\in[0,n-2]\right\},$$

where γ and d are given by (2.3).

Theorem 3.2. *Let $F\in V(\tau)$, conditions* (1.3), (2.1) *and* (2.2) *be fulfilled, where $\beta_i\le 1$, $d_i\in(-\infty,0]$ $(i=1,\dots,m)$. Then for equation* (1.1) *to have Property* **B**, *it is sufficient that the condition*

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-1+\frac{\gamma}{m}}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\left\{\frac{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod\limits_{i=1}^m(1+\gamma_i+\lambda)\Big)^{\frac{1}{m}}e^{-\frac{\lambda d}{m}}}{\Big(\prod\limits_{i=1}^m\big(\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda}\big)\Big)^{\frac{1}{m}}} : \lambda\in[n-3,n-2]\right\}$$

holds when n is even and the condition

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-1+\frac{\gamma}{m}}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\left\{\frac{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod\limits_{i=1}^m(1+\gamma_i+\lambda)\Big)^{\frac{1}{m}}e^{-\frac{\lambda d}{m}}}{\Big(\prod\limits_{i=1}^m\big(\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda}\big)\Big)^{\frac{1}{m}}} : \lambda\in[0,1]\cup[n-3,n-2]\right\}$$

holds when n is odd, where γ and d are given by (2.3).

Theorem 3.3. *Let $F \in V(\tau)$, conditions* (1.3), (2.1) *and* (2.2) *be fulfilled, where $\alpha_i \geq 1$, $d_i \in [0,+\infty)$ $(i = 1,\dots,m)$. Then for equation* (1.1) *to have Property* **B**, *it is sufficient that the condition*

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-1+\frac{\gamma}{m}}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\left\{\frac{\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\frac{\lambda d}{m}}\Big(\prod\limits_{i=1}^m(1+\gamma_i+\lambda)\Big)^{\frac{1}{m}}}{\Big(\prod\limits_{i=1}^m\big(\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda}\big)\Big)^{\frac{1}{m}}} : \lambda\in[1,2]\right\}$$

holds when n is even and the condition

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-1+\frac{\gamma}{m}}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\left\{\frac{\lambda(\lambda-1)\cdots(\lambda-n+1)e^{-\frac{\lambda d}{m}}\Big(\prod\limits_{i=1}^m(1+\gamma_i+\lambda)\Big)^{\frac{1}{m}}}{\Big(\prod\limits_{i=1}^m\big(\beta_i^{1+\gamma_i+\lambda}-\alpha_i^{1+\gamma_i+\lambda}\big)\Big)^{\frac{1}{m}}} : \lambda\in[0,1]\right\}$$

holds when n is odd, where d and γ are given by (2.3).

Theorem 3.4. *Let $F \in V(\tau)$, conditions* (1.3), (2.1) *and* (2.5) *be fulfilled. Then for equation* (1.1) *to have Property* **B**, *it is sufficient that*

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-2}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\left\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m\alpha_i e^{d_i}\Big)^{-\frac{\lambda}{m}} : \lambda\in[0,n-2]\right\}.$$

Theorem 3.5. *Let $F \in V(\tau)$, conditions* (1.3), (2.4) *and* (2.5) *be fulfilled, where $\alpha_i \leq 1$ and $d_i \in (-\infty,0]$ $(i = 1,\dots,m)$. Then for equation* (1.1) *to have Property* **B**, *it is sufficient that the condition*

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-2}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\left\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m\alpha_i e^{d_i}\Big)^{-\frac{\lambda}{m}} : \lambda\in[n-3,n-2]\right\}$$

holds when n is even and the condition

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-2}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\bigg\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m \alpha_i e^{d_i}\Big)^{-\frac{\lambda}{m}} : \lambda\in[0,1]\cup[n-3,n-2]\bigg\}$$

holds when n is odd.

Theorem 3.6. *Let $F\in V(\tau)$, conditions* (1.3), (2.4) *and* (2.5) *be fulfilled, where $\alpha_i>1$ and $d_i\in[0,+\infty)$ $(i=1,\dots,m)$. Then for equation* (1.1) *to have Property* **B**, *it is sufficient that the condition*

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-2}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\bigg\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m \alpha_i e^{d_i}\Big)^{-\frac{\lambda}{m}} : \lambda\in[1,2]\bigg\}$$

holds when n is even and the condition

$$\liminf_{t\to+\infty} t\int_t^{+\infty} s^{n-2}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds$$
$$> \frac{1}{m}\max\bigg\{\lambda(\lambda-1)\cdots(\lambda-n+1)\Big(\prod_{i=1}^m \alpha_i e^{d_i}\Big)^{-\frac{\lambda}{m}} : \lambda\in[0,1]\bigg\}$$

holds when n is odd.

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