Seminar of I. Vekua Institute of Applied Mathematics REPORTS, Vol. 42, 2016

HIGHER ORDER DIFFERENCE EQUATIONS WITH PROPERTIES **A** AND **B**

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Abstract. The following higher order difference equation

$$\Delta^{(n)}u(k) + p(k)|u(\sigma(k))|^{\lambda}\mathrm{sign}(u(\sigma(k))) = 0$$

is considered, where $n \geq 2$, $0 < \lambda < 1$, $p : N \to R$, $\sigma : N \to N$, $\sigma(k) \geq k+1$.

Necessary conditions are obtained for the above equation to have monotone solutions. The obtained results are also new for the oscillation of solutions.

Keywords and phrases: Property **A**, Property **B**, oscillation.

AMS subject classification (2010): 34K11.

1. Introduction

Consider the higher order difference equation

$$\Delta^{(n)}u(k) + p(k)|u(\sigma(k))|^{\lambda}\mathrm{sign}(u(\sigma(k))) = 0, \tag{1.1}$$

where $n \geq 2$, $0 < \lambda < 1$, $p : N \to R$, $\sigma : N \to N$, $\sigma(k) \geq k+1$.

Here

$$\Delta^{(0)}u(k) = u(k), \quad \Delta^{(1)}u(k) = u(k+1) - u(k), \quad \Delta^{(i)}u(k) = \Delta^{(1)} \circ \Delta^{(i-1)}u(k) \quad (i = 2, \dots, n).$$

It will always be assumed that either the condition

$$p(k) \geq 0 \quad \text{for} \quad k \in N, \tag{1.2}$$

or

$$p(k) \leq 0 \quad \text{for} \quad k \in N \tag{1.3}$$

holds.

For each $k \in N$ denote $N_k = \{k, k+1, \dots\}$.

Definition 1.1. Let $k_0 \in N$. A function $u : N_{k_0} \to R$ is said to be a proper solution of equation (1.1), if it satisfies (1.1) on N_{k_0} and

$$\sup\{|u(k)| : k \geq s\} > 0 \quad \text{for any} \quad s > k_0.$$

Definition 1.2. Let $k_0 \in N$. A proper solution $u : N_{k_0} \to R$ of equation (1.1) is said to be oscillatory if for any $k \in N_{k_0}$ there are $k_1, k_2 \in N_{k_0}$ such that $u(k_1)u(k_2) < 0$. Otherwise the solution is called nonoscillatory.

Definition 1.3. We say that equation (1.1) has Property **A** if any its proper solutions either is oscillatory or satisfies

$$|\Delta^{(i)}u(k)| \downarrow 0 \quad \text{for} \quad k \uparrow +\infty \quad (i = 0, 1, \dots, n-1), \tag{1.4}$$

when n is odd.

Definition 1.4. We say that equation (1.1) has Property **B** if any of its proper solutions is oscillatory or satisfies either (1.4) or

$$|\Delta^{(i)}u(k)| \uparrow +\infty \quad \text{for} \quad k \uparrow +\infty \quad (i = 0, 1, \dots, n-1), \tag{1.5}$$

when n is even, either is oscillatory or satisfies (1.5) when n is odd.

For a functional differential equation, similar problems were considered in [1–4] (see also the references therein). Oscillatory properties for first and second order difference equations are studied in [5–9].

In the present paper we give sufficient conditions for equation (1.1) to have properties **A** and **B**.

2. Necessary condition of the existence of monotone solutions

For any $k_0 \in N$ denote by $U_{k_0,l}$ the set of solutions $u : N_{k_0} \to R$ of equation (1.1) which satisfies the condition:

$$\begin{aligned} \Delta^{(i)}u(k) > 0 \quad &\text{for} \quad k \geq k_0 \quad i = 0, \dots, l-1, \\ (-1)^i \Delta^{(i)}u(k) \geq 0 \quad &\text{for} \quad k \geq k_0 \quad i = l, \dots, n. \end{aligned}$$

Theorem 2.1. *Let $0 < \lambda < 1$, $k_0 \in N$, condition (1.3) ((1.4)) be fulfilled, $l \in \{1, 2, \dots, n-1\}$, $l + n$ be odd ($l + n$ be even) and $U_{k_0,l} \neq \varnothing$.*

Moreover, if

$$\sum_{k=1}^{+\infty} k^{n-l}(\sigma(k))^{\lambda(l-1)}|p(k)| = +\infty \tag{2.1}$$

then for any $\delta \in [0; \lambda]$ and $i \in N$ we have

$$\sum_{k=1}^{+\infty} k^{n-l-1+\lambda-\delta}(\sigma(k))^{\lambda(l-1)}[\rho_{l,i}(\sigma(k))]^{\delta}|p(k)| < +\infty,$$

where

$$\rho_{l,1}(k) = \left(\frac{1-\lambda}{l!(n-l)!} \sum_{i=1}^{k-1} \sum_{j=i}^{+\infty} j^{n-l-1}(\sigma(j))^{\lambda(l-1)}|p(j)| \right)^{\frac{1}{1-\lambda}}, \tag{2.2}$$

$$\rho_{l,s}(k) = \frac{1-\lambda}{l!(n-l)!} \sum_{i=1}^{k} \sum_{j=i}^{+\infty} j^{n-l-1}(\sigma(j))^{\lambda(l-1)}|p(j)|(\rho_{l,s-1}(\sigma(j)))^{\lambda} \quad (s = 2, 3, \dots). \tag{2.3}$$

3. Sufficient conditions of nonexistence of monotone solutions

Theorem 3.1 *Let conditions* (1.2) ((1.3)) (2.1) *be fulfilled,* $l \in \{1, \dots, n-1\}$, *let* $l+n$ *be odd* ($l+n$ *be even*) *and for any* $\delta \in [0, \lambda]$ *and* $i \in N$

$$\sum_{k=i}^{+\infty} k^{n-l-1+\lambda-\delta}(\sigma(k))^{\lambda(l-1)}(\rho_{l,i}(\sigma(k)))^{\delta}|p(k)| = +\infty \tag{3.1}$$

then for any $k_0 \in N$, $U_{l,k_0} = \varnothing$, *where* $\rho_{l,i}$ *is defined by* (2.2) *and* (2.3).

Theorem 3.2. *Let conditions* (1.2) ((1.3)) (2.1), *for any* $\gamma \in (0; 1)$

$$\liminf_{k\to+\infty} k^{\gamma} \sum_{j=k}^{+\infty} j^{n-l-1}(\sigma(j))^{\lambda(l-1)}|p(j)| > 0$$

be fulfilled, $l \in \{1, \dots, n-1\}$, *let* $l+n$ *be odd* ($l+n$ *be even*) *and for any* $\alpha \in (1; +\infty)$

$$\liminf_{k\to+\infty} \frac{\sigma(k)}{k^{\alpha}} > 0.$$

Moreover, if either

$$\alpha\lambda \geq 1,$$

or

$$\alpha\lambda < 1 \quad \text{and} \quad \sum_{k=1}^{+\infty} k^{n-l-1+\frac{\alpha\lambda(1-\lambda)}{1-\alpha\lambda}-\varepsilon}(\sigma(k))^{\lambda(l-1)}|p(k)| = +\infty$$

is fulfilled. Then for any $k_0 \in N$, $U_{l,k_0} = \varnothing$.

4. Difference equations with property A

Theorem 4.1. *Let conditions* (1.2) (2.1) *be fulfilled,* $l \in \{1, \dots, n-1\}$, *let* $l+n$ *be odd and for any* $\delta \in [0, \lambda]$ *and let* $k \in N$ (3.1) *be fulfilled. Moreover, if*

$$\sum_{k=1}^{n} k^{n-1}p(k) = +\infty, \tag{4.1}$$

when n *is odd, then Equation* (1.1) *has Property* **A**.

Theorem 4.2. *Let conditions* (1.2) *and*

$$\liminf_{k\to+\infty} \frac{(\sigma(k))^{\lambda}}{k} > 0$$

be fulfilled. Then for the equation (1.1) *to have Property* **A**, *it is sufficient that*

$$\sum_{k=1}^{+\infty} k^{n-2+\lambda}p(k) = +\infty.$$

Theorem 4.3. *Let conditions* (1.2) *and*

$$\limsup_{k\to+\infty}\frac{(\sigma(k))^{\lambda}}{k}<+\infty$$

be fulfilled. Then for equation (1.1) *to have Property* **A**, *it is sufficient that conditions* (4.1) *and*

$$\sum_{k=1}^{+\infty}k^{\lambda}(\sigma(k))^{\lambda(n-2)}p(k)=+\infty$$

be fulfilled.

5. Difference equations with property B

Theorem 5.1. *Let conditions* (1.3), (2.1) *be fulfilled,* $l\in\{1,\dots,n-1\}$, $l+n$ *is even and for any* $\delta\in[0,\lambda]$ *and let* $k\in N$ (3.1) *be fulfilled. Moreover, if*

$$\sum_{k=1}^{+\infty}k^{n-1}|p(k)|=+\infty, \tag{5.1}$$

when n *is even, then equation* (1.1) *has Property* **B**.

Theorem 5.2. *Let conditions* (1.3) *and*

$$\liminf_{k\to+\infty}\frac{(\sigma(k))^{\lambda}}{k}>0$$

be fulfilled. Then for equation (1.1) *to have Property* **B**, *it is sufficient that condition*

$$\sum_{k=1}^{+\infty}k^{n-2+\lambda}|p(k)|=+\infty$$

be fulfilled.

Theorem 5.3. *Let conditions* (1.3) *and*

$$\limsup_{k\to+\infty}\frac{(\sigma(k))^{\lambda}}{k}<+\infty$$

be fulfilled. Then for equation (1.1) *to have Property* **B**, *it is sufficient that conditions* (5.1),

$$\sum_{k=1}^{+\infty}k^{\lambda+1}(\sigma(k))^{\lambda(n-3)}|p(k)|=+\infty$$

and

$$\sum_{k=1}^{+\infty}(\sigma(k))^{\lambda(n-1)}|p(k)|=+\infty$$

be fulfilled.

R E F E R E N C E S

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Received 12.09.2016; revised 11.10.2016; accepted 21.11.2016.

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