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ONE PROBLEM OF THE BENDING OF A PLATE FOR A CURVILINEAR QUADRANGULAR DOMAIN WITH A RECTILINEAR CUT

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Abstract. In the present paper we consider the problem of bending of a plate for a curvilinear quadrangular domain with a rectilinear cut. It is assumed that the external boundary of the domain composed of segments (parallel to the abscissa axis) and arcs of one and the same circumference. The internal boundary is the rectilinear cut (parallel to the Ox-axis). The plate is bent by normal moments applied to rectilinear segments of the boundary, the arcs of the boundary are free from external forces, while the cut edges are simply supported. The problem is solved by the methods of conformal mappings and boundary value problems of analytic functions. The sought complex potentials which determine the bending of the midsurface of the plate are constructed effectively (in the analytical form). Estimates are given of the behavior of these potentials in the neighborhood of the corner points.

Keywords and phrases: The bending of a plate, conformal mapping, Riemann-Hilbert problem for circular ring.

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1. Statement of the problem

Let a homogeneous Isotropic plate on a plane $z = x + iy$ of a complex variable occupy the doubly-connected domain S, the external boundary of the domain is composed of segments (parallel to the abscissa axis) and arcs of one and the same circumference. The internal boundary is the rectilinear cut (parallel to the Ox-axis).

We will assume that normal bending moments M_n act on each rectilinear sections $L_0^{(1)} = A_1A_1$, $L_0^{(2)} = A_3A_4$ of the external boundary, the arcs $L_0^{(3)} = A_2A_3$, $L_0^{(k)} = A_4A_1$ of the boundary are free from external forces, while the cut $L_1 = B_1B_2$ edges are simply supported and for better clearness, we consider the symmetric case. We denote by $\alpha^0\pi$ the value of internal (with respect to the domain S) vertex angles $A_k (k = 1, ..., 4)$ (we mean the angles between the segments $L_0^{(1)}$, $L_0^{(2)}$ and the tangent arcs $L_0^{(3)}$ and $L_0^{(4)}$) and we will choose as the positive direction on the boundary $L = L_0 \cup L_1$ ($L_0 = \bigcup\limits_{k=1}^{4} L_0^{(k)}$, $L_1 = \bigcup\limits_{m=1}^{2} L_1^{(m)}$, $L_1^{(1)} = B_1B_2$, $L_1^{(2)} = B_2B_1$) which leaves the region S on the left. Let $\alpha(t)$ and $\beta(t)$ be the angles lying between the Ox-axis and the outer normals to the contours L_0 and L_1 at the point $t \in L$, where

$$\alpha(t) = \begin{cases} \dfrac{\pi}{2}(2k-1), & t \in L_0^{(k)}, \ k = 1, 2, \\ \arg t, & t \in L_0^{(k)}, \ k = 3, 4, \end{cases} \qquad \beta(t) = \begin{cases} \dfrac{\pi}{2}, & t \in L_1^{(1)}, \\ -\dfrac{\pi}{2}, & t \in L_1^{(2)}. \end{cases}$$

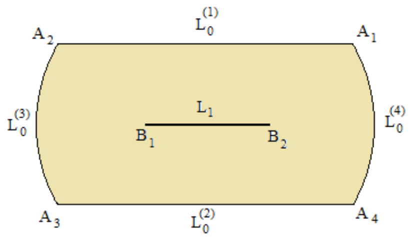

Fig. 1

The problem consists in defining the bending deflection of the middle surface of the plate and establishing the situations of the concentration of stresses near the angular points which in turn depend on the behavior of Kolosov-Muskhelishvili potentials at these points.

Analogous problems of plane elasticity and plate bending for finite doubly-connected domains bounded by polygons are considered in [1, 4].

2. Solution of the problem

Let us recall some results concerning the conformal mapping of a doubly-connected domain $S^{(0)}$ onto the circular ring $D_0\{1<|\zeta|<R_0\}$. The derivative of the function $\omega(\varsigma)$ is the solution of the Riemann-Hilbert problems for the circular ring [5]

$$\mathrm{Re}[i\sigma e^{-i\nu_0(\sigma)}\omega'(\sigma)]=0,\quad \sigma\in l, \tag{1}$$

where $l=l_0\cup l_1,\ l_0=\{|\sigma|=R\},\ l_1=\{|\sigma|=1\},\ \nu_0(\sigma)=\alpha[\omega(\sigma)]=\alpha_0(\sigma),\quad \sigma\in l_0$, $\nu_0(\sigma)=\beta[\omega(\sigma)]=\beta_0(\sigma),\ \ \sigma\in l_1$.

To solve the problem (1) (with respect to the function $\omega'(\zeta)$) of the class $h(b_1,b_2)$ [6] (the index of the given class problem (1) is equal to zero), it is necessary and sufficient that the condition

$$\prod_{k=1}^{4}R^2a_k^{\alpha_k^0-1}\cdot\prod_{m=1}^{2}b_m=1,\ \ (a_k=\omega^{-1}(A_k),\ b_m=\omega^{-1}(B_m)), \tag{2}$$

be fulfilled, and a solution itself is given by the formula

$$\omega'(\zeta)=K^0e^{\gamma(\zeta)}B(\zeta), \tag{3}$$

where

$$\gamma(\zeta)=\frac{1}{2\pi i}\sum_{j=-\infty}^{\infty}\int_{l_0}\frac{\ln(R^2\sigma^{-2}e^{2i\alpha_0(\sigma)}}{\sigma-R^{2j}\zeta}d\sigma,\qquad B(\zeta)=\prod_{j=-\infty}^{\infty}\prod_{m=1}^{2}(R^{2j}\zeta-b_m), \tag{4}$$

with k^0 as an arbitrary real constant.

Based on the results given in [6, §78], we conclude that the function $e^{\gamma(\varsigma)}$ near the points $a_k(k=\overline{1,4})$ can be written in the form

$$e^{\gamma(\varsigma)}=\prod_{k=1}^{4}(\zeta-a_k)^{\alpha_k^0-1}\Omega^0(\zeta), \tag{5}$$

where Ω^0 is the function holomorphic near the point a_k and tending to definite nonzero limits as $\varsigma\to a_k$.

Thus, for a conformally mapping function bounded at the points a_k from (4) we obtain the formula

$$\omega'(\zeta) = K^0 \prod_{k=1}^{4} (\zeta - a_k)^{\alpha_k^0 - 1} \Omega^0(\zeta) B(\varsigma). \tag{6}$$

Let us now return to the considered problem. According to the approximate theory of the bending of a plate, the bending deflection $w(x, y)$ of the midsurface of the plate in the case considered satisfies the biharmonic equation

$$\Delta^2 w(x, y) = 0, \quad z = x + iy \in S$$

and the boundary conditions

$$\begin{aligned}
&M_n(t) = f(t), \quad \frac{\partial w}{\partial s} = 0, \ \ t \in L_0^{(1)} \cup L_0^{(2)}, \\
&M_n(t) = 0, \ \frac{\partial w}{\partial n} = 0, \ \ t \in L_0^{(3)} \cup L_0^{(4)}, \\
&w(t) = 0, \ \ M_n(t) = 0, \ \ t \in L_1, \ \ N(t) = 0, \ \ t \in L_0 \cup L_1,
\end{aligned} \tag{7}$$

where $M_n(t)$ is the normal bending moments, $N(t)$ is the shearing force.

Using the well-known formulae [6-8] we have

$$\begin{aligned}
&\frac{\partial w}{\partial n} + i\frac{\partial w}{\partial s} = e^{-i\nu(t)}[\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}], \\
&2D_0(\sigma - 1)d[\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}] = \left[M_n(t) + i\int_0^s N(t)ds\right] dt, \\
&\nu(t) = \alpha(t), \ \ t \in L_0, \ \ \nu(t) = \beta(t), \ \ t \in L_1, \ \ \varkappa = (\sigma + 3)(1 - \sigma)^{-1},
\end{aligned} \tag{8}$$

where σ is Poisson ratio, D_0 is the cylindrical stiffness of the plate.

By virtue of condition (7) and formula (8) with respect to the required functions $\varphi(z)$ and $\psi(z)$ we obtain the boundary problems

$$\begin{aligned}
&\operatorname{Re}\left[ie^{-i\nu(t)}(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)})\right] = 0, \\
&\operatorname{Re}\left[ie^{-i\nu(t)}(\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)})\right] = F_0^{(1)}(t), \ \ t \in L_0^{(1)} \cup L_0^{(2)},
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\operatorname{Re}\left[ie^{-i(\nu(t)+\frac{\pi}{2})}(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)})\right] = 0, \\
&\operatorname{Re}\left[ie^{-i(\nu(t)+\frac{\pi}{2})}(\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)})\right] = F_0^{(2)}(t), \ \ t \in L_0^{(3)} \cup L_0^{(4)},
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\operatorname{Re}\left[ie^{-i\nu(t)}(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)})\right] = 0, \\
&\operatorname{Re}\left[ie^{-i\nu(t)}(\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)})\right] = c^{(1)}(t), \ \ t \in L_1,
\end{aligned} \tag{11}$$

where

$$F_0^{(1)}(t) = -[2D_0(\sigma - 1)]^{-1} \int_0^s M_n(t)ds + c^{(0)}(t), \ \ t \in L_0^{(1)} \cup L_0^{(2)},$$

$$F_0^{(2)}(t) = \operatorname{Re}[2D_0(\sigma - 1)]^{-1} t^{-1} c^{(0)}(t), \ \ t \in L_0^{(3)} \cup L_0^{(4)},$$

$$c^{(0)}(t) = c_k^{(0)} = const, \ \ t \in L_0^{(k)} \ \ (k = \overline{1,4}),$$

$$c^{(1)}(t) = c_k^{(1)} = const, \ \ t \in L_1^{(k)} \ \ (k = 1,2).$$

The constant $c_k^{(j)}$ $(j = 0,1)$ are unknown in advance and must be determined when solving the problem in such a away that the function $\varphi(z)$ and $\bar{z}\varphi'(z) + \psi(z)$ extend continuously into to domain $S \cup L$.

These boundary problems are in turn divided into two problems

$$\mathrm{Re}\left[ie^{-i\Delta(t)}\varphi(t)\right] = F(t), \ \ t \in L_0 \cup L_1, \tag{12}$$

$$\mathrm{Re}\left[ie^{-i\Delta(t)}(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)})\right] = 0, \ \ t \in L_0^{(1)} \cup L_0^{(2)}, \tag{13}$$

where $\Delta(t) = \alpha(t),\ t \in L_0^{(1)} \cup L_1^{(2)};\ \Delta(t) = \dfrac{\pi}{2} + \arg t,\ t \in L_0^{(3)} \cup L_1^{(4)};\ \Delta(t) = \beta(t),\ t \in L_1;$ $F(t) = F_0^{(1)},\ t \in L_0^{(1)} \cup L_1^{(2)};\ F(t) = F_0^{(2)},\ t \in L_0^{(3)} \cup L_1^{(4)};\ F(t) = c^{(1)}(t),\ t \in L_1.$

Let us consider problem (12). After the conformal mapping of the domain S onto the circular ring D, this problem for the function $\chi(\zeta) = \zeta^{-1}\varphi_0(\zeta)$ $(\varphi_0(\zeta) = \varphi\left[\omega(\zeta)\right])$ reduces to the Riemann-Hilbert problem for a circular ring

$$\mathrm{Re}\left[i\sigma e^{-i\Delta_0(\sigma)}\chi(\sigma)\right] = F_0(\sigma), \ \ \sigma \in l, \tag{14}$$

where $\Delta_0(\sigma) = \Delta[\omega(\sigma)]$, $F_0(\sigma) = F[\omega(\sigma)]$, $\sigma \in l$.

Let us consider the homogeneous problem corresponding to problem (14)

$$\mathrm{Re}\left[i\sigma e^{-i\Delta_0(\sigma)}\chi(\sigma)\right] = 0, \ \ \sigma \in l, \tag{15}$$

Although problem (15) is different from problem (1) we can use it [5] and its solution is given by the formula

$$\chi(\zeta) = \omega'(\zeta)T(\zeta), \tag{16}$$

where $T(\zeta) = \prod\limits_{j=-\infty}^{\infty}\prod\limits_{k=1}^{4}(R^{2j}\zeta - a_k)^{-\frac{1}{2}}$, $\omega'(\zeta)$ is defined by formula (6).

Thus we have obtained the factorization coefficient of problem (15) in the form

$$e^{2i\Delta_0(\sigma)}\frac{\bar{\sigma}}{\sigma} = \frac{\omega'(\sigma)T(\sigma)}{\overline{\omega'(\sigma)}\cdot\overline{T(\sigma)}}, \ \ \sigma \in l.$$

With the obtained results taken into account, from the boundary conditions (14) for the function

$$\Omega(\zeta) = i\varphi_0(\zeta)[\zeta\omega'(\zeta)T(\zeta)]^{-1} \tag{17}$$

we obtain the Dirichlet problem for a circular ring

$$\mathrm{Re}\left[\Omega(\sigma)\right] = F_0(\sigma)e^{i\Delta_0(\sigma)}[\sigma\omega'(\sigma)T(\sigma)]^{-1}, \ \ \sigma \in l. \tag{18}$$

A solvability condition of problem (18) has the form

$$\int\limits_{l}\frac{F_0(\sigma)e^{i\Delta_0(\sigma)}}{\sigma^2\omega'(\sigma)T(\sigma)}d\sigma = 0, \tag{19}$$

and its solution is given by the formula

$$\Omega(\zeta) = \frac{1}{\pi i} \sum_{j=-\infty}^{\infty} \int_l \frac{F_0(\sigma) e^{i\Delta_0(\sigma)} d\sigma}{(\sigma - R^{2j}\zeta)\sigma\omega'(\sigma)T(\sigma)} + ic_0^*, \tag{20}$$

where c_0^* is an arbitrary real constant.

Thus, using (17) and (20), for the function $\varphi_0(\zeta)$ we obtain the formula

$$\varphi_0(\zeta) = \omega'(\zeta)T(\zeta)M(\zeta), \tag{21}$$

where

$$M(\zeta) = -\frac{\zeta}{\pi}\left[\sum_{j=-\infty}^{\infty} \int_l \frac{F_0(\sigma) e^{i\Delta_0(\sigma)} d\sigma}{(\sigma - R^{2j}\zeta)\sigma\omega'(\sigma)T(\sigma)} + c_0^*\right]. \tag{22}$$

Since the function $\omega'(\zeta)T(\zeta)$ at the points a_k $(k = \overline{1,4})$ has singularities of the form $|\zeta - a_k|^{\alpha_k^0 - \frac{3}{2}}$, for the function $\varphi_0(\zeta)$ to be continuously extendable into the domain $D \cup l$ it is necessary and sufficient for the following conditions to be satisfied

$$M(a_k) == 0, \quad k = \overline{1,4}. \tag{23}$$

Since $\varphi'(z) = \frac{\varphi_0'(\zeta)}{\omega'(\zeta)}$, from (21) we have

$$\varphi'(z) = \frac{\omega''(\zeta)}{\omega'(\zeta)} T(\zeta)M(\zeta) + [T(\zeta)M(\zeta)]'. \tag{24}$$

Bearing in mind both the behavior of the Cauchy type integral in the neighborhood on the points density discontinuity [6] and that of the conformally mapping fuction in the neighborhood of angular points [9], we conclude that near the points b_k $(k = 1, 2)$

$$\begin{aligned} &\omega(\zeta) = B + (\zeta - b)^2[N_0 + N_1(\zeta - b) + \cdots], \\ &\frac{\omega''(\zeta)}{\omega'(\zeta)} = \frac{1}{\zeta - b} + E_1 + E_2(\zeta - b) + \cdots, \\ &T(\zeta)M(\zeta) = \frac{k_0}{\zeta - b} + k_1 + k_2(\zeta - b) + \cdots, \end{aligned} \tag{25}$$

where b is one of the points b_k, B is the preimage of the point b, N_0,...,k_2,... are some constants.

Thus, using (24) and (25), near a point B we have the estimates

$$\left|\varphi'(z)\right| < M_1 \left|z - B\right|^{-\frac{1}{2}}, \quad \left|\varphi''(z)\right| < M_2 \left|z - B\right|^{-\frac{3}{2}}, \quad M_1, M_2 = const.$$

By a similar reasoning to the above, it is proved that $\varphi'(z)$ is almost bounded (i.e. has singularities of logarithmic type $\ln(z - A)$) near the points A_k $(k = \overline{1,4})$.

After finding the function $\varphi(z)$, the definition of the function $\psi(z)$ by (13) reduces to the following problem which is analogous to problem (12)

$$\operatorname{Re}\left[ie^{i\Delta(t)}R(t)\right] = \Gamma(t), \quad t \in L, \tag{26}$$

where

$$R(z) = \psi(z) + P(z)\varphi'(z),$$

$$\Gamma(t) = F(t) + \mathrm{Re}\left[ie^{i\Delta(t)}(P(t) - \bar{t})\varphi'(t)\right], \quad t \in L,$$

and $P(z)$ is an interpolation polynomial satisfying the condition $P(B_k) = \bar{B}_k$ $(k = 1, 2)$, $\bar{B}_k$ is a number conjugate to B_k.

The use of the polynomial $P(z)$ makes bounded the right-hand part of the boundary condition (26) so that the solution of this problem can be constructed in an analogous manner as above (see problem (12)), while the solvability condition (with the assumption that the function $\psi(z)$ is continuous up to the boundary) will be analogous to conditions (19) and (23).

All these conditions are represented as an inhomogeneous system with real coefficient with respect to 8 constants $c_k^{(0)}$ $(k = \overline{1,4})$, $c_m^{(1)}$ $(m = 1, 2)$, c_0^*, c_0^{**} (c_0^{**} is a real constant which occurs when solving problem (26)). For the definition of these constant we have 8 equations. It is proved that the obtained system is uniquely solvable and therefore the problem posed has a unique solution.

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