Seminar of I. Vekua Institute
of Applied Mathematics
REPORTS, Vol. 42, 2016

ABOUT ONE METHOD OF CONSTRUCTION OF APPROXIMATE SOLUTIONS OF SOME BOUNDARY VALUE PROBLEMS

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Abstract. A simple algorithm for construction of the approximate solution of some classical and nonlocal boundary value problems of the mathematical physics is considered. The efficiency of the offered algorithm for construction of the approximate solutions of problems is shown on the examples of two-dimensional classical and nonlocal boundary value problems of the theory of elasticity and for two-dimensional equations of Laplace and Helmholtz.

Keywords and phrases: Boundary value problems, approximate solution, nonlocal problems.

AMS subject classification (2010): 35J25, 35J55, 65N99.

1. Introduction

In this work a simple algorithm for construction of the approximate solution of some boundary value problems of the mathematical physics is considered. The mentioned algorithm has been offered in [1]. We may call a considered method a semi-analytical method. From the approximate methods known in the literature it is the closest to a method of fundamental solutions [2-4] and a boundary elements method [5-9].

In the work the main relations of the offered method for the problems of the two-dimensional equations of Laplace and Helmholtz and for problems of the plane theory of thermoelasticity are obtained. By means of this method the approximate solutions for several classical boundary value problems and nonlocal problems of Bitsadze-Samarskii type [10-21] are constructed and exact solutions of these problems are known in advance. The relevant exact and approximate solutions are compared with each other and appropriate conclusions are drawn.

2. Problems for the Laplaces two dimensional equation

Let Oxy be a rectangular cartesian coordinate system on the plane. We consider the Laplace equation

$$\Delta u = 0, \tag{1}$$

where $\Delta(\cdot) = (\cdot)_{,xx} + (\cdot)_{,yy}$ is a two-dimensional laplacian, $(\cdot)_{,x} \equiv \dfrac{\partial(\cdot)}{\partial x}$, $(\cdot)_{,y} \equiv \dfrac{\partial(\cdot)}{\partial y}$; $u(x, y)$ is a scalar function.

First we consider the simply connected domain Ω with a sufficiently smooth boundary L . The domain Ω covers the origin of coordinates. On a contour L the $2N + 1$ points with coordinates of $(x_1, y_1), (x_2, y_2), ..., (x_{2N+1}, y_{2N+1})$ are more or less evenly distributed (Fig. 1).

The approximate solution is sought in the form of

$$\bar{u} = a_0 + \sum_{n=1}^{N} r^n(x,y)[a_n \cos(n\theta(x,y)) + b_n \sin(n\theta(x,y))], \tag{2}$$

where $a_0, a_1, \cdots, a_n, b_1, \cdots, b_n$ are sought-for real coefficients; $r(x,y) = \sqrt{x^2+y^2}$,

$$\theta(x,y) = \begin{cases} \arctan\dfrac{y}{x}, & x>0, \\ \arctan\dfrac{y}{x}+\pi, & x<0,\ y\geq 0, \\ \arctan\dfrac{y}{x}-\pi, & x<0,\ y<0, \\ \dfrac{\pi}{2}, & x=0,\ y>0, \\ -\dfrac{\pi}{2}, & x=0,\ y<0. \end{cases}$$

The partial derivatives of $\bar{u}(x,y)$ are expressed by the formulas

$$\begin{aligned} \bar{u}_{,x} &= \sum_{n=1}^{N} nr^{n-1}(x,y)[a_n \cos((n-1)\theta(x,y)) + b_n \sin((n-1)\theta(x,y))], \\ \bar{u}_{,y} &= \sum_{n=1}^{N} nr^{n-1}(x,y)[-a_n \sin((n-1)\theta(x,y)) + b_n \cos((n-1)\theta(x,y))]. \end{aligned} \tag{3}$$

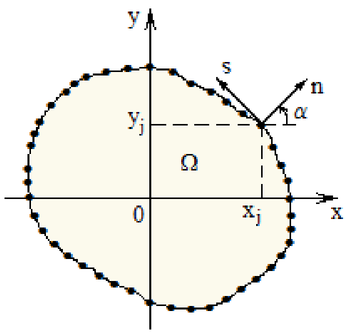

Fig. 1. The simply connected domain Ω

The algorithm of construction of the approximate solution is stated on the example of the classical mixed boundary value problem. The contour L is divided into two contours L_1 and L_2 so that by $L_1 \bigcap L_2 = \varnothing$ and $\overline{L}_1 \bigcap \overline{L}_2 = L$ (Fig. 1). Let us assume that the contour L_1 includes points of (x_1,y_1), $(x_2,y_2), \cdots, (x_{N_1}, y_{N_1})$ and the contour L_2 includes points of (x_{N_1+1}, y_{N_1+1}), $(x_{N_1+2}, y_{N_1+2}), \cdots, (x_{2N+1}, y_{2N+1})$. On the contour L_1 the value of the sought-for function is set, and on the contour L_2 - of the value of its normal derivative

$$\begin{cases} u|_{L_1} = f_1(x,y), & (x,y) \in L_1 \\ u_{,n}|_{L_2} = f_2(x,y), & (x,y) \in L_2, \end{cases} \tag{4}$$

where $f_1(x,y)$ and $f_2(x,y)$ are the functions defined on the boundary; $(\cdot)_{,n}$ derivative in the direction $\vec{n} = (\cos\alpha, \sin\alpha)$, i. e.

$$u_{,n} = u_{,x}\cos\alpha + u_{,y}\sin\alpha. \tag{5}$$

External unit normal in a point (x_j, y_j) on the boundary is designated through $(\cos\alpha_j, \sin\alpha_j)$.

When $j = 1, 2, \cdots, N_1$ in the formula (2) x and y are replaced through x_j and y_j respectively. The expressions obtained $f_1(x_j, y_j)$ are equated to the corresponding values of the boundary conditions (4). Similarly, when $j = N_1+1, N_1+2, \cdots, 2N+1$ in the formula (3) x and y are replaced through x_j and y_j. The expressions received are substituted in (4), where instead of α value α_j is substituted. The resulting expressions are equated to the corresponding values $f_2(x_j, y_j)$ of the boundary conditions (4).

Thus, we obtain the system of the linear algebraic $2N+1$ equations with $2N+1$ unknown $a_0, a_1, ..., a_N, b_1, ..., b_N$

$$\begin{cases} a_0 + \sum\limits_{n=1}^{N}(A_{1nj}a_n + A_{2nj}b_n) = f_1(x_j, y_j), & j = 1, 2, \cdots, N_1, \\ \sum\limits_{n=1}^{N}(B_{1nj}a_n + B_{2nj}b_n) = f_2(x_j, y_j), & j = N_1+1, N_1+2, \cdots, 2N+1, \end{cases} \tag{6}$$

where

$$A_{1nj} = r^n(x_j, y_j)\cos(n\theta(x_j, y_j)),$$
$$A_{2nj} = r^n(x_j, y_j)\sin(n\theta(x_j, y_j)),$$
$$B_{1nj} = nr^{n-1}(x_j, y_j)[\cos((n-1)\theta(x_j, y_j))\cos\alpha_j + \sin((n-1)\theta(x_j, y_j))\sin\alpha_j],$$
$$B_{2nj} = nr^{n-1}(x_j, y_j)[-\sin((n-1)\theta(x_j, y_j))\cos\alpha_j + \cos((n-1)\theta(x_j, y_j))\sin\alpha_j].$$

After solving the system (6), its solution $(a_0, a_1, ..., a_N, b_1, ..., b_N)$ is substituted in the formula (2) and thus we've got the approximate solution of a boundary value problem (1), (4).

Example 1. As an example we consider a classical problem of Dirichlet in elliptic domain $V = \{(x,y)|\ x^2 + 4y^2 < 1\}$. The boundary of domain V is the ellipse of S, which is set parametrically $x = \cos t,\ y = 0.5\sin t,\ 0 \le t < 2\pi$. Thus, the following problem is considered

$$\Delta u = 0 \quad in\ \ V,$$
$$u|_S = 0.5(x^2 + y^2)|_{(x,y)\in S}.$$

The exact solution of this problem is the following function

$$u = 0.2 + 0.3(x^2 - y^2).$$

On the boundary S the points $\left(\cos\frac{\pi}{36}(j-1), 0.5\sin\frac{\pi}{36}(j-1)\right)$, $j = 1, 2, ..., 71$ are marked (Fig. 2). The approximate solution is sought in the form (2), where $N = 35$. Meeting the boundary conditions in the marked points, we've got the system of the algebraic 71 equations with 71 unknown.

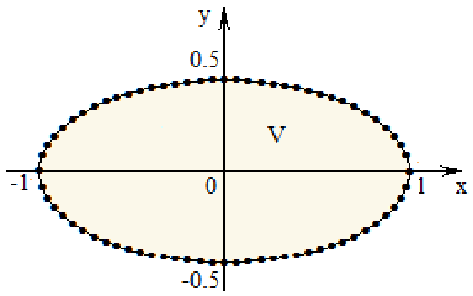

Fig. 2. The domain V with the points marked on the boundary

After solving this system, the resulting solution is substituted in (2) ($N = 35$) and we've got the approximate solution.

The appropriate program is made in the Maple12. Numerical results are specified in Table 1.

Tab. 1. Numerical results for the problem 1

(x, y)	$\bar{u}(x, y)$	$u(x, y)$	$\lvert\bar{u}(x, y) - u(x, y)\rvert$
(0.01, 0)	0.2000300000	0.20003	0
(0.1, 0)	0.2030000000	0.20300	0
(0.5, 0)	0.2750000000	0.27500	0
(0.9, 0)	0.4429999995	0.44300	$5.0 \cdot 10^{-10}$
(0.2, −0.2)	0.2000000000	0.20000	0
(0, 0.3)	0.1730000000	0.17300	0
(0.8, 0.1)	0.3890000001	0.38900	10^{-10}

As Table 1 shows the constructed approximate solution may be called the exact solution of the problem of Dirichlet.

The approximate solutions for multi-connected domains are constructed analogously. For simplicity the doubly connected domain Ω, bounded by the simple closed contours L_1 and L_2 is considered from which the last one embraces the latter and the previous embraces the origin of coordinates. On these contours the points $2(2N + 1)$ with the coordinates $(x_1, y_1), (x_2, y_2), ..., (x_{2(2N+1)}, y_{2(2N+1)})$ are more or less evenly distributed (Fig. 3).

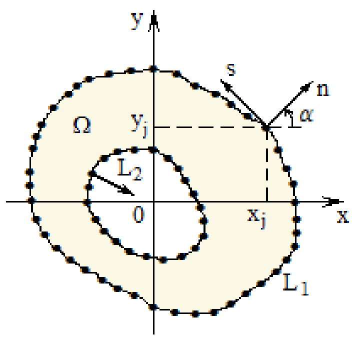

Fig. 3. The doubly connected domain Ω

The approximate solution is sought in the following form

$$\bar{u} = a\ln r(x,y) + a_0 + \sum_{n=1}^{N} r^{-n}(x,y)[a_n\cos(n\theta(x,y)) + b_n\sin(n\theta(x,y))] \\ + r^n(x,y)[c_n\cos(n\theta(x,y) + d_n\sin(n\theta(x,y))]. \tag{7}$$

Partial derivatives of $\bar{u}(x,y)$ function are expressed by means of the formulas

$$\bar{u}_{,x} = \frac{ax}{r^2(x,y)} + \sum_{n=1}^{N} -nr^{-n-1}(x,y)[a_n\cos((n+1)\theta(x,y)) \\ + b_n\sin((n+1)\theta(x,y))] + nr^{n-1}(x,y)[c_n\cos((n-1)\theta(x,y)) \\ + d_n\sin((n-1)\theta(x,y))], \tag{8}$$

$$\bar{u}_{,y} = \frac{ay}{r^2(x,y)} + \sum_{n=1}^{N} -nr^{-n-1}(x,y)[a_n\sin((n+1)\theta(x,y)) \\ - b_n\cos((n+1)\theta(x,y))] + nr^{n-1}(x,y)[-c_n\sin((n-1)\theta(x,y)) \\ + d_n\cos((n-1)\theta(x,y))], \tag{9}$$

Using the formulas (7)-(9), (5) of the simply connected domain considered above, the boundary conditions are satisfied point-wise in the points selected on the boundary. As a result the we've got a system of the linear algebraic $4N+2$ equations with $4N+2$ unknowns $a, a_0, a_1, ..., a_N, b_1, ..., b_N, c_1, ..., c_N, d_1, ..., d_N$.

The considered way can be applied to construct the approximate solution of rather a wide class of tasks for harmonic functions. The example of construction of the approximate solution of nonlocal problem of Bitsadze-Samarskii for the doubly connected domain bounded by the rectangular contours is given below.

Example 2. Let the domain V represent the doubly connected domain $V = V_1\backslash\overline{V}_2$, where $V_1 = \{-2 < x < 3, -2 < y < 2\}$, $V_2 = \{-1 < x < 1, -1 < y < 1\}$ (Fig. 4). We consider below the nonlocal problem of Bitsadze-Samarskii

$$\Delta u = 0 \quad in \;\; V,$$

$$u(-2,y) = -\frac{2}{4+y^2} - y^2 + 20, \quad -2 \le y \le 2,$$

$$u(x,\pm 2) = \frac{x}{x^2+4} + x^2 - 5x + 2, \quad -2 < x \le 3,$$

$$u(3,y) - u(2,y) = \frac{3}{9+y^2} - \frac{2}{4+y^2}, \quad -2 < y < 2,$$

$$u_{,x}(-1,y) = \frac{y^2-1}{(y^2+1)^2} - 7, \quad -1 \le y < 1,$$

$$u_{,y}(x,\pm 1) = \mp\Big(\frac{2x}{(x^2+1)^2} + 2\Big), \quad -1 \le x < 1,$$

$$u_{,x}(1,y) = \frac{y^2-1}{(y^2+1)^2} - 3, \quad -1 < y \le 1.$$

The exact solution of this problem is as follows

$$u(x,y) = \frac{x}{x^2+y^2} + x^2 - 5x - y^2 + 6.$$

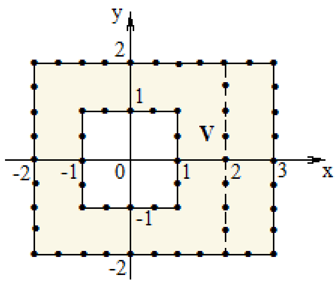

Fig. 4. Doubly connected domain V, in which nonlocal problem is solved

On an external contour beginning from the point (3, 0), with a step 0.5 points 36 are marked. Analogously, on an internal contour beginning from a point (1, 0), with the same frequency 16 more points are marked. On an internal contour two more points with coordinates $(0.75, -1.0)$ and $(-0.75, 1.0)$ are marked. In fig. 4 also 7 points are marked on the segment inside the body where nonlocal conditions are set. The approximate solution sought in the form (7), where $N = 13$. Boundary and nonlocal conditions are satisfied in the marked points.

The solution of the nonlocal problem is tabulated to solution of the problem of system of the linear algebraic 54 equations with 54 unknown. After solving this system, the resulting solution is substituted in (7) ($N = 13$) and we've got the approximate solution.

The appropriate program is made in the Maple 12. Numerical results are presented in Table 2.

Tab. 2. Numerical results for a problem 2

(x,y)	$\bar{u}(x,y)$	$u(x,y)$	$\lvert\bar{u}(x,y) - u(x,y)\rvert$
(2.0, 0)	0.5000066859	0.5	$6.66859 \cdot 10^{-6}$
(1.6, 1.8)	-2.404136519	-2.404137931	$1.412 \cdot 10^{-6}$
(0.4, 1.74)	1.257888466	1.257886259	$2.203 \cdot 10^{-6}$
$(-1.43, -2.25)$	9.931201879	9.931201249	$6.3 \cdot 10^{-7}$
$(0.7, -1.23)$	1.826596858	1.826593235	$3.623 \cdot 10^{-6}$
$(-1.5, 1.5)$	13.16666710	13.16666667	$4.3 \cdot 10^{-7}$
$(3.0, -2.0)$	-3.769230771	-3.769230769	$2.0 \cdot 10^{-9}$

3. Problems of the plane theory of thermoelasticity

Let consider the plane deformation parallel to the plane Oxy for the homogeneous transversely isotropic thermoelastic body.

If the plane of an isotropie is parallel to the Oxy plane then the homogenous system of the equations of thermoelastic equilibrium in displacements has the form [22, 23, 25]

$$\begin{cases} \mu\Delta u + \dfrac{1}{2}\dfrac{E_1E_2}{(1-\nu_1)E_2 - 2\nu_2^2E_1}(u_{,x} + v_{,y})_{,x} - \beta T_{,x} = 0, \\ \mu\Delta v + \dfrac{1}{2}\dfrac{E_1E_2}{(1-\nu_1)E_2 - 2\nu_2^2E_1}(u_{,x} + v_{,y})_{,y} - \beta T_{,y} = 0, \end{cases} \tag{10}$$

where μ are shear modulus $\mu = \dfrac{E_1}{2(1+\nu_1)}$; $\nu_1,\ \nu_2$ and E_1, E_2 Poisson's coefficients and Young's modulus in the Oxy and in the direction of perpendicular thereto, respectively. u and v are components of the displacement vector along axes x and y, respectively; β constant depending on the thermal properties of material $\beta = \dfrac{E_1E_2(\alpha_1 + \nu_2\alpha_2)}{(1-\nu_1)E_2 - 2\nu_2^2E_1}$; α_1, α_2 are the coefficients of the linear thermal expansion; T is the temperature changes in the elastic body satisfying the Laplace equation

$$\Delta T = 0. \tag{11}$$

Duhamel-Neumann relations has the form

$$\begin{gathered} \sigma_{xx} = \frac{2\mu}{(1-\nu_1)E_2 - 2\nu_2^2E_1}[(E_2 - \nu_2^2E_1)u_{,x} + (\nu_1E_2 + \nu_2^2E_1)v_{,y}] - \beta T, \\ \sigma_{yy} = \frac{2\mu}{(1-\nu_1)E_2 - 2\nu_2^2E_1}[(\nu_1E_2 + \nu_2^2E_1)u_{,x} + (E_2 - \nu_2^2E_1)v_{,y}] - \beta T, \\ \sigma_{xy} = \sigma_{yx} = \mu(u_{,y} + v_{,x}), \\ \sigma_{zz} = \frac{\nu_2E_1E_2}{(1-\nu_1)E_2 - 2\nu_2^2E_1}(u_{,x} + v_{,y}) - \beta T, \end{gathered} \tag{12}$$

where $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{zz}$ are components of the stresses tensor. Other components of a tensor of stresses in case of plane deformation equal to zero.

Next, we construct the general representation of the system of equations (10) by means of harmonic functions (Kolosov-Muskhelishvilis formula).

The first equation of the system (10) is differentiated by x, the second - by y and are added up. Given the fact, that we've got the T harmonic function

$$\Delta[(c+\mu)(u_{,x} + v_{,y})] = 0, \tag{13}$$

where denotation is entered

$$c := \frac{1}{2}\frac{E_1E_2}{(1-\nu_1)E_2 - 2\nu_2^2E_1}.$$

If the second equation of the system (10) is differentiated by x , and the first equation is differentiated by y and to consider their difference, we'll obtain

$$\Delta[\mu(v_{,x} - u_{,y})] = 0. \tag{14}$$

The notation is introduced

$$\theta := (c+\mu)(u_{,x}+v_{,y}), \quad \omega := \mu(v_{,x}-u_{,y}). \tag{15}$$

Thus, according to (13) and (14), θ and ω are harmonic functions

$$\Delta\theta = 0, \ \ \Delta\omega = 0. \tag{16}$$

According to notation (15)

$$\begin{cases} u_{,x}+v_{,y} = \dfrac{\theta}{c+\mu} \\ v_{,x}-u_{,y} = \dfrac{\omega}{\mu} \end{cases} \tag{17}$$

From (17) there follows

$$\Delta u = \frac{\theta_{,x}}{c+\mu} - \frac{\omega_{,y}}{\mu}, \quad \Delta v = \frac{\theta_{,y}}{c+\mu} + \frac{\omega_{,x}}{\mu}. \tag{18}$$

Formulas (18) are substituted in the system (10) and the notation introduced in this section are accounted

$$\begin{cases} (\theta-\beta T)_{,x} - \omega_{,y} = 0, \\ (\theta-\beta T)_{,y} + \omega_{,x} = 0. \end{cases} \tag{19}$$

As θ and ω are harmonic functions, from (19) we have

$$\theta = a\varphi + \beta T = 0.5[(a\varphi^* + \beta T^*)_{,x} + (a\tilde{\varphi} + \beta\tilde{T})_{,y}], \tag{20}$$

$$\omega = 0.5a(-\varphi^*_{,y} + \tilde{\varphi}_{,x}), \tag{21}$$

where a is any real constant other than zero; $\varphi^*, \tilde{\varphi}$ and $T^*,\ \tilde{T}$ are the mutually conjugate harmonic functions

$$\varphi^*_{,x} = \tilde{\varphi}_{,y} = \varphi, \ \ \varphi^*_{,y} = -\tilde{\varphi}_{,x},$$

$$T^*_{,x} = \tilde{T}_{,y} = T, \ \ T^*_{,y} = -\tilde{T}_{,x},$$

Relations (20) and (21) are substituted in system (17)

$$\begin{cases} \left(u - \dfrac{a}{2(c+\mu)}\varphi^* - \dfrac{\beta}{\beta(c+\mu)}T^*\right)_{,x} - \left(v - \dfrac{a}{2(c+\mu)}\tilde{\varphi} - \dfrac{\beta}{2(c+\mu)}\tilde{T}\right)_{,y} = 0, \\ v_{,x} - u_{,y} = \dfrac{a}{2\mu}(-\varphi^*_{,y} + \tilde{\phi}_{,x}) = 0. \end{cases} \tag{22}$$

The first equation of system (22) is identically satisfied, if

$$u = \Phi_{,y} + \frac{a}{2(c+\mu)}\varphi^* + \frac{\beta}{2(c+\mu)}T^*, \ \ v = -\Phi_{,x} + \frac{a}{2(c+\mu)}\tilde{\varphi} + \frac{\beta}{2(c+\mu)}\tilde{T}. \tag{23}$$

The equalities (23) are substituted in the second equation (22) as a result of which we've got the equation relating to the function Φ

$$\Delta\Phi = \frac{ca}{2\mu(c+\mu)}(\varphi^*_{,y} - \tilde{\varphi}_{,x}) + \frac{\beta}{2(c+\mu)}(-T^*_{,y} + \tilde{T}_{,x}). \tag{24}$$

The general solution of equation (24) is presented in the form

$$\Phi = \frac{ca}{4\mu(c+\mu)}(y\varphi^* - x\tilde{\varphi}) + b\psi + \frac{\beta}{4(c+\mu)}(-yT^* + x\tilde{T}). \tag{25}$$

where ψ is an arbitrary harmonic function, b is any real constant other than zero.

Constants a and b may be represented as follows

$$a = \frac{c+\mu}{c}, \qquad b = \frac{1}{2\mu},$$

and the formula (25) is substituted in the ratio (23)

$$2\mu u = \frac{c+2\mu}{2c}\varphi^* + 0.5(y\varphi^*_{,y} - x\tilde{\varphi}_{,y}) + \psi_{,y} + \frac{\mu\beta}{2(c+\mu)}(T^* - yT^*_{,y} + x\tilde{T}_{,y}), \tag{26}$$

$$2\mu v = \frac{c+2\mu}{2c}\tilde{\varphi} + 0.5(x\tilde{\varphi}_{,x} - y\varphi^*_{,x}) - \psi_{,x} + \frac{\mu\beta}{2(c+\mu)}(\tilde{T} - x\tilde{T}_{,x} + yT^*_{,x}). \tag{27}$$

By substituting (26) and (27) in the formulas (12) we've obtained the following expressions for stress tensor components

$$\begin{aligned}
\sigma_{xx} &= \varphi + 0.5(y\varphi^*_{,xy} - x\tilde{\varphi}_{,xy}) + \psi_{,xy} - \frac{\beta\mu}{2(c+\mu)}(2T + yT^*_{,xy} - x\tilde{T}_{,xy}),\\
\sigma_{yy} &= \varphi - 0.5(y\varphi^*_{,xy} - x\tilde{\varphi}_{,xy}) - \psi_{,xy} - \frac{\beta\mu}{2(c+\mu)}(2T - yT^*_{,xy} + x\tilde{T}_{,xy}),\\
\sigma_{xy} &= 0.5(y\varphi^*_{,yy} + x\tilde{\varphi}_{,xx}) + \psi_{,yy} - \frac{\beta\mu}{2(c+\mu)}(yT^*_{,yy} + x\tilde{T}_{,xx}),\\
\sigma_{zz} &= 2\nu_2\varphi - \frac{(1-2\nu_2)c+\mu}{c+\mu}\beta T.
\end{aligned} \tag{28}$$

For simplification of representations (26) - (28) the following notation is introduced

$$\phi = \varphi - \frac{\mu\beta}{c+\mu}T, \quad \phi^* = \varphi^* - \frac{\mu\beta}{c+\mu}T^*, \quad \tilde{\phi} = \tilde{\varphi} - \frac{\mu\beta}{c+\mu}\tilde{T}. \tag{29}$$

ϕ is a harmonic function, and ϕ^* and $\tilde{\phi}$ are the mutually conjugate harmonic functions

$$\phi^*_{,x} = \tilde{\phi}_{,y} = \phi, \qquad \phi^*_{,y} = -\tilde{\phi}_{,x}.$$

From (29) functions $\varphi, \varphi^*, \tilde{\varphi}$ are defined and are substituted in the formulas (26) - (28). As a result we obtain displacement representations

$$2\mu u = \frac{c+2\mu}{2c}\phi^* + 0.5(y\phi^*_{,y} - x\tilde{\phi}_{,y}) + \psi_{,y} + \frac{\beta\mu}{c}T^*, \tag{30}$$

$$2\mu v = \frac{c+2\mu}{2c}\tilde{\phi} + 0.5(x\tilde{\phi}_{,x} - y\phi^*_{,x}) - \psi_{,x} + \frac{\beta\mu}{c}\tilde{T}. \quad (31)$$

The following representations are fair for stresses

$$\begin{aligned} \sigma_{xx} &= \phi + 0.5(y\phi_{,y} - x\phi_{,x}) + \psi_{,xy}, \\ \sigma_{yy} &= \phi - 0.5(y\phi_{,y} - x\varphi_{,x}) - \psi_{,xy}, \\ \sigma_{xy} &= -0.5(y\phi_{,x} + x\phi_{,x}) + \psi_{,yy}, \\ \sigma_{zz} &= 2\nu_2\phi - (1-2\nu_2)\beta T. \end{aligned} \quad (32)$$

The analogs of formulas of Kolosov-Muskhelishvili [24] of (30)-(32) plane theories of thermoelasticity for transversely isotropic bodies may be used both for construction of exact solutions of boundary value problems and for construction of approximate solutions of a wide class of problems.

In case of finite simply connected domain the harmonic functions $\phi^*, \tilde{\phi}, \phi$ are represented by the following finite series

$$\begin{aligned} \phi^* &= a_0 + \sum_{n=1}^{N} r^n(x,y)[a_n\cos(n\theta(x,y)) + b_n\sin(n\theta(x,y))], \\ \tilde{\phi} &= b_0 + \sum_{n=1}^{N} r^n(x,y)[a_n\sin(n\theta(x,y)) - b_n\cos(n\theta(x,y))], \\ \phi &= \sum_{n=1}^{N} nr^{n-1}(x,y)[a_n\cos((n-1)\theta(x,y)) + b_n\sin((n-1)\theta(x,y))]. \end{aligned} \quad (33)$$

As the formulas (30), (31) show the constants a_0 , b_0 correspond to rigid displacement of a body, therefore they are equal to zero $a_0 = b_0 = 0$. The harmonic function ψ is represented as

$$\psi = \sum_{n=1}^{N} r^n(x,y)[c_n\cos(n\theta(x,y)) + d_n\sin(n\theta(x,y))]. \quad (34)$$

Analogously, the harmonic functions $T^*, \tilde{T}, T$ are also represented as

$$\begin{aligned} T^* &= t_0 + \sum_{n=1}^{N_T} r^n(x,y)[t_n\cos(n\theta(x,y)) + \tau_n\sin(n\theta(x,y))], \\ \tilde{T} &= \tau_0 + \sum_{n=1}^{N_T} r^n(x,y)[t_n\sin(n\theta(x,y)) - \tau_n\cos(n\theta(x,y))], \\ T &= \sum_{n=1}^{N_T} nr^{n-1}(x,y)[t_n\cos((n-1)\theta(x,y)) + \tau_n\sin((n-1)\theta(x,y))]. \end{aligned} \quad (35)$$

For construction of the approximate solution of problems, the representations (33)-(35) are substituted in the formulas (30)-(32), if necessary formulas of transformation of components of a vector and a tensor of the second rank are used and the conditions set are satisfied point-wise. The problem is tabulated to the solution of square system of the linear algebraic equations for required expansion coefficients (33)-(35).

The example of a nonlocal problem of Bitsadze-Samarskii in case of the plane theory of elasticity for rectangular domain is given below.

Example 3. We consider the domain $V = \{-2.5 < x < 2.5, -2 < y < 2\}$ (Fig. 5). In the domain V it is required to find such solution of system (10) (where $c = 3, \mu = 1, T = 0$ is accepted), which satisfies the following conditions (see [1])

$$u = -5.5y^2 + 14.375, \quad x = -2.5, -2 \leq y \leq 2,$$

$$v = 7.0y, \quad x = -2.5, -2 \leq y \leq 2,$$

$$\sigma_{yy}|_{y=2} = -2.0x - 1, \quad -2.5 < x < 2.5,$$

$$\sigma_{yx}|_{y=2} - \sigma_{yx}|_{y=1} = -14.0, \quad -2.5 < x < 2.5,$$

$$u = -5.5y^2 + 16.875, \quad x = 2.5, -2 \leq y \leq 2,$$

$$v = -8.0y, \quad x = 2.5, -2 \leq y \leq 2,$$

$$u = 2.5x^2 + 0.5x - 22.0, \quad y = -2, -2.5 < x < 2.5,$$

$$v = 6.0x + 1.0, \quad y = -2, -2.5 < x < 2.5.$$

The exact solution of this problem is as follows

$$u = 2.5x^2 - 5.5y^2 + 0.5x,$$

$$v = -0.3xy - 0.5y.$$

The boundary counter of the considered domain is divided by points into 72 equal segments. 19 points are also distributed evenly on a segment inside the domain where nonlocal conditions are set. The approximate solutions are sought as follows

$$\bar{u} = 0.5 \sum_{n=1}^{36} r^{n-1} \Big\{ \Big[\frac{5}{6} r \cos(n\theta) - \frac{n}{2} y \sin((n-1)\theta) - \frac{n}{2} x \cos((n-1)\theta) \Big] a_n$$

$$+ \Big[\frac{5}{6} r \sin(n\theta) + \frac{n}{2} y \cos((n-1)\theta) - \frac{n}{2} x \sin((n-1)\theta) \Big] b_n$$

$$-n \sin((n-1)\theta) c_n + n \cos((n-1)\theta) d_n \Big\},$$

$$\bar{v} = 0.5 \sum_{n=1}^{36} r^{n-1} \Big\{ \Big[\frac{5}{6} r \sin(n\theta) + \frac{n}{2} y \sin((n-1)\theta) - \frac{n}{2} x \cos((n-1)\theta) \Big] a_n$$

$$- \Big[\frac{5}{6} r \cos(n\theta) + \frac{n}{2} y \cos((n-1)\theta) + \frac{n}{2} x \sin((n-1)\theta) \Big] b_n$$

$$-n \cos((n-1)\theta) c_n - n \sin((n-1)\theta) d_n \Big\}.$$

The components of the stress tensor σ_{yy} and σ_{yx} appearing in the reference condition are presented in the form of the following finite rows

$$\sigma_{yy} = \sum_{n=1}^{36} n r^{n-2} \Big\{ \Big[r\cos((n-1)\theta) + \frac{n-1}{2} y \sin((n-2)\theta) + \frac{n-1}{2} x \cos((n-2)\theta) \Big] a_n$$

$$+ \Big[r\sin((n-1)\theta) - \frac{n-1}{2} y \cos((n-2)\theta) + \frac{n-1}{2} x \sin((n-2)\theta) \Big] b_n$$

$$+(n-1)\sin((n-2)\theta) c_n - (n-1)\cos((n-2)\theta) d_n \Big\},$$

$$\sigma_{yx} = \sum_{n=1}^{36} \frac{n(n-1)}{2} r^{n-2} \Big\{ \Big[-y\cos((n-2)\theta) + x\sin((n-2)\theta) \Big] a_n$$

$$-\Big[y\sin((n-2)\theta) + x\cos((n-2)\theta) \Big] b_n$$

$$-2\cos((n-2)\theta) c_n - 2\sin((n-2)\theta) d_n \Big\}.$$

In the last four formulas the coordinates of points marked on the boundary and inside the domain are substituted and the corresponding boundary and nonlocal conditions are satisfied on them. As a result we obtained the system consisting of the 144-linear algebraic equations and containing 144 unknowns ($a_1, ..., a_{36}, b_1, ..., b_{36}, c_1, ..., c_{36}, d_1, ..., d_{36}$). After solving this system by means of the formulas given above one can easily find components of a vector of displacement and a tensor of stresses.

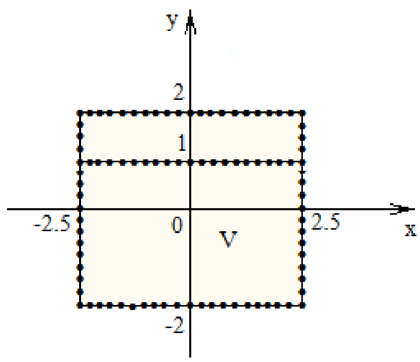

Fig. 5. The domain V in which the nonlocal problem of the plane theory of elasticity is solved

The appropriate program is made in the Maple 12. Numerical results are presented in Table 3, where $\bar{u}$ and $\bar{v}$ denote the approximate values of components of the displacement vector.

Tab. 3. Numerical results for a problem 3

(x, y)	$\bar{u}(x, y)$	$u(x, y)$	$\lvert\bar{u}(x, y) - u(x, y)\rvert$
(0, 0)	$4.166195286 \cdot 10^{-8}$	0	$4.166195286 \cdot 10^{-9}$
$(-1.0, 1.0)$	-3.499999843	-3.500	$1.57 \cdot 10^{-7}$
$(1.5, -1.5)$	-5.999999978	-6.000	$2.2 \cdot 10^{-8}$
$(1.2, -0.8)$	0.6800000324	0.680	$3.24 \cdot 10^{-8}$
$(-1.7, 1.5)$	-5.999999651	-6.000	$3.49 \cdot 10^{-7}$
$(2.2, -1.4)$	2.420000034	2.420	$3.4 \cdot 10^{-8}$
(1.25, 1.75)	-12.31250013	-12.31250	$1.3 \cdot 10^{-7}$

(x, y)	$\bar{v}(x, y)$	$v(x, y)$	$\lvert\bar{v}(x, y) - v(x, y)\rvert$
(0, 0)	$-1.190721638 \cdot 10^{-7}$	0	$1.190721638 \cdot 10^{-7}$
$(-1.0, 1.0)$	2.499999817	2.500	$1.83 \cdot 10^{-7}$
$(1.5, -1.5)$	7.499999968	7.500	$3.2 \cdot 10^{-8}$
$(1.2, -0.8)$	3.279999938	3.280	$6.2 \cdot 10^{-8}$
$(-1.7, 1.5)$	6.899999694	6.900	$3.06 \cdot 10^{-7}$
$(2.2, -1.4)$	9.939999959	9.940	$4.1 \cdot 10^{-8}$
$(1.25, 1.75)$	-7.437500134	-7.43750	$1.34 \cdot 10^{-7}$

As numerical results show the considered method gives the good approximate solution for nonlocal mixed boundary value problem of the plane theory of elasticity.

4. Problems for the Helmholtzs two dimensional equation

Let on the plane Oxy there be a domain Ω (shown in Fig. 3). In this domain the following equation of Helmholtz is considered

$$\Delta\omega - \zeta^2\omega = 0 \quad in \quad \Omega, \tag{36}$$

where ζ is any real constant other than zero.

The approximate solution is sought as follows

$$\begin{aligned}\bar{\omega} &= a_0 I_0(\zeta r(x, y)) + b_0 K_0(\zeta r(x, y)) \\ &+ \sum_{n=1}^{N} \{I_n(\zeta r(x, y))[a_n \cos(n\theta(x, y)) + b_n \sin(n\theta(x, y))] \\ &+ K_n(\zeta r(x, y))[c_n \cos(n\theta(x, y)) + d_n \sin(n\theta(x, y))]\},\end{aligned} \tag{37}$$

where $I_n(\zeta r)$ and $K_n(\zeta r)$ are modified Bessel functions of n order according to [26].

Partial derivatives of functions $\bar{\omega}(x, y)$ are expressed by means of the formulas

$$\begin{aligned}\bar{\omega}_{,x} &= \frac{\zeta x}{r}(a_0 I_1(\zeta r) - b_0 K_1(\zeta r) \\ &+ \frac{\zeta x}{2r} \sum_{n=1}^{N} \{(I_{n-1}(\zeta r) + I_{n+1}(\zeta r))[a_n \cos((n-1)\theta(x, y)) + b_n \sin((n-1)\theta(x, y))] \\ &- (K_{n-1}(\zeta r) + K_{n+1}(\zeta r))[c_n \cos((n-1)\theta(x, y)) + d_n \sin((n-1)\theta(x, y))]\},\end{aligned} \tag{38}$$

$$\begin{aligned}\bar{\omega}_{,y} &= \frac{\zeta y}{r}(a_0 I_1(\zeta r) - b_0 K_1(\zeta r))\\ &+\frac{\zeta y}{2r}\sum_{n=1}^{N}\{(I_{n-1}(\zeta r)+I_{n+1}(\zeta r))[-a_n\sin((n-1)\theta(x,y))\\ &+b_n\cos((n-1)\theta(x,y))]+(K_{n-1}(\zeta r)+K_{n+1}(\zeta r))[c_n\sin((n-1)\theta(x,y))\\ &-d_n\cos((n-1)\theta(x,y))]\}.\end{aligned} \tag{39}$$

By means of the formulas (37)-(39) one can construct the approximate solutions of various boundary value problems or boundary value contact problems for Helmholtz's equation (36).

An example of nonlocal problem of Bitsadze-Samarskii for the Helmholtz's equation is given below.

Example 4. The Helmholtz equation in a rectangle $V = \{-3 < x < 3, -2 < y < 2\}$(Fig. 6) is given as an example to find such a function ω satisfying the following conditions

$$\Delta\omega - \frac{\pi^2}{12}\omega = 0 \quad in \quad V, \tag{40}$$

$$\omega(-3,y) - \sqrt{2}\omega(-1.5,y) + \omega(0,y) = 0, \;\; -2 < y < 2,$$

$$\omega(x,\pm 2) = e^{\pm\frac{2\pi}{3}}\sin\frac{\pi x}{6}, \;\; -3 \le x \le 3,$$

$$\omega(3,y) = e^{\frac{\pi y}{3}}, \;\; -2 < y < 2.$$

It is easy to verify that the exact solution of the problem set is as follows

$$\omega(x,y) = e^{\frac{\pi y}{3}}\sin\frac{\pi x}{6}.$$

The approximate solution of the considered nonlocal problem is sought in the form of the sum

$$\begin{aligned}\bar{\omega} = a_0 I_0\Big(\frac{\sqrt{3}\pi}{6}r(x,y)\Big) + \sum_{n=1}^{39}\Big\{I_n\Big(\frac{\sqrt{3}\pi}{6}r(x,y)\Big)[a_n\cos(n\theta(x,y))\\ +b_n\sin(n\theta(x,y))]\Big\}.\end{aligned} \tag{41}$$

Beginning from a point (-3, 0) on the boundary of the considered rectangle with a step 0.25, 79 points are evenly distributed. 15 points are evenly distributed on each piece inside the domain where nonlocal conditions are set. After satisfying the given boundary conditions and nonlocal conditions we've obtained the system of the linear algebraic 79 equations with 79 unknowns. The solution of this system $(a_0, a_1, ..., a_{39}, b_1, ..., b_{39})$ is substituted in formula (41) representing the approximate solution of the stated problem. The constructed approximate solution satisfies the Helmholtz equation in the domain V and satisfies the boundary conditions and nonlocal conditions in the respective points marked in advance.

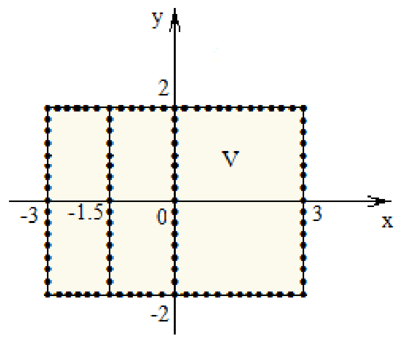

Fig. 6. Domain V , in which the nonlocal problem for Helmholtz's equation is considered

The appropriate program is made in the Maple 12. Numerical results are presented in the table 4.

Tab. 4. Numerical results for the problem 4.

(x, y)	$\bar{\omega}(x, y)$	$\omega(x, y)$	$\lvert\bar{\omega}(x, y) - \omega(x, y)\rvert$
$(-3.0, -1.5)$	-0.2078795840	-0.2078795765	$7.5 \cdot 10^{-9}$
$(-1.75, 1.75)$	-4.958529035	-4.958529038	$3.0 \cdot 10^{-9}$
$(0, -1.5)$	$-3.564486401 \cdot 10^{-10}$	0	$3.564486401 \cdot 10^{-10}$
$(0.5, -2.0)$	0.03187219544	0.03187219654	$1.1 \cdot 10^{-9}$
$(1.0,\ 1.5)$	2.405238691	2.405238689	$2.0 \cdot 10^{-9}$
$(1.5,\ 1.25)$	2.618033198	2.618032200	$2.0 \cdot 10^{-9}$
$(3.0,\ 1.5)$	4.810477384	4.810477377	$7.0 \cdot 10^{-9}$

As the table shows the constructed approximate solution of the nonlocal problem is a good approximation to the exact solution of this problem.

5. Conclusion. In the work we propose the simple method of the approximate solution of boundary value problems of mathematical physics. The approximate solutions of such two-dimensional classical and nonlocal boundary value problems for Laplace's and Helmholtz's equations and the theory of elasticity, the exact solutions of which are known in advance, are constructed by the proposed method.

We believe that by means of the considered algorithm it is possible to receive quite good approximate solutions of some boundary value problems of mathematical physics.

Acknowledgement. The designated project has been fulfilled by a financial support of Shota Rustaveli National Science Foundation (Grant SRNSF/FR/358/5-109/14). Any idea in this publication is possessed by the author and may not represent the opinion of Shota Rustaveli National Science Foundation itself.

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Received 04.05.2016; revised 14.11.2016; accepted 21.11.2016.

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