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INVERSE PROBLEM ABOUT TRANSITION IN A FIXED POINT FOR LINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In the paper the following inverse problem is considered: find such initial functions that the value of corresponding solution at given moment is equal to a fixed vector. On the basis of necessary conditions an algorithm is provided for the approximate solution of the inverse problem.

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Let $\mathbb{R}^n$ be an n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$ with

$$| x |^2 = \sum_{i=1}^{n} (x^i)^2.$$

Let $K_1 \subset \mathbb{R}^n, K_2 \subset \mathbb{R}^n$ be convex compact sets, let $\tau(t), t \in \mathbb{R}$ and $\eta(t), t \in \mathbb{R}$ be continuously differentiable scalar functions (delay functions) satisfying the conditions

$$\tau(t) < t,\ \eta(t) < t,\ \dot{\tau}(t) > 0,\ \dot{\eta}(t) > 0.$$

Let $t_0 < t_1$ be given numbers with $\tau(t_1) > t_0$ and $\eta(t_1) > t_0$. By Δ_1 and Δ_2 we denote, respectively, the sets of measurable initial functions $\varphi : [\hat{\tau}, t_0] \to K_1$ and $g : [\hat{\tau}, t_0] \to K_2$, where $\hat{\tau} = t_0 - \max\{\tau(t_0), \eta(t_0)\}$.

To each element (initial data) $w = (\varphi(t), g(t)) \in W = \Delta_1 \times \Delta_2$ we assign the linear neutral functional differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)x(\tau(t)) + C(t)\dot{x}(\eta(t)) \tag{1}$$

with the initial condition

$$\begin{cases} x(t) = \varphi(t), t \in [\hat{\tau}, t_0], \quad (\varphi(t_0) = \varphi(t_0-)), \\ \dot{x}(t) = g(t), t \in [\hat{\tau}, t_0), \end{cases} \tag{2}$$

where $A(t), B(t), C(t), t \in [t_0, t_1]$, are given continuous matrix functions with appropriate dimensions.

Definition. Let $w = (\varphi(t), g(t)) \in W$, a function $x(t) = x(t; w) \in \mathbb{R}^n, t \in [\hat{\tau}, t_1]$ is called a solution of differential equation (1) with the initial condition (2) or a solution corresponding to the element w if $x(t)$ satisfies the initial condition (2) is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere.

For every element $w \in W$ there exists a unique solution $x(t; w)$ defined on the interval $[\hat{\tau}, t_1]$.

Introduce the set

$$Y = \Big\{ y \in \mathbb{R}^n : \exists w \in W,\ x(t_1; w) = y \Big\}.$$

The inverse problem. Let $y \in Y$ be a given vector. Find element $w \in W$ such that the following condition holds

$$x(t_1; w) = y.$$

The vector y, as a rule, by distinct error is beforehand given. Thus instead of the vector y we have $\hat{y}$ (so called observed vector) which is an approximation to the y and in general, $\hat{y} \notin Y$. Therefore it is natural to change the posed inverse problem by the following approximate problem.

The approximate inverse problem. Find an element $w \in W$ such that the deviation

$$\frac{1}{2}|x(t_1; w) - \hat{y}|^2$$

takes the minimal value.

It is clear that the approximate inverse problem is equivalent to the following optimization problem:

$$\dot{x}(t) = A(t)x(t) + B(t)x(\tau(t)) + C(t)\dot{x}(\eta(t)) \tag{3}$$

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0], \dot{x}(t) = g(t), t \in [\hat{\tau}, t_0), \tag{4}$$

$$J(w) = \frac{1}{2}|x(t_1; w) - \hat{y}|^2 \to \min, w \in W. \tag{5}$$

Problem (3)-(5) is called an optimal control problem corresponding to the inverse problem.

Theorem 1.([1]) *There exists an optimal element $w_0 = (\varphi_0(t), g_0(t))$ for problem (3)-(5).*

Theorem 2.([1]) *Let $w_0 = (\varphi_0(t), g_0(t)) \in W$ be an optimal element. Then the following conditions hold:*
1) the condition for the initial function $\varphi_0(t)$

$$\psi(\gamma(t))B(\gamma(t))\dot{\gamma}(t)\varphi_0(t) = \max_{\varphi \in K_1} \psi(\gamma(t))B(\gamma(t))\dot{\gamma}(t)\varphi,$$

$$t \in [\tau(t_0), t_0],$$

where $\gamma(t)$ is the inverse function of $\tau(t)$;
2) the condition for the initial function $g_0(t)$

$$\psi(\rho(t))C(\rho(t))\dot{\rho}(t)g_0(t) = \max_{g \in K_2} \psi(\rho(t))C(\rho(t))\dot{\rho}(t)g,$$

$$t \in [\eta(t_0), t_0].$$

where $\rho(t)$ is the inverse function of $\eta(t)$.
Here $(\psi(t), \chi(t))$ is solution of the system

$$\begin{cases} \dot{\chi}(t) = -\psi(t)A(t) - \psi(\gamma(t))B(\gamma(t))\dot{\gamma}(t), \\ \psi(t) = \chi(t) + \psi(\rho(t))C(\rho(t))\dot{\rho}(t) \end{cases} \tag{6}$$

with the initial condition

$$\psi(t_1) = \chi(t_1) = -(x_0(t_1) - \hat{y})^T, \psi(t) = \chi(t) = 0, t > t_1.$$

Let the optimal element $w_0 = (\varphi_0(t), g_0(t))$ be unique and conditions *1)* and *2)* give the unique initial functions $\varphi(t)$ and $g(t)$, respectively.

The algorithm. Let $\varphi_1(t) \in \Delta_1$ and $g_1(t) \in \Delta_2$ be starting approximation of the initial functions. We construct the sequences

$$\{\varphi_k(t)\}, \{g_k(t)\}, \{x_k(t)\}, \{\psi_k(t)\}, \{\chi_k(t)\}$$

by the following process:

3) for given $\varphi_1(t)$ and $g_1(t)$ find $x_1(t)$: the solution of the differential equation (3) with the initial condition

$$x(t) = \varphi_1(t), t \in [\tau(t_0), t_0], \dot{x}(t) = g_1(t), t \in [\eta(t_0), t_0);$$

4) find $\psi_1(t)$ and $\chi_1(t)$: the solution of the differential equation (6) with the initial condition

$$\psi(t_1) = \chi(t_1) = -(x_1(t_1) - \hat{y}), \psi(t) = \chi(t) = 0, t > t_1;$$

5) find the next iterations $\varphi_2(t)$ and $g_2(t)$ from *1)* and *2)*, respectively.

6) if

$$| J(w_1) - J(w_2) | \le \varepsilon$$

stop, where $w_1 = (\varphi_1(t), g_1(t)), w_2 = (\varphi_2(t), g_2(t))$ and ε is a given number.

If

$$| J(w_1) - J(w_2) | > \varepsilon$$

go to 3).

Theorem 3. *The following relations are valid:*

$$\lim_{k\to\infty} \varphi_k(t) = \varphi_0(t) \text{ weakly in the space } L[\tau(t_0), t_0];$$

$$\lim_{k\to\infty} g_k(t) = g_0(t) \text{ weakly in the space } L[\sigma(t_0), t_0];$$

$$\lim_{k\to\infty} x_k(t) = x_0(t) \text{ uniformly for } t \in [t_0, t_1];$$

$$\lim_{k\to\infty} \sup_{[t_0,t_1]} | \psi_k(t) - \psi(t) | = 0;$$

$$\lim_{k\to\infty} \chi_k(t) = \chi(t) \text{ uniformly for } t \in [t_0, t_1].$$

Moreover, $w_0 = (\varphi_0(t), g_0(t))$ is an optimal element, $x_0(t) = x(t; w_0)$ is an optimal trajectory, $(\psi(t), \chi(t))$ is the solution of equation (6) corresponding to w_0.

Theorem 3 is proved by the scheme given in [2].

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R E F E R E N C E S

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