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ON ESTIMATION OF THE INCREMENT OF SOLUTION FOR A CONTROLLED FUNCTIONAL DIFFERENTIAL EQUATION CONSIDERING DELAY PARAMETER PERTURBATION

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Abstract. The estimation of the increment of solution is obtained with respect to small parameter for nonlinear delay functional differential equation with the continuous initial condition. Moreover, value of the increment is calculated at the initial moment. This estimation plays an important role in proving the variation formulas of solution.

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Let R^n_x be the n-dimensional vector space of points $x = (x^1, ..., x^n)^T$, where T is the sign of transposition; suppose that $O \subset R^n_x$ and $V \subset R^r_u$ are open sets. Let the n-dimensional function $f(t, x, y, u)$ satisfy the following conditions: for almost all $t \in I = [a, b]$, the function $f(t, \cdot) : O^2 \times V \to R^n_x$ is continuously differentiable; for any $(x, y, u) \in O^2 \times V$, the functions

$$f(t, x, y, u), f_x(t, x, y, u), f_y(t, x, y, u), f_u(t, x, y, u)$$

are measurable on I; for arbitrary compacts $K \subset O, U \subset V$ there exists a function $m_{K,U}(t) \in L(I, [0, \infty))$, such that for any $(x, y, u) \in K^2 \times U$ and for almost all $t \in I$ the following inequality is fulfilled

$$| f(t, x, y, u) | + | f_x(t, x, y, u) | + | f_y(t, x, y, u) | + | f_u(t, x, y, u) | \leq m_{K,U}(t).$$

Furthermore, let $0 < \tau_1 < \tau_2$ be given numbers and let E_φ be the space of continuous functions $\varphi : I_1 \to R^n_x$, where $I_1 = [\hat{\tau}, b], \hat{\tau} = a - \tau_2; \Phi = \{\varphi \in E_\varphi : \varphi(t) \in O, t \in I_1\}$ is a set of initial functions; let E_u be the space of bounded measurable functions $u : I \to R^r_u$ and let $\Omega = \{u \in E_u : clu(I) \subset V\}$ be a set of control functions, where $u(I) = \{u(t) : t \in I\}$ and $clu(I)$ is closer of the set $u(I)$.

To each element $\mu = (t_0, \tau, \varphi, u) \in \Lambda = (a, b) \times (\tau_1, \tau_2) \times \Phi \times \Omega$ we assign the controlled delay functional differential equation

$$\dot{x}(t) = f(t, x(t), x(t - \tau), u(t)) \tag{1}$$

with the initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0]. \tag{2}$$

Condition (2) is said to be a continuous initial condition since always $x(t_0) = \varphi(t_0)$.

Definition 1. Let $\mu = (t_0, \tau, \varphi, u) \in \Lambda$. A function $x(t) = x(t; \mu) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b)$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to μ and defined on the interval $[\hat{\tau}, t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let $\mu_0 = (t_{00}, \tau_0, \varphi_0, u_0) \in \Lambda$ be a fixed element. In the space $E_\mu = R^1_{t_0} \times R^1_\tau \times E_\varphi \times E_u$ we introduce the set of variations:

$$V = \{\delta\mu = (\delta t_0, \delta\tau, \delta\varphi, \delta u) \in E_\mu - \mu_0 : \mid \delta t_0 \mid \leq \alpha, \mid \delta\tau \mid \leq \alpha,$$

$$\delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \delta u = \sum_{i=1}^{k} \lambda_i \delta u_i, \mid \lambda_i \mid \leq \alpha, i = \overline{1, k}\},$$

where $\delta\varphi_i \in E_\varphi - \varphi_0, \delta u_i \in E_u - u_0, i = \overline{1, k}$ are fixed functions ; $\alpha > 0$ is a fixed number.

Theorem 1([1]). *Let $x_0(t)$ be the solution corresponding to $\mu_0 = (t_{00}, \tau_0, \varphi_0, u_0) \in \Lambda$ and defined on $[\hat{\tau}, t_{10}], t_{10} \in (t_{00}, b)$ and let $K_0 \subset O$ and $U_0 \subset V$ be compact sets containing neighborhoods of sets $\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$ and $cl u_0(I)$, respectively. Then there exist numbers $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that, for any $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$, we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$. In addition, a solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ corresponds to this element. Moreover,*

$$\begin{cases} x(t; \mu_0 + \varepsilon\delta\mu) \in K_0, t \in [\hat{\tau}, t_{10} + \delta_1], \\ u_0(t) + \varepsilon\delta u(t) \in U_0, t \in I. \end{cases} \tag{3}$$

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\hat{\tau}, t_{10} + \delta_1]$.

Theorem 1 allows one to define the increment of the solution $x_0(t) = x(t; \mu_0)$:

$$\begin{cases} \Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \\ (t, \varepsilon, \delta\mu) \in [\hat{\tau}, t_{10} + \delta_1] \times [0, \varepsilon_1] \times V. \end{cases}$$

Theorem 2. *Let the following conditions hold:*

1. the function $\varphi_0(t), t \in I_1$ is absolutely continuous and the function $\dot{\varphi}_0(t)$ is bounded;

2. there exist compact sets $K_0 \subset O$ and $U_0 \subset V$ containing neighborhoods of sets $\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$ and $cl u_0(I)$, respectively, such that the function $f(t, x, y, u)$ is bounded on the set $I \times K_0^2 \times U_0$;

3. there exist the limits

$$\lim_{t \to t_{00}-} \dot{\varphi}_0(t) = \dot{\varphi}_0^-, \lim_{w \to w_0} f(w, u_0(t)) = f^-,$$

where $w = (t, x, y) \in (a, t_{00}] \times O^2, w_0 = (t_{00}, \varphi_0(t_{00}), \varphi_0(t_{00} - \tau_0))$. *Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1]$ and $\delta_2 \in (0, \delta_1]$ such that*

$$\max_{t \in [\hat{\tau}, t_{10} + \delta_2]} \mid \Delta x(t; \varepsilon\delta\mu) \mid \leq O(\varepsilon\delta\mu) \tag{4}$$

for arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V^-$, *where* $V^-=\{\delta\mu\in V:\delta t_0\le 0\}$. *Moreover,*

$$\Delta x(t_{00};\varepsilon\delta\mu)=\varepsilon\Big[\delta\varphi(t_{00})+\{\dot\varphi_0^- - f^-\}\delta t_0\Big]+o(\varepsilon\delta\mu).$$

Here the symbols $O(t;\varepsilon\delta\mu), o(t;\varepsilon\delta\mu)$ stand for quantities that have the corresponding order of smallness with respect to ε uniformly with respect to t and $\delta\mu$.

Theorem 3. *Let the conditions 1 and 2 of Theorem 2 hold and there exist the limits*

$$\lim_{t\to t_{00}+}\dot\varphi_0(t)=\dot\varphi_0^+,\ \lim_{w\to w_0}f(w,u_0(t))=f^+, w\in[t_{00},b)\times O^2.$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1]$ *and* $\delta_2\in(0,\delta_1]$ *such that inequality (4) is valid for arbitrary* $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V^+$, *where* $V^+=\{\delta\mu\in V:\delta t_0\ge 0\}$. *Moreover,*

$$\Delta x(t_{00}+\varepsilon\delta t_0;\varepsilon\delta\mu)=\varepsilon\Big[\delta\varphi(t_{00})+\{\dot\varphi_0^+ - f^+\}\delta t_0\Big]+o(\varepsilon\delta\mu).$$

Theorems 2 and 3 are proved by the scheme given in [2,3].

Theorem 4. *Let the conditions of Theorems 2 and 3 hold. Moreover,*

$$\dot\varphi_0^- - f^- = \dot\varphi_0^+ - f^+ := \hat f.$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1]$ *and* $\delta_2\in(0,\delta_1]$ *such that inequality (4) is valid for arbitrary* $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V$ *and*

$$\Delta x(t_{00}+\varepsilon\delta t_0;\varepsilon\delta\mu)=\varepsilon\Big[\delta\varphi(t_{00})+\hat f\delta t_0\Big]+\gamma(\varepsilon\delta\mu), \tag{5}$$

where

$$\gamma(\varepsilon\delta\mu)=\begin{cases} o(\varepsilon\delta\mu)+\hat O(\varepsilon\delta\mu) \ \ for \ \ \delta t_0\le 0,\\ o(\varepsilon\delta\mu) \ \ for \ \ \delta t_0\ge 0.\end{cases}$$

Here $\hat O(\varepsilon\delta\mu)=0$ *for* $\delta t_0=0$.

Proof. It is clear that inequality (4) holds for arbitrary $(\varepsilon,\delta\mu)\in[0,\varepsilon_2]\times V$ and formula (5) is valid for $\delta t_0\ge 0$.

Let $\delta t_0\le 0$ then

$$\Delta x(t_{00}+\varepsilon\delta t_0;\varepsilon\delta\mu)-\Delta x(t_{00};\varepsilon\delta\mu)=\int_{t_{00}}^{t_{00}+\varepsilon\delta t_0}\dot\Delta x(t;\varepsilon\delta\mu)dt$$

$$=\int_{t_{00}}^{t_{00}+\varepsilon\delta t_0}[f(t,x(t;\mu_0+\varepsilon\delta\mu),x(t-\tau;\mu_0+\varepsilon\delta\mu),u(t))-\dot\varphi_0(t)]dt=\hat O(\varepsilon\delta\mu),$$

(see (3) and the conditions *1* and *2*), i.e.

$$\Delta x(t_{00}+\varepsilon\delta t_0;\varepsilon\delta\mu)=\Delta x(t_{00};\varepsilon\delta\mu)+\hat O(\varepsilon\delta\mu)$$
$$=\varepsilon\Big[\delta\varphi(t_{00})+\hat f\delta t_0\Big]+o(\varepsilon\delta\mu)+\hat O(\varepsilon\delta\mu).$$

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