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ON HIGHER ORDER "ALMOST LINEAR" FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PROPERTY A AND B

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Abstract. An operator differential equation is considered. A particular case of this equations is the ordinary differential equation

$$u^{(n)}(t)+p(t)\big|u(t)\big|^{\mu(t)}\operatorname{sign}u(t)=0,$$

where $p\in L_{\rm loc}(R_+;R)$, $\mu\in C(R_+;(0,+\infty))$. This equation is "almost linear" if the condition $\liminf\limits_{t\to+\infty}\mu(t)=1$ holds, while if $\liminf\limits_{t\to+\infty}\mu(t)\neq 1$ or $\limsup\limits_{t\to+\infty}\mu(t)\neq 1$, then the equation is an essentially nonlinear differential equation. "Almost linear" differential equations are considered and sufficient condition are established for oscillation of solutions.

Keywords and phrases: Property A, property B, oscillation.

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Introduction

This work deals with study of oscillatory properties of solutions of a functional-differential equation

$$u^{(n)}(t)+F(u)(t)=0, \tag{1.1}$$

where $F:C(R_+;R)\to L_{\rm loc}(R_+;R)$ is a continuous mapping. Let $\tau\in C(R_+;R_+)$, $\lim\limits_{t\to+\infty}\tau(t)=+\infty$. Denote by $V(\tau)$ the set of continuous mappings F satisfying the condition: $F(x)(t)=F(y)(t)$ holds for any $t\in R_+$ and $x,y\in C(R_+;R)$ provided that $x(s)=y(s)$ for $s\geq\tau(t)$. For any $t_0\in R_+$, we denote by $H_{t_0,\tau}$ the set of all functions $u\in C(R_+;R)$ satisfying $u(t)\neq 0$ for $t\geq t_*$, where $t_*=\min\{t_0,\tau_*(t_0)\}$, $\tau_*(t)=\inf\{\tau(s):s\geq t\}$. Throughout the work whenever the notation $V(\tau)$ and $H_{t_0,\tau}$ occurs, it will be understood, unless specified otherwise that the function τ satisfies the conditions stated above.

It will always be assumed that either the condition

$$F(u)(t)\,u(t)\geq 0 \quad \text{for} \quad t\geq t_0, \quad u\in H_{t_0,\tau}, \tag{1.2}$$

or the condition

$$F(u)(t)\,u(t)\leq 0 \quad \text{for} \quad t\geq t_0, \quad u\in H_{t_0,\tau} \tag{1.3}$$

is fulfilled.

A function $u:[t_0,+\infty)\to R$ is said to be a proper solution of equation (1.1), if it is locally absolutely continuous along with its derivatives up to the order $n-1$ inclusive, $\sup\{|u(s)|:s\geq t\}>0$ for $t\geq t_0$ and there exists a function $\overline{u}\in C(R_+;R)$ such that $\overline{u}(t)\equiv u(t)$ on $[t_0,+\infty)$ and the equality

$$\overline{u}^{(n)}(t)+F(\overline{u})(t)=0$$

holds for $t \in [t_0, +\infty)$. A proper solution $u : [t_0, +\infty) \to R$ of equation (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution u is said to be nonoscillatory.

Definition 1.1 We say that equation (1.1) has Property **A** if any of its proper solutions is oscillatory when n is even either is oscillatory or satisfies

$$\big|u^{(i)}(t)\big| \downarrow 0 \quad \text{for} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.4}$$

when n is odd.

Definition 1.2 We say that equation (1.1) has Property **B** if any of its proper solutions either is oscillatory or satisfies either (1.4) or

$$\big|u^{(i)}(t)\big| \uparrow +\infty, \quad \text{for} \quad t \uparrow +\infty \quad (i = 0, \dots, n-1) \tag{1.5}$$

when n is even and either is oscillatory or satisfies (1.5) when n is odd.

The ordinary differential equation with deviating argument

$$u^{(n)}(t) + p(t)\big|u(\sigma(t))\big|^{\mu(t)} \operatorname{sign} u(\sigma(t)) = 0 \tag{1.6}$$

is a particular case of equation (1.1), where $p \in L_{\rm loc}(R_+; R)$, $\mu \in C(R_+; (0, +\infty))$. In the case $\lim\limits_{t\to+\infty} \mu(t) = 1$, we call differential equation (1.6) “almost linear”, while if $\liminf\limits_{t\to+\infty} \mu(t) \neq 1$ or $\limsup\limits_{t\to+\infty} \mu(t) \neq 1$, then we call equation (1.6) essentially nonlinear generalized Emden-Fowler type differential equation.

Everywhere below we assume that the inequality

$$\big|F(u)(t)\big| \geq \sum_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \big|u(s)\big|^{\mu(s)} d_s\, r_i(s,t) \quad \text{for} \quad t \geq t_0, \quad u \in H_{t_0,\tau}, \tag{1.7}$$

holds, where

$$\begin{gathered} \mu \in C(R_+; (0, +\infty)), \quad \tau_i, \sigma_i \in C(R_+; R_+), \quad \tau_i(t) \leq \sigma_i(t) \\ \text{for} \quad t \in R_+, \quad \lim_{t\to+\infty} \tau_i(t) = +\infty \quad (i = 1, \dots, m), \end{gathered} \tag{1.8}$$

$r_i : R_+ \times R_+ \to R_+$ are nondecreasing in the first argument and Lebesgue integrable in the second argument on any finite subsegment of $[0, +\infty)$.

Study of oscillatory properties of differential equation of type (1.1) begin in 1990. Namely, in [1,2] for the first time a new approach was used for establishing oscillatory properties. Investigation of “almost linear” (essentially nonlinear) differential equations, in our opinion for the first time, was carried out [3,4] ([5–7]).

In the present paper the both cases of Properties **A** and **B** will be studied for “almost linear” differential equations.

2. Necessary conditions of the existence of monotone solutions

Let $t_0 \in R_+$, $\ell \in \{1, \dots, n-1\}$. By U_{ℓ,t_0} we denote the set of proper solutions of equation (1.1) satisfying the conditions

$$\begin{gathered} u^{(i)}(t) > 0 \quad \text{for} \quad t \geq t_0 \quad (i = 0, \dots, \ell-1), \\ (-1)^{i+\ell} u^{(i)}(t) \geq 0 \quad \text{for} \quad t \geq t_0 \quad (i = \ell, \dots, n-1). \end{gathered} \tag{2.1$_\ell$}$$

Theorem 2.1 *Let* $F \in V(\tau)$*, conditions* (1.2) ((1.3))*,* (1.6)*,* (1.7) *be fulfilled,* $\ell \in \{1,\dots,n-1\}$*,* $\ell + n$ *be odd* ($\ell + n$ *be even*)*,*

$$\int_0^{+\infty} t^{n-\ell} \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} s^{(\ell-1)\mu(s)} d_s r_i(s,t) = +\infty, \tag{2.2$_\ell$}$$

$$\int_0^{+\infty} t^{n-\ell-1} \sum_{i=1}^m \int_{\tau_i(t)}^{\sigma_i(t)} s^{\ell\mu(s)} d_s r_i(s,t) = +\infty, \tag{2.3$_\ell$}$$

and

$$\liminf_{t\to+\infty} \mu(t) > 0. \tag{2.4}$$

Moreover, let $U_{\ell,t_0} \neq \varnothing$ *for some* $t_0 \in R_+$*. Then there exist* $\lambda \in [\ell-1,\ell]$ *such that*

$$\limsup_{\varepsilon\to 0+} \Big(\liminf_{t\to+\infty} g_\ell(t,\lambda,\varepsilon) \Big) \le (\ell-1)!\,(n-\ell-1)!,$$

where

$$g_\ell(t,\lambda,\varepsilon) = t^{\ell-\lambda+h_{2\varepsilon}(\lambda)} \int_t^{+\infty} s^{-n}(s-t)^{n-\ell-1}\big(\overline{\sigma}(s)\big)^{-h_\varepsilon(\lambda)}$$

$$\times \int_{t_0}^{s} (s-\xi)^{\ell-1}\xi^{n-\ell} \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{\lambda+h_{1\varepsilon}(\lambda)} d_{\xi_1} r_i(\xi_1,\xi)\, d\xi\, ds, \tag{2.4}$$

$$\overline{\sigma}(t) = \max\big\{ \max(s,\sigma_1(s),\dots,\sigma_m(s)) : 0 \le s \le t \big\},$$

$$h_{1\varepsilon}(\lambda) = \begin{cases} 0 & \text{for } \lambda = \ell, \\ \varepsilon & \text{for } \lambda \in [\ell-1,\ell), \end{cases}$$

$$h_{2\varepsilon}(\lambda) = \begin{cases} 0 & \text{for } \lambda = \ell-1, \\ \varepsilon & \text{for } \lambda \in (\ell-1,\ell], \end{cases} \qquad h_\varepsilon(\lambda) = h_{1\varepsilon}(\lambda) + h_{2\varepsilon}(\lambda). \tag{2.5}$$

Theorem 2.2 *Let the conditions of Theorem* 2.1 *be fulfilled and*

$$\liminf_{t\to+\infty} \frac{t}{\sigma_i(t)} > 0 \quad (i = 1,\dots,m). \tag{2.7}$$

Then there exist $\lambda \in [\ell-1,\ell]$ *such that*

$$\limsup_{\varepsilon\to 0+} \Big(\liminf_{t\to+\infty} g_{\ell,1}(t,\lambda,\varepsilon) \Big) \le (\ell-1)!\,(n-\ell-1)!,$$

where

$$g_{\ell,1}(t,\lambda,\varepsilon) = t^{\ell-\lambda+h_{2\varepsilon}(\lambda)} \int_t^{+\infty} s^{-n-h_\varepsilon(\lambda)}(s-t)^{n-\ell-1} \int_{t_0}^{s} (s-\xi)^{\ell-1}\xi^{n-\ell}$$

$$\times \sum_{i=1}^m \int_{\tau_i(\xi)}^{\sigma_i(\xi)} \xi_1^{(\lambda+h_{1\varepsilon}(\lambda))\mu(\xi_1)} d_{\xi_1} r_i(\xi_1,\xi)\, d\xi\, ds, \tag{2.8}$$

$h_{1\varepsilon}$, $h_{2\varepsilon}$ and h_ε *are given by* (2.6).

3. Sufficient conditions of nonexistence of monotone solutions

Theorem 3.1 *Let $F \in V(\tau)$, conditions* (1.2) ((1.3)), (1.6), (1.7), (2.2_ℓ)–(2.4) *be fulfilled, $\ell \in \{1,\dots,n-1\}$, with $\ell+n$ odd ($\ell+n$ even), and for any $\lambda \in [\ell-1,\ell]$*

$$\limsup_{\varepsilon\to0+}\Big(\liminf_{t\to+\infty} g_\ell(t,\lambda,\varepsilon)\Big) > (\ell-1)!\,(n-\ell-1)!. \qquad (3.1_\ell)$$

Then for any $t_0 \in R_+$, $U_{\ell,t_0}=\varnothing$, where g_ℓ, $h_{1\varepsilon}$, $h_{2\varepsilon}$ and h_ε are defined by (2.5) and (2.6).

Theorem 3.2 *Let $F \in V(\tau)$, conditions* (1.2) ((1.3)), (1.6), (1.7), (2.2_ℓ)–(2.4) *and* (2.7) *be fulfilled, $\ell \in \{1,\dots,n-1\}$, with $\ell+n$ odd ($\ell+n$ even) and for any $\lambda \in [\ell-1,\ell]$*

$$\limsup_{\varepsilon\to0+}\Big(\liminf_{t\to+\infty} g_{\ell1}(t,\lambda,\varepsilon)\Big) > (\ell-1)!\,(n-\ell-1)!. \qquad (3.2_\ell)$$

Then for any $t_0 \in R_+$, $U_{\ell,t_0}=\varnothing$, where $g_{\ell1}$, $h_{1\varepsilon}$, $h_{2\varepsilon}$ and h_ε are defined by (2.6) *and* (2.8).

4. Functional differential equation with property A

Relying on the results obtained in Section 3, in Sections 4 and 5 we establish sufficient conditions for equation (1.1) to have Properties **A** and **B**.

Theorem 4.1 *Let $F \in V(\tau)$, conditions* (1.2), (1.6), (1.7) *and* (2.4) *be fulfilled and for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd and $\lambda \in [\ell-1,\ell]$ conditions* (2.2_ℓ), (2.3_ℓ) *and* (3.1_ℓ) *hold. If moreover,* (2.3_0) *holds when n is odd, then equation* (1.1) *has Property* **A**.

Theorem 4.2 *Let $F \in V(\tau)$, conditions* (1.2), (1.6), (1.7), (2.4), (2.7) *be fulfilled and for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd and $\lambda \in [\ell-1,\ell]$ conditions* (2.2_ℓ), (2.3_ℓ) *and* (3.2_ℓ) *hold. If moreover,* (2.3_0) *holds when n is odd, then equation* (1.1) *has Property* **A**.

Theorem 4.3 *Suppose $F \in V(\tau)$, condition* (1.2) *be fulfilled and for large $t_0 \in R_+$*

$$\big|F(u)(t)\big| \ge \sum_{i=1}^m p_i(t)\int_{\alpha_i t}^{\beta_i t} |u(s)|^{1-\frac{d}{\ln s}}ds \ \ \text{for}\ \ t\ge t_0,\ \ u\in H_{t_0,\tau} \qquad (4.1)$$

and

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t s^{n+1}\Big(\prod_{i=1}^m p_i(s)\Big)^{\frac{1}{m}}ds > \frac{1}{m}\max\bigg(\prod_{i=1}^m\Big(\beta_i^{1+\lambda}-\alpha_i^{1+\lambda}\Big)^{-\frac{1}{m}}\times$$
$$\times e^{\lambda d}(1+\lambda)\lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda\in[0,n-1]\bigg).$$

Then equation (1.1) *has Property* **A**, *where*

$$p_i \in L_{\rm loc}(R_+;R_+),\ \ 0<\alpha_i<\beta_i<+\infty\ \ (i=1,\dots,m),\ \ d\in[0,+\infty). \qquad (4.2)$$

Theorem 4.4 *Suppose $F \in V(\tau)$, condition (1.2) be fulfilled and for large $t_0 \in R_+$*

$$\big|F(u)(t)\big| \ge \sum_{i=1}^{m} p_i(t)\big|u(\alpha_i t)\big|^{1-\frac{d}{\ln t}} \ \text{ for } \ t \ge t_0, \ \ u \in H_{t_0,\tau} \tag{4.3}$$

and

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t s^n\Big(\prod_{i=1}^{m} p_i(s)\Big)^{\frac{1}{m}} ds >$$
$$> \frac{1}{m}\max\bigg(\Big(\prod_{i=1}^{m}\alpha_i^{-\frac{\lambda}{m}}\Big)e^{\lambda d}\lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda \in [0, n-1]\bigg).$$

Then equation (1.1) has Property **A**, *where*

$$p_i \in L_{\mathrm{loc}}(R_+; R_+), \ \ \alpha_i \in (0, +\infty) \ \ (i = 1, \dots, m), \quad d \in [0, +\infty). \tag{4.4}$$

5. Functional differential equation with property B

Theorem 5.1 *Let $F \in V(\tau)$, conditions (1.3), (1.6), (1.7), (2.4) be fulfilled and for any $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ even and $\lambda \in [\ell-1, \ell]$ conditions (2.2_ℓ), (2.3_ℓ) and (3.1_ℓ) hold. If moreover, (2.3_0) when n is even, and (2.2_n) hold then equation (1.1) has Property* **B**.

Theorem 5.2 *Let $F \in V(\tau)$, conditions (1.3), (1.6), (1.7), (2.4), (2.7) be fulfilled and for any $\ell \in \{1, \dots, n-1\}$ with $\ell+n$ even and $\lambda \in [\ell-1, \ell]$ conditions (2.2_ℓ), (2.3_ℓ) and (3.2_ℓ) hold. If moreover, (2.3_0) when n is even, and (2.2_n) hold then equation (1.1) has Property* **B**.

Theorem 5.3 *Suppose $F \in V(\tau)$, conditions (1.3), (4.1), (4.2) be fulfilled and*

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t s^{n+1}\Big(\prod_{i=1}^{m} p_i(s)\Big)^{\frac{1}{m}} ds > \frac{1}{m}\max\Big(-\prod_{i=1}^{m}\big(\beta_i^{1+\lambda} - \alpha_i^{1+\lambda}\big)^{-\frac{1}{m}} \times$$
$$\times e^{\lambda d}(1+\lambda)\lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda \in [0, n-1]\Big).$$

Then equation (1.1) has Property **B**.

Theorem 5.4 *Suppose $F \in V(\tau)$, conditions (1.3), (4.3), (4.4) be fulfilled and*

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t s^n\Big(\prod_{i=1}^{m} p_i(s)\Big)^{\frac{1}{m}} ds >$$
$$> \frac{1}{m}\max\Big(-\prod_{i=1}^{m}\alpha_i^{-\frac{\lambda}{m}} \cdot e^{\lambda d} \cdot \lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda \in [0, n-1]\Big).$$

Then equation (1.1) has Property **B**.

R E F E R E N C E S

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